The Riemann-Silberstein vector in the Dirac algebra

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It is shown that the Riemann-Silberstein vector, defined as $\mathbf{E} + i\mathbf{B}$, appears naturally in the $SL(2, C)$ algebraic representation of the electromagnetic field. Accordingly, a compact form of the Maxwell equations is obtained in terms of Dirac matrices, in combination with the null-tetrad formulation of general relativity. The formalism is fully covariant; an explicit form of the covariant derivatives is presented in terms of the Fock coefficients.

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1. Introduction

The Riemann-Silberstein (RS) vector is defined as the complex sum of the electric and magnetic field vectors: $\mathbf{E} + i\mathbf{B}$. It appeared in 1907 in an article by Silberstein [1], and was applied many years later by various authors to different problems [2–4] (see, in particular, Ref. [3] for a historical account and full bibliography).

The RS vector appears conspicuously in electromagnetism because it describes the electromagnetic field in a particular representation of the Lorentz group, namely an irreducible representation of the $SL(2, C)$ group. This is evident if the method of spin coefficients is applied to the Maxwell equations.

As for the applications of spinor algebra, several formulation of the Maxwell and Einstein equations have been proposed following the pioneering article of Newman and Penrose [5]. A particulary compact formulation was worked out by Plebanski [6] in the seventies, based on the use of a null tetrad as a system of reference (see Ernst [7] for its relation with other authors formulations).

The aim of the present article is to further elucidate the role of the RS vector in the context of spinorial calculus. For this purpose, the null tetrad formalism of general relativity is used in combination with Dirac spinors, *i.e.* four-components spinors, and the related matrices of the Dirac algebra. The RS can thus be identified as the spinorial image of the electromagnetic field in this particular representation. Being fully covariant, our approach is valid in any Riemannian space-time. Furthermore, it generalizes to the Maxwell equations a previous work on the Dirac equation in curved space-time [8]. The result is a particularly compact and covariant form of the Maxwell equations that can be used in combination with the Dirac equation in problems of general relativity.

2. Maxwell equations and Dirac matrices

The Dirac matrices γ^α are such that

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = -2g^{\alpha\beta}, \qquad (1)$$

where $g^{\alpha\beta}$ is the metric tensor (signature $\{- + ++\}$ and $c = 1$, in the following)

In the chiral gauge, for instance, they take the form

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & -I \\ -I & \mathbf{0} \end{pmatrix}, \qquad (2)$$

$$\gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \qquad (3)$$

where σ^i are the usual Pauli matrices.

Let A_μ be the electromagnetic potential and $f_{\alpha\beta} = \partial_\beta A_\alpha - \partial_\alpha A_\beta$ the electromagnetic tensor. In flat space and Cartesian coordinates ($f_{01} = E_x,\ f_{12} = -B_z$, etc.), we have

$$\gamma^\mu A_\mu \equiv \not{A} = \begin{pmatrix} \mathbf{0} & A^0 + \boldsymbol{\sigma} \cdot \mathbf{A} \\ A^0 - \boldsymbol{\sigma} \cdot \mathbf{A} & \mathbf{0} \end{pmatrix} \qquad (4)$$

and it follows from the definition of $\mathbf{E} = -(\partial/\partial t)\mathbf{A} - \nabla A^0$ and $\mathbf{B} = \nabla \times \mathbf{A}$ in terms of A^μ –which are equivalent to the homogeneous Maxwell equations–:

$$-\not{\partial}\not{A} = \begin{pmatrix} \boldsymbol{\sigma} \cdot (\mathbf{E} + i\mathbf{B}) & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\sigma} \cdot (\mathbf{E} - i\mathbf{B}) \end{pmatrix} \equiv \mathbb{F}, \qquad (5)$$

if the Lorentz gauge,

$$\nabla_\mu A^\mu = \frac{\partial}{\partial t} A^0 + \nabla \cdot \mathbf{A} = 0,$$

is used.

