RESEARCH Revista Mexicana de Física **64** (2018) 539–547 SEPTEMBER-OCTOBER 2018

Coupled reaction-diffusion waves in a chemical system via fractional derivatives in Liouville-Caputo sense

Khaled M. Saad[a,b,*], J.F. Gómez-Aguilar[c,**]
[a]*Department of Mathematics, Faculty of Arts and Sciences, Najran University, Saudi Arabia.*
**e-mail: khaledma_sd@hotmail.com*
[b]*Department of Mathematics, College of Applied Science, Taiz University, Taiz, Yemen.*
[c]*CONACyT-Tecnológico Nacional de México/CENIDET, Interior Internado Palmira S/N, Col. Palmira, 62490, Cuernavaca, Morelos, México.*
***e-mail: jgomez@cenidet.edu.mx*

Received 19 April 2018; accepted 26 May 2018

In this paper, we have generalized the fractional cubic isothermal auto-catalytic chemical system (FCIACS) with Liouville-Caputo, Caputo-Fabrizio-Caputo, and Atangana-Baleanu-Caputo fractional time derivatives, respectively. We apply the Homotopy Analysis Transform Method (HATM) to compute the approximate solutions of FCIACS using these fractional derivatives. We study the convergence analysis of HATM by computing the residual error function. Also, we find the optimal values of h so we assure the convergence of the approximate solutions. Finally we show the behavior of the approximate solutions graphically. The results obtained are very effectiveness and accuracy.
2010 Mathematics Subject Classification: 34A34, 35R11, 65M12, 26A33, 34A08, 35-XX.

Keywords: Fractional isothermal auto-catalytic chemical systems; HATM; Liouville-Caputo fractional derivative; Caputo-Fabrizio fractional derivative; Atangana-Baleanu fractional derivative; h-curves.

PACS: 02.30.Jr; 03.65.Fd; 04.20.Jb

1. Introduction

Travelling reaction-diffusion waves occur in many situations of interest in chemistry, biology, physics, engineering. In some cases, such waves are isolated events, travelling independently of other chemical processes. Many chemical systems exhibit chemical waves, *i.e.*, reactants are converted into products as the front propagates through the reaction mixture, in which autocatalytic reaction couples with molecular diffusion to give constant waveform and constant velocity fronts [1-3].

In this paper we consider reaction-diffusion travelling waves that can be initiated in a coupled isothermal chemical system governed by cubic autocatalysis. We assumed that reactions take place along semipermeable membrane interfaces with the reaction on one interface (region I). The cubic isothermal, auto-catalytic reaction step in region (I) is given by

$$U + V \to 2V(\text{rate } r_1uv^2), \tag{1}$$

with the step of the linear decay

$$V \to W(\text{rate } r_2v), \tag{2}$$

where u and v are the concentrations of the reactant U and auto-catalyst V, r_1 and r_2 are the rate constants and W is some inert product of reaction. The non-dimensional equations are given by

$$\frac{\partial u_1}{\partial \xi} = \frac{\partial^2 u_1}{\partial \varsigma^2} - u_1v_1^2 + \gamma(u_2 - u_1), \tag{3}$$

$$\frac{\partial v_1}{\partial \xi} = \frac{\partial^2 v_1}{\partial \varsigma^2} + u_1v_1^2 - kv_1, \tag{4}$$

$$\frac{\partial u_2}{\partial \xi} = \frac{\partial^2 u_2}{\partial \varsigma^2} - u_2v_2^2 + \gamma(u_1 - u_2), \tag{5}$$

$$\frac{\partial v_2}{\partial \xi} = \frac{\partial^2 v_2}{\partial \varsigma^2} + u_2v_2^2, \tag{6}$$

with the boundary conditions

$$\lim_{\varsigma \to \pm\infty} u_i(\varsigma, \xi) = 1, \qquad \lim_{\varsigma \to \pm\infty} v_i(\varsigma, \xi) = 0. \tag{7}$$

In the above equations we assume that cubic autocatalytic is only present in the other region II with the same rate r_1. The two regions were considered to be coupled through a linear diffusive interchange of the auto-catalytic V. The parameters γ and k refer to the couple between the two regions and the strength of the auto-catalyst decay [4].

In fractional differentiation analysis, there are many different definitions of fractional derivatives. The Liouville-Caputo fractional derivative involve the convolution of the local derivative of a given function with power law function [5]. Recently, Caputo and Fabrizio in [6], proposed a novel fractional derivative based on the exponential decay law without singularities [7-10]. Atangana and Baleanu in [11], introduced a fractional derivative based in the Mittag-Leffler law (this function is of course the more generalized exponential function) and permits describe complex physical problems that follows at the same time the power and exponential decay law [12-14].

Many numerical methods for solving fractional differential equations have been developed over the past few years, such as homotopy analysis method, proposed by Liao, has been successfully applied to solving many problems in physics and engineering [15-18]. The homotopy analysis method is based on construction of a homotopy which continuously deforms an initial guess approximation to the exact solution of the given problem. Another powerful methods for finding exact solutions have been found in [19-24].

In this paper we obtain analytical approximate solutions to fractional cubic isothermal auto-catalytic chemical system model by applying Liouville-Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional order derivatives in Liouville-Caputo sense using HATM.

2. Basic definitions

Fractional calculus unifies and generalizes the notions of integer-order differentiation. Now, we give some basic definitions and properties of FC theory.

Definition 1. The Liouville-Caputo operator (C) with order $(\alpha > 0)$ is defined as follows [5]

$$ {}^{C}\left({}_0D_\theta^\beta(\cdot)\right) = \frac{1}{\Gamma(m-\beta)}\int_0^\theta (\theta-t)^{m-\beta-1}D^{(m)}(\cdot)\,dt, \quad m-1<\beta\le m, \tag{8}$$

for $m\in\mathbb{N}$, $t>0$, $f\in C_\mu^m$, $\mu\ge -1$.

Definition 2. The Caputo-Fabrizio fractional order derivative in the Liouville-Caputo sense (CFC) with order $(\alpha > 0)$ is given by [6]

$$ {}^{CFC}\left({}_0D_\theta^\beta(\cdot)\right) = \frac{M(\beta)}{m-\beta} \times \int_0^\theta \exp\left(-\frac{\beta}{m-\beta}(\theta-t)\right)D^{(m)}(\cdot)dt, \quad m-1<\beta\le m, \tag{9}$$

where $M(\beta)$ is a constant of normalization that depend of β, which satisfies that, $M(0)=M(1)=1$.

Definition 3. The Atangana-Baleanu fractional derivative in the Liouville-Caputo sense (ABC) with order $(\alpha > 0)$ is defined as follows [11]

$$ {}^{ABC}\left({}_0D_\theta^\beta(\cdot)\right) = \frac{M(\beta)}{m-\beta}\int_0^\theta E_\beta \times\left(-\frac{\beta}{m-\beta}(\theta-t)\right)D^{(m)}(\cdot)dt, \quad m-1<\beta\le m, \tag{10}$$

where

$$E_\beta(\Xi)=\sum_{k=0}^{\infty}\frac{\Xi^k}{\Gamma(\beta k+1)}$$

is the Mittag-Leffler function and $M(\beta)=M(0)=M(1)=1$.

If, $0<\beta\le 1$, then we define the Laplace transform for the Liouville-Caputo, Caputo-Fabrizio-Caputo and the Atangana-Baleanu-Caputo fractional derivatives, respectively as follows

$$\mathcal{L}\left({}_0^CD_\xi^\beta\{u(\varsigma,\xi)\}\right) = s^\beta\mathcal{L}\{u(\varsigma,\theta)\} - \sum_{k=0}^{m-1}u^{(k)}(\varsigma,0^+)s^{\beta-k-1}, \tag{11}$$

$$\mathcal{L}\left({}_0^{CFC}\mathcal{D}_\xi^\beta\{u(\varsigma,\xi)\}\right) = M(\beta) \times\left(\frac{s\mathcal{L}\{u(\varsigma,\xi)\}(s)-u(\varsigma,0)}{s+\beta(1-s)}\right), \tag{12}$$

$$\mathcal{L}\left({}_0^{ABC}D_\xi^\beta\{u(\varsigma,\xi)\}\right) = M(\beta) \times\left(\frac{s^\beta\mathcal{L}\{u(\varsigma,\xi)\}(s)-s^{\beta-1}\{u(\varsigma,\xi)\}}{s^\beta(1-\beta)+\beta}\right). \tag{13}$$

Considering these fractional order derivatives, we develop a new model FCIACS by replacing partial derivatives with respect to ξ by time fractional derivatives of order β. Then the set of the Eqs. (3)-(6) become

$${}_0^{(\cdot)}D_\xi^\beta u_1 = u_{1,\varsigma\varsigma}-u_1v_1^2+\gamma(u_2-u_1), \tag{14}$$

$${}_0^{(\cdot)}D_\xi^\beta v_1 = v_{1,\varsigma\varsigma}+u_1v_1^2-kv_1, \tag{15}$$

$${}_0^{(\cdot)}D_\xi^\beta u_2 = u_{2,\varsigma\varsigma}-u_2v_2^2+\gamma(u_1-u_2), \tag{16}$$

$${}_0^{(\cdot)}D_\xi^\beta v_2 = v_{2,\varsigma\varsigma}+u_2v_2^2, \tag{17}$$

where the operator ${}_0^{(\cdot)}D_\xi^\beta$ can be of type Liouville-Caputo ${}_0^CD_\xi^\beta$, Caputo-Fabrizio-Caputo ${}_0^{CFC}D_\xi^\beta$, and Atangana-Baleanu-Caputo ${}_0^{ABC}D_\xi^\beta$ time fractional derivatives with order β.

3. Solution of the problem

In this section, we apply the HATM [25-26] on FCIACS model. We take the initial conditions to satisfy the boundary conditions, namely

$$u_i(\varsigma,0)=1-a_i\ \exp(-\varsigma^2), \tag{18}$$

$$v_i(\varsigma,0)=b_i\ \exp(-\varsigma^2), \tag{19}$$

where $i=1,2$.

As we know that HAM is based on a particular type of continuous mapping [27-32]

$$u_i(\varsigma,\xi) \to \phi_i(\varsigma,\xi;\varrho), \qquad v_i(\varsigma,\xi) \to \psi_i(\varsigma,\xi;\varrho), \tag{20}$$

such that, as the embedding parameter q increases from 0 to 1; $\phi_i(\varsigma,\xi;\varrho)$, $\psi_i(\varsigma,\xi;\varrho)$ varies from the initial iteration to the exact solution.

Involving Eqs. (8)–(13) and [27-32] we can present the following nonlinear operators as

$$\begin{aligned}
\mathcal{N}_1(\phi_1(\varsigma,\xi;\varrho)) &= \mathcal{L}\left(\phi_1(\varsigma,\xi;\varrho)\right) - \frac{1}{s}u_1(\varsigma,0) \\
&- \Omega^{(\cdot)}(s,\beta)\mathcal{L}\Big(\phi_{1,\varsigma\varsigma}(\varsigma,\xi;\varrho) \\
&- \phi_1(\varsigma,\xi;\varrho)\psi_1^2(\varsigma,\xi;\varrho) + \gamma(\phi_2(\varsigma,\xi;\varrho) \\
&- \phi_1(\varsigma,\xi;\varrho)\Big), \\
\mathcal{M}_1(\psi_1(\varsigma,\xi;\varrho)) &= \mathcal{L}\left(\psi_1(\varsigma,\xi;\varrho)\right) \\
&- \frac{1}{s}v_1(\varsigma,0) - \Omega^{(\cdot)}(s,\beta)\mathcal{L}\big(\psi_{1,\varsigma\varsigma}(\varsigma,\xi;\varrho) \\
&- k\psi_1(\varsigma,\xi;\varrho) + \phi_1(\varsigma,\xi;\varrho)\psi_1^2(\varsigma,\xi;\varrho)\Big), \\
\mathcal{N}_2(\phi_2(\varsigma,\xi;\varrho)) &= \mathcal{L}\left(\phi_2(\varsigma,\xi;\varrho)\right) \\
&- \frac{1}{s}u_2(\varsigma,0) - \Omega^{(\cdot)}(s,\beta)\mathcal{L}\Big(\phi_{2,\varsigma\varsigma}(\varsigma,\xi;\varrho) \\
&- \phi_2(\varsigma,\xi;\varrho)\psi_2^2(\varsigma,\xi;\varrho) \\
&+ \gamma(\phi_1(\varsigma,\xi;\varrho) - \phi_2(\varsigma,\xi;\varrho)\Big), \\
\mathcal{M}_2(\psi_2(\varsigma,\xi;\varrho)) &= \mathcal{L}\left(\psi_2(\varsigma,\xi;\varrho)\right) - \frac{1}{s}v_2(\varsigma,0) \\
&- \Omega^{(\cdot)}(s,\beta)\mathcal{L}\Big(\psi_{2,\varsigma\varsigma}(\varsigma,\xi;\varrho) \\
&+ \phi_2(\varsigma,\xi;\varrho)\psi_2^2(\varsigma,\xi;\varrho)\Big),
\end{aligned} \tag{21}$$

where $\Omega^{(\cdot)}(s,\beta)$ can be of type Liouville-Caputo

$$\Omega^{(C)}(s,\beta) = \frac{1}{s^\beta},$$

Caputo-Fabrizio-Caputo

$$\Omega^{(CFC)}(s,\beta) = \frac{\beta(1-s)+s}{sM(\beta)}$$

and Atangana-Baleanu-Caputo

$$\Omega^{(ABC)}(s,\beta) = \frac{\beta\left(-1+s^{-\beta}\right)+1}{M(\beta)}.$$

Using the embedding parameter ϱ, we develop the following set of equations

$$\begin{aligned}
&(1-\varrho)\mathcal{L}(\phi_i(\varsigma,\xi;\varrho) - u_{i,0}(\varsigma,\xi)) \\
&\qquad = \varrho hH(\varsigma,\xi)\mathcal{N}_i(\phi_i(\varsigma,\xi;\varrho)), \\
&(1-\varrho)\mathcal{L}(\psi_i(\varsigma,\xi;\varrho) - v_{i,0}(\varsigma,\xi)) \\
&\qquad = \varrho hH(\varsigma,\xi)\mathcal{M}_i(\psi_i(\varsigma,\xi;\varrho)),
\end{aligned} \tag{22}$$

with initial conditions

$$\phi_i(\varsigma,0;\varrho) = u_{i,0}(\varsigma,0), \qquad \psi_i(\varsigma,0;\varrho) = v_{i,0}(\varsigma,0),$$

where $h \neq 0$ is the auxiliary parameter and $H(\varsigma,\xi) \neq 0$ is the auxiliary function.

Expanding in Taylor series $\phi_i(\varsigma,\xi;\varrho)$ and $\psi_i(\varsigma,\xi;\varrho)$ with respect to ϱ, we get

$$\begin{aligned}
\phi_i(\varsigma,\xi;\varrho) &= u_{i,0}(\varsigma,\xi) + \sum_{j=1}^{\infty} u_{i,j}(\varsigma,\xi)\varrho^j, \\
\psi_i(\varsigma,\xi;\varrho) &= v_{i,0}(\varsigma,\xi) + \sum_{j=1}^{\infty} v_{i,j}(\varsigma,\xi)\varrho^j,
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
u_{i,j}(\varsigma,\xi) &= \frac{1}{j!}\frac{\partial^j\phi_i(\varsigma,\xi;\varrho)}{\partial\varrho^j}|_{\varrho=0}, \\
v_{i,j}(\varsigma,\xi) &= \frac{1}{j!}\frac{\partial^j\psi_i(\varsigma,\xi;\varrho)}{\partial\varrho^j}|_{\varrho=0}.
\end{aligned} \tag{24}$$

If we let $\varrho = 1$ in Eq. (23), the series become

$$\begin{aligned}
u_i(\varsigma,\xi) &= u_{i,0}(\varsigma,\xi) + \sum_{j=1}^{\infty} u_{i,j}(\varsigma,\xi), \\
v_i(\varsigma,\xi) &= v_{i,0}(\varsigma,\xi) + \sum_{j=1}^{\infty} v_{i,j}(\varsigma,\xi).
\end{aligned} \tag{25}$$

Considering [25-26], the mth-order deformation equation is constructed of the following manner

$$\begin{aligned}
\mathcal{L}(u_{i,j}(\varsigma,\xi) - \mathcal{X}_j u_{i,(j-1)}(\varsigma,\xi)) &= hH(\varsigma,\xi)R_j^{(\cdot)}(u_i), \\
\mathcal{L}(v_{i,j}(\varsigma,\xi) - \mathcal{X}_j v_{i,(j-1)}(\varsigma,\xi)) &= hH(\varsigma,\xi)R_j^{(\cdot)}(v_i),
\end{aligned} \tag{26}$$

and

$$\mathcal{X}_j = \begin{cases} 0 & \text{if } j \leq 1, \\ 1 & \text{if } j > 1. \end{cases}$$

with initial conditions $u_{i,j}(\varsigma,0) = 0$ and $v_{i,j}(\varsigma,0) = 0$, for $j > 1$

$$\begin{aligned}
R_j^{(\cdot)}(u_1) &= \mathcal{L}\left(u_{1,(j-1)}(\varsigma,\xi)\right) - \frac{1}{s}u_1(\varsigma,0)(1-\mathcal{X}_j) \\
&- \Omega^{(\cdot)}(s,\beta)\mathcal{L}\Big(u_{1,(j-1),\varsigma\varsigma}(\varsigma,t) \\
&- u_{1,(j-1)}(\varsigma,\xi)v_{1,(j-1)}^2(\varsigma,\xi) + \gamma(u_{2,(j-1)}(\varsigma,\xi) \\
&- u_{1,(j-1)}(\varsigma,\xi))\Big),
\end{aligned}$$

$$
\begin{aligned}
R_j^{(\cdot)}(v_1) &= \mathcal{L}_{(j-1)}\left(v_{1,(j-1)}(\varsigma,\xi)\right) - \frac{1}{s}v_1(\varsigma,0)(1-\mathcal{X}_j) \\
&\quad - \Omega^{(\cdot)}(s,\beta)\mathcal{L}\Big(v_{1,(j-1),\varsigma\varsigma}(\varsigma,\xi) + u_{1,(j-1)}(\varsigma,\xi)v^2_{1,(j-1)}(\varsigma,\xi) - kv_{1,(j-1)}(\varsigma,\xi)\Big). \\
R_j^{(\cdot)}(u_2) &= \mathcal{L}\left(u_{2,(j-1)}(\varsigma,\xi)\right) - \frac{1}{s}u_2(\varsigma,0)(1-\mathcal{X}_j) \\
&\quad - \Omega^{(\cdot)}(s,\beta)\mathcal{L}\Big(u_{2,(j-1),\varsigma\varsigma}(\varsigma,t) - u_{2,(j-1)}(\varsigma,\xi)v^2_{2,(j-1)}(\varsigma,\xi) + \gamma(u_{1,(j-1)}(\varsigma,\xi) - u_{2,(j-1)}(\varsigma,\xi))\Big), \\
R_j^{(\cdot)}(v_2) &= \mathcal{L}_{(j-1)}\left(v_{2,(j-1)}(\varsigma,\xi)\right) - \frac{1}{s}v_2(\varsigma,0)(1-\mathcal{X}_j) \\
&\quad - \Omega^{(\cdot)}(s,\beta)\mathcal{L}\Big(v_{2,(j-1),\varsigma\varsigma}(\varsigma,\xi) + u_{2,(j-1)}(\varsigma,\xi)v^2_{2,(j-1)}(\varsigma,\xi)\Big). \qquad (27)
\end{aligned}
$$

Applying inverse Laplace transform, we have

$$
u_{i,j} = \mathcal{X}_j u_{i,(j-1)} + h\mathcal{L}^{-1}R_j^{(\cdot)}(u_i), \qquad v_{i,j} = \mathcal{X}_j v_{i,(j-1)} + h\mathcal{L}^{-1}R_j^{(\cdot)}(v_i). \qquad (28)
$$

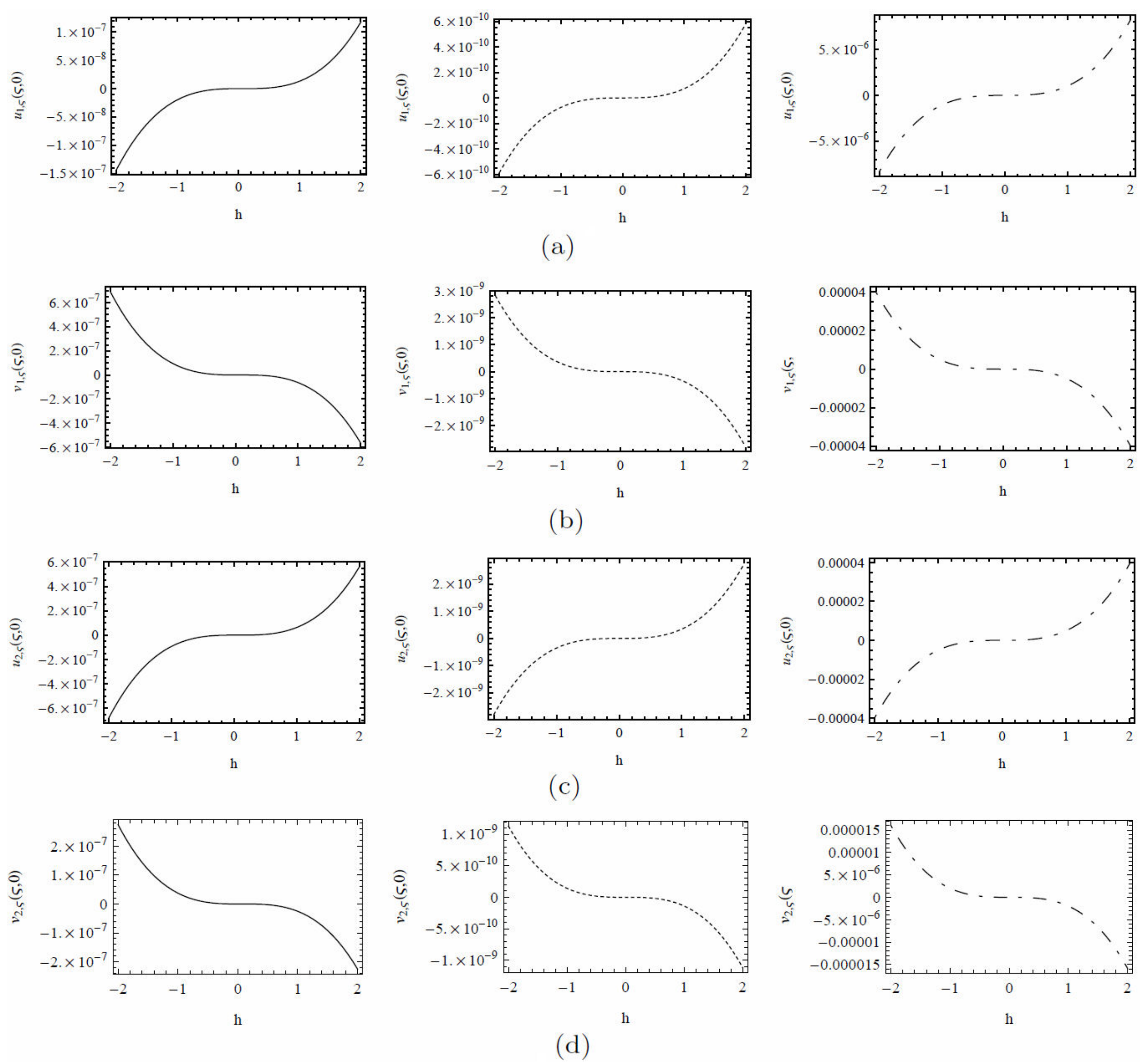

FIGURE 1. Plotting the h-curves for 5-terms of HATM solutions using the C, CFC and ABC operators with $\beta = 0.7$, $k = 0.01$, $\gamma = 0.4$, $\varsigma = 6$, $\xi = 0$, $a_1 = 0.2$, $b_1 = 0.1$, $a_2 = 1$ and $b_2 = 0.4$. Solid line (C), Dotted line (CFC), and Dash - Dotted line (ABC).

4. Numerical results

In this section we evaluate the first approximations for the Liouville-Caputo, Caputo-Fabrizio-Caputo and Atangana-Baleanu-Caputo operators respectively. The intervals of convergence obtained by the h-curves, the averaged residual error, and the residual error function were evaluated. Furthermore, we will show the behavior of the HATM solutions for different values of fractional derivative β.

We take the initial approximation as

$$u_{i,0}(\varsigma,\xi)=u_{i,0}(\varsigma,0),\qquad v_{i,0}(\varsigma,\xi)=v_{i,0}(\varsigma,0). \tag{29}$$

For $j=1$, we obtain the first approximation as following

$$\begin{aligned}u^{(\cdot)}_{i,1}(\varsigma,\xi)=h_i\mathcal{L}^{-1}\Bigg(&\mathcal{L}\left(u_{i,0}(\varsigma,\xi)\right)-\frac{1}{s}u_i(\varsigma,0)(1-\mathcal{X}_1)\\&-\Omega^{(\cdot)}(s,\beta)\mathcal{L}\left(u_{i,0,\varsigma\varsigma}(\varsigma,\xi)-u_{i,0}(\varsigma,\xi)v^2_{i,0}(\varsigma,\xi)\right)\Bigg)\\&+(-1)^i\gamma(u_{1,0}(\varsigma,\xi)-u_{2,0}(\varsigma,\xi)),\end{aligned} \tag{30}$$

$$\begin{aligned}v^{(\cdot)}_{i,1}(\varsigma,\xi)=h_i\mathcal{L}^{-1}\Bigg(&\mathcal{L}(v_{i,0}(\varsigma,\xi))-\frac{1}{s}v(\varsigma,0)(1-\mathcal{X}_j)\\&-\Omega^{(\cdot)}(s,\beta)\mathcal{L}(v_{i,0,\varsigma\varsigma}(\varsigma,\xi)-(2-i)kv_0(\varsigma,\xi)\\&+u_{i,0}(\varsigma,\xi)v^2_{i,0}(\varsigma,\xi))\Bigg).\end{aligned} \tag{31}$$

We can obtain the first approximation via Liouville-Caputo, Caputo-Fabrizio-Caputo and Atangana-Baleanu-Caputo operators, with $\Omega^{C}(s,\beta)$, $\Omega^{CFC}(s,\beta)$ and $\Omega^{ABC}(s,\beta)$, respectively.

And by the similar procedure we can evaluate the rest of the approximations. We therefore have HATM solutions of Eqs. (14)-(17)

$$u^{(\cdot)}_{i,m}(\varsigma,\xi)=u_{i,0}(\varsigma,\xi)+\sum_{j=1}^{m}\frac{u_{i,j}(\varsigma,\xi)}{n^j}, \tag{32}$$

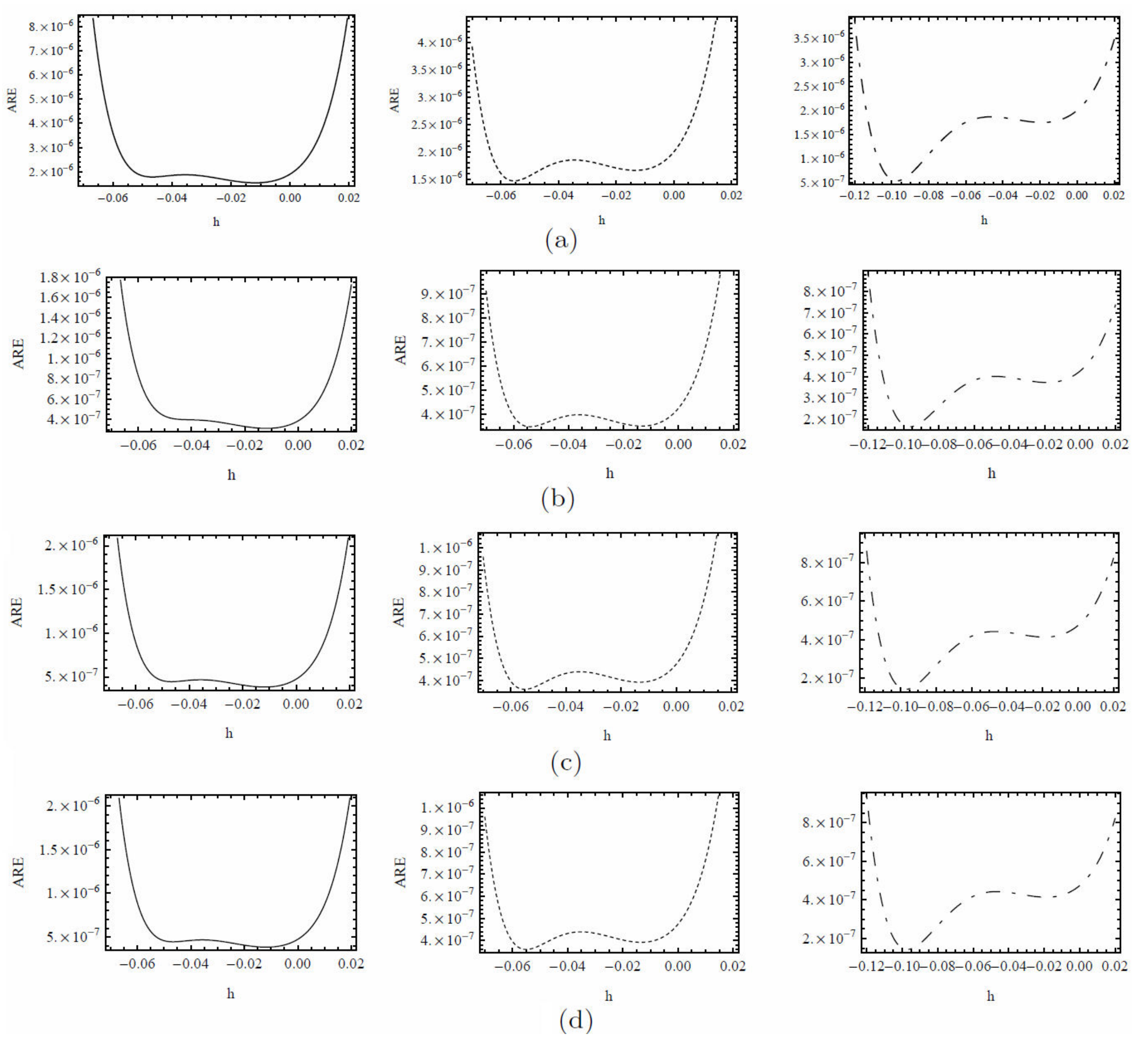

FIGURE 2. Plotting the average residual error for 5-terms of HATM solutions using the Liouville-Caputo, Caputo-Fabrizio-Caputo and Atangana-Baleanu-Caputo operators arranged from left to right with $\beta=0.7$, $0\le\varsigma,\xi\le 10$, $k=0.001$, $\gamma=0.4$, $a_1=0.002$, $b_1=0.002$, $a_2=0.001$, $b_2=0.001$.

TABLE I. The average residual error for 4-terms of HATM solutions with $\beta = 0.7, 0 \leq \varsigma, \xi \leq 10, k = 0.001, \gamma = 0.4, a_1 = 0.002, b_1 = 0.002, a_2 = 0.001, b_2 = 0.001$, using the C, CFC and ABC operators, respectively.

Operators	Optimal value of h_{u_1}	Minimum of $E_{u_1}(h)$
C	-0.011844	1.542×10^{-6}
CFC	-0.055422	1.465×10^{-6}
ABC	-0.0973563	5.431×10^{-7}
Operators	Optimal value of h_{v_1}	Minimum of $E_{v_1}(h)$
C	-0.011811	1.542×10^{-6}
CFC	-0.055427	1.466×10^{-6}
ABC	-0.097383	5.438×10^{-7}

TABLE II. The average residual error for 5-terms of HATM solutions with $\beta = 0.7, 0 \leq \varsigma, \xi \leq 10, k = 0.001, \gamma = 0.4, a_1 = 0.002, b_1 = 0.002, a_2 = 0.001, b_2 = 0.001$, using the C, CFC and ABC operators, respectively.

Operators	Optimal value of h_{u_2}	Minimum of $E_{u_2}(h)$
C	-0.011900	3.133×10^{-7}
CFC	-0.053997	3.486×10^{-7}
ABC	-0.096408	1.633×10^{-7}
Operators	Optimal value of h_{v_2}	Minimum of $E_{v_2}(h)$
C	-0.011807	3.834×10^{-7}
CFC	-0.055014	3.592×10^{-7}
ABC	-0.097085	1.432×10^{-7}

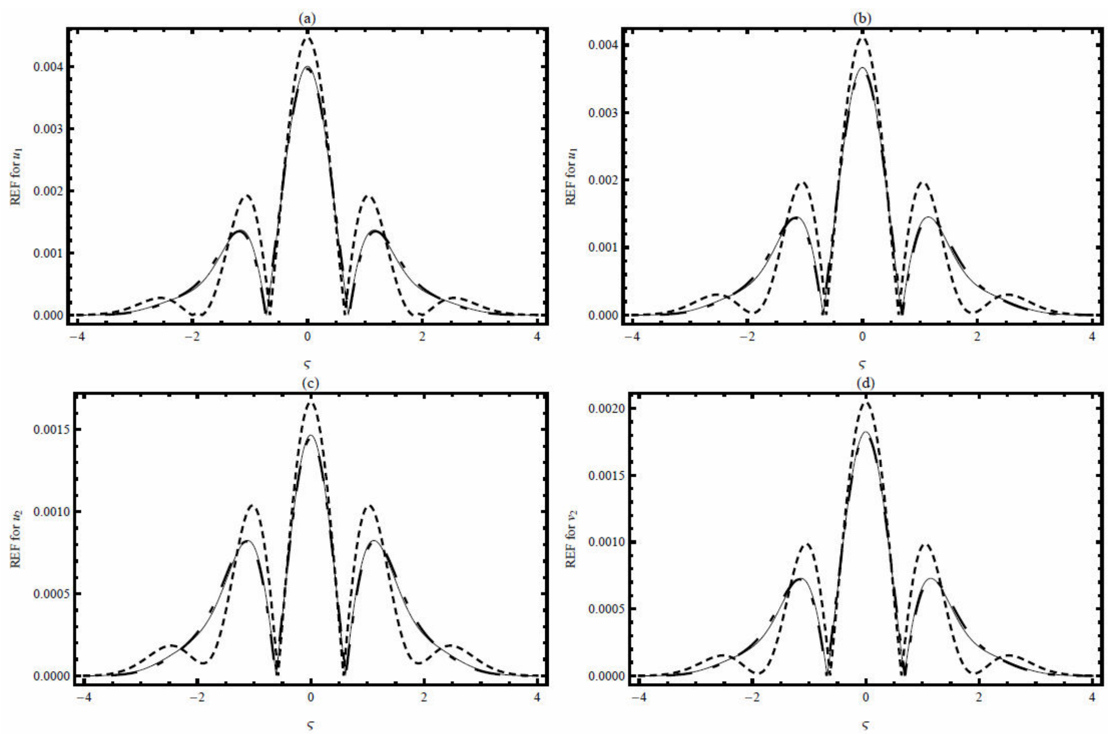

FIGURE 3. Plotting the residual error functions for 4-terms of HATM solutions with $\beta = 0.7$, $\xi = 20$, $k = 0.001$, $\gamma = 0.4$, $a_1 = 0.002$, $b_1 = 0.002$, $a_2 = 0.001$, $b_2 = 0.001$. Solid line (C), Dotted line (CFC), and Dash - Dotted line (ABC).

$$v_{i,m}^{(\cdot)}(\varsigma,\xi) = v_{i,0}(\varsigma,\xi) + \sum_{j=1}^{m} \frac{v_{i,j}(\varsigma,\xi)}{n^j}. \qquad (33)$$

Figures 1(a)-(d) shows the numerical solutions for $u_{i,\xi}(\varsigma,0)$, $v_{i,\xi}(\varsigma,0)$ against h with $\beta = 0.7$ $k = 0.01, \gamma = 0.4, \varsigma = 6, \xi = 0$, $a_1 = 0.2$, $b_1 = 0.1$, $a_2 = 1$ and $b_2 = 0.4$. We plot the h-curves of 5-terms of HATM solutions (32)–(33) with the aim to observe the intervals of convergence. In these figures, the straight line that parallels the h-axis provides the valid region of the convergence [30]. Now, we compute the optimal values of the convergence-control parameters by the minimum of the averaged residual errors [33-38].

$$E_{u_i}(h) = \frac{1}{(\Xi+1)(\Upsilon+1)} \times \sum_{s=0}^{\Xi}\sum_{j=0}^{\Upsilon}\left[\mathcal{N}\left(\sum_{k=0}^{m} u_{i,k}\left(\frac{10s}{\Xi}, \frac{10j}{\Upsilon}\right)\right)\right]^2, \qquad (34)$$

$$E_{v_i}(h) = \frac{1}{(\Xi+1)(\Upsilon+1)} \times \sum_{s=0}^{\Xi}\sum_{j=0}^{\Upsilon}\left[\mathcal{M}\left(\sum_{k=0}^{m} v_{i,k}\left(\frac{10s}{\Xi}, \frac{10j}{\Upsilon}\right)\right)\right]^2, \qquad (35)$$

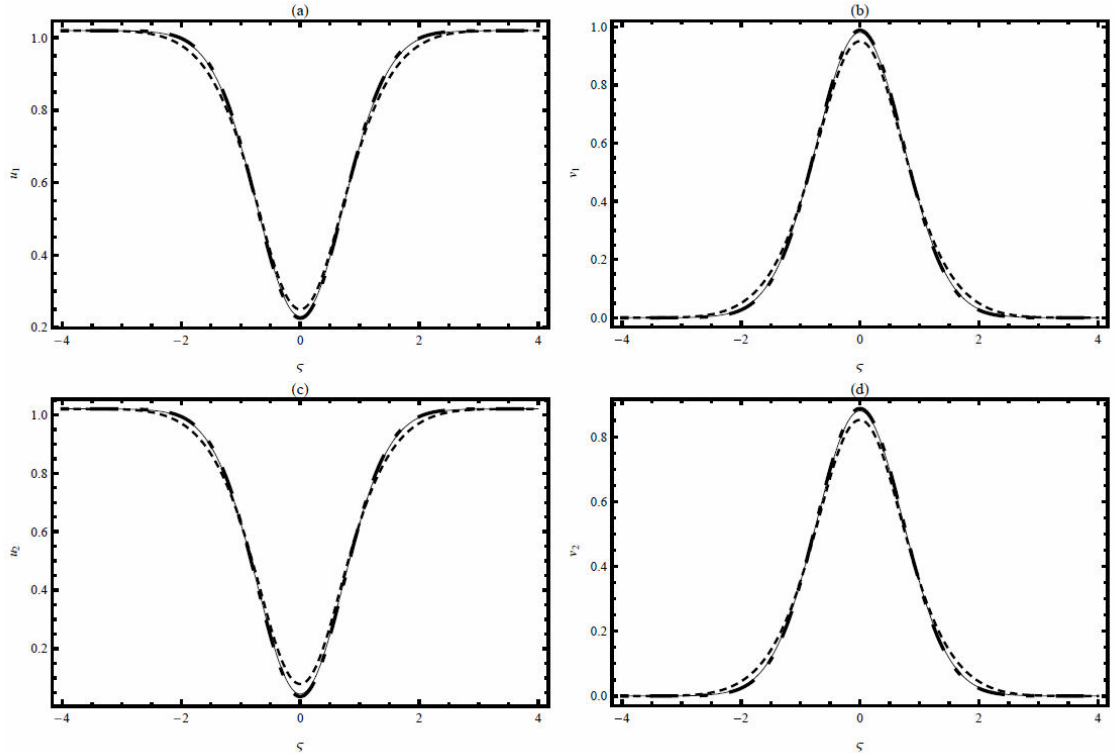

FIGURE 4. The plot of 4-terms of HATM solutions using LC, CFC and ABC operators with $\beta = 0.4$, $k = 0.01$, $\gamma = 0.6$, $\xi = 15$, $a_1 = 0.8$, $b_1 = 1$, $a_2 = 1$, $b_2 = 0.9$. Solid line (C), Dash line (CFC) and Dash-Dot-Dash line (ABC).

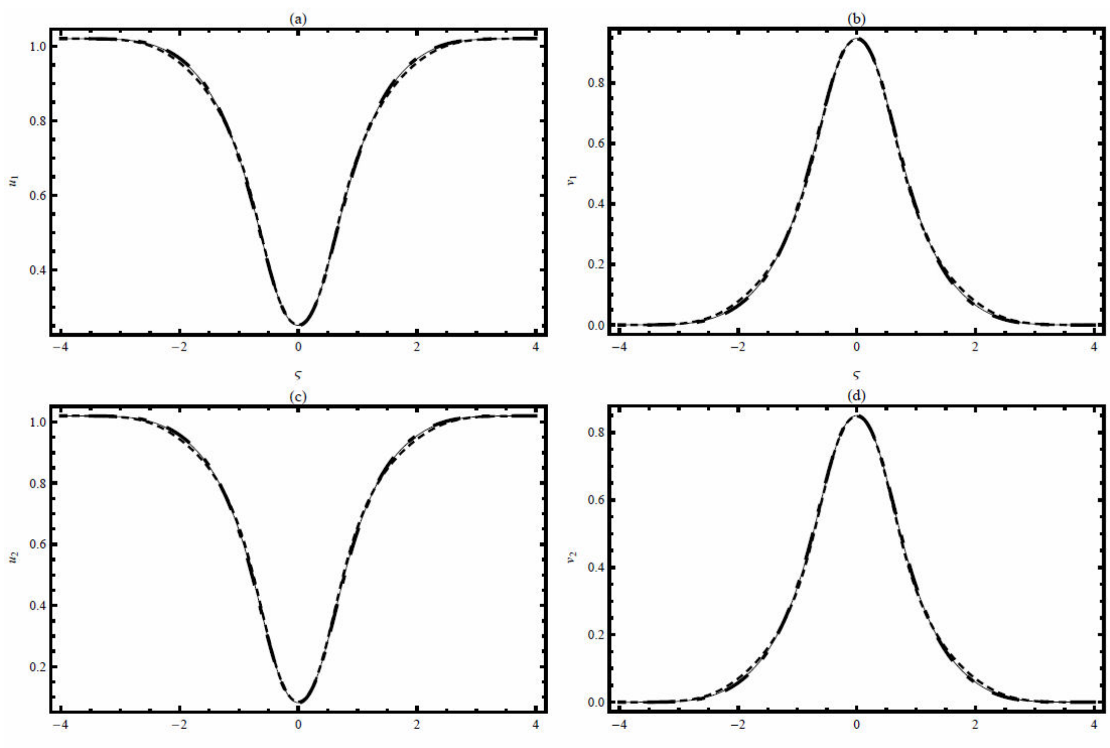

FIGURE 5. The plot of 4-terms of HATM solutions using LC,CFC and ABC operators with $\beta = 0.9$, $k = 0.01$, $\gamma = 0.6$, $\xi = 15$, $a_1 = 0.8$, $b_1 = 1$, $a_2 = 1$, $b_2 = 0.9$. Solid line: (C), Dash line: (CFC), and Dash-Dot-Dash line:(ABC).

corresponding to a nonlinear algebraic equations

$$\frac{dE_{u_i}(h)}{dh} = 0, \qquad \frac{dE_{v_i}(h)}{dh} = 0. \tag{36}$$

Figure 2(a)-(d) and Tables I-II show the averaged residual error for the Liouville-Caputo (C), Caputo-Fabrizio-Caputo (CFC), and Atangana-Baleanu-Caputo (ABC) operators. These figures show the $E_{u_i}(h)$ and $E_{v_i}(h)$ for 4-terms obtained with HATM. Solutions we set into (34) - (35) $\Xi = 10$ and $\Upsilon = 10$ with $k = 0.001$, $\gamma = 0.4$, $a_1 = 0.002, b_1 = 0.002, a_2 = 0.001$ and $b_2 = 0.001$. Using the command "Minimize" of Mathematica we plotting the residual error against h to get the optimal values h. From Fig. 2 and Tables I-II, we observe the average residual error of order $10^{-6} - 10^{-7}$. This observation assures that the HATM solutions for C, CFC and ABC are converging very rapidly. Figure 3(a)-(d) shows the residual errors functions with C, CFC and ABC operators for (14)-(15) at $\beta = 0.7$ It can be seen from these figures the order of REF are very small for all operators. Of Course, we can not say which the better?, due to the operators have a different kernel.

Finally we plot the HATM solutions for C, CFC and ABC fractional derivatives for different values of β. Figures 4-5 show the behavior of the new models with C, CFC and ABC operators for $\beta = 0.4$, and 0.9. From these figures, we noted that these new operators identical as the fractional order approaches from the integer order.

5. Conclusion

In this paper, HATM was employed analytically to compute the approximate solutions of FCIACS using the Liouville-Caputo, Caputo-Fabrizio-Caputo and Atangana-Baleanu-Caputo fractional derivatives. The interval of the convergence of HATM and optimal value of h were compute. Also the residual error functions were obtained. The order of the average residual error and residual error functions indicate that the approximations that have been calculated by HATM with C, CFC and ABC fractional derivatives to the accuracy and effectiveness of our results.

Acknowledgments

José Francisco Gómez Aguilar acknowledges the support provided by CONACyT: Cátedras CONACyT para jóvenes investigadores 2014. José Francisco Gómez Aguilar acknowledges the support provided by SNI-CONACyT.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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