Accordingly, the inhomogeneous Maxwell equations take the compact form

$$-\not{\partial}\mathbb{F} = \not{\partial}\not{\partial}\not{A} = 4\pi \not{J}, \qquad (6)$$

where $J^\mu = (\rho, \mathbf{J})$ is the electromagnetic current

We thus see that the RS vector, defined as

$$\mathbf{F} \equiv \mathbf{E} + i\mathbf{B}, \qquad (7)$$

appears naturally in the above representation of the electromagnetic field.

Defining

$$\sigma^{\alpha\beta} = \frac{1}{2}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha),$$

it simply follows that

$$\frac{1}{2} f_{\mu\nu}\sigma^{\mu\nu} = \mathbb{F} = \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{F} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\sigma}\cdot\mathbf{F}^* \end{pmatrix}. \qquad (8)$$

The components of $f_{\mu\nu}$ will be identified in the context of the null-tetrad formalism (see below).

2.1. The RS vector

Defining the invariants of the field:

$$\mathbf{E}^2 - \mathbf{B}^2 \equiv \mathcal{E}^2 - \mathcal{B}^2, \quad \mathbf{E}\cdot\mathbf{B} \equiv \mathcal{E}\mathcal{B},$$

we have $\mathbf{F}^2 = (\mathcal{E} + i\mathcal{B})^2$, and it follows that

$$\mathbb{F}^2 = \begin{pmatrix} (\mathcal{E}+i\mathcal{B})^2 I & \mathbf{0} \\ \mathbf{0} & (\mathcal{E}-i\mathcal{B})^2 I \end{pmatrix}. \qquad (9)$$

Thus $\mathbb{F}^2$ is totally diagonal.

In the particular case of a null-electromagnetic field, $\mathcal{E} = 0 = \mathcal{B}$, the matrix $\mathbb{F}$ turns out to be nilpotent of degree 2: $\mathbb{F}^2 = 0$.

In the general case, the matrix $\mathbb{F}$ satisfies the equation

$$\mathbb{F}^4 - 2(\mathcal{E}^2 - \mathcal{B}^2)\mathbb{F}^2 + (\mathcal{E}^2 + \mathcal{B}^2)^2\mathbb{I} = 0, \qquad (10)$$

implying that the eigenvalues λ of $\mathbb{F}$ are

$$\lambda^2 = (\mathcal{E} \pm i\mathcal{B})^2.$$

Accordingly we have in general 4 eigenvalues, $\lambda_{(i)} = \pm(\mathcal{E} \pm i\mathcal{B})$, with 4 eigenfunctions $\psi_{(i)}$ such that

$$\mathbb{F}\psi_{(i)} = \lambda_{(i)}\psi_{(i)} \quad i = (1...4).$$

Furthermore, since $\mathbb{F}^2$ is completely diagonal, its eigenvectors can be taken as any set of four linearly independent spinors $u_{(i)}$, namely

$$\mathbb{F}^2 u_{(i)} = \lambda_{(i)}^2 u_{(i)}.$$

It then follows that, in general,

$$\psi_{(i)} = (\mathbb{F} + \lambda_{(i)})u_{(i)}, \qquad (11)$$

which can be interpreted as a generalization to Dirac spinors of the two-components Bloch spinors (if the $u_{(i)}$ are chosen as constant units spinors).

3. Null tetrad formalism

The null-tetrad is a set of null-vectors e^a_α defining one-forms $e^a = e^a_\mu\, dx^\mu$, such that e^1 and e^2 are complex conjugates to each other, e^3 and e^4 are real, and

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta = \eta_{ab}\, e^a e^b,$$

where

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

As shown in Ref. [8], a convenient choice of the Dirac matrices in the null-tetrad formalism is

$$\gamma^1 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^4 = \sqrt{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad (12)$$

satisfying the condition

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab}\mathbb{I}.$$

With the above choice of Dirac matrices, it follows that in (standard) Cartesian coordinates

$$\partial\!\!\!/ = \gamma^n\partial_n = \sqrt{2}\begin{pmatrix} 0 & 0 & \partial_4 & \partial_2 \\ 0 & 0 & \partial_1 & -\partial_3 \\ -\partial_3 & -\partial_2 & 0 & 0 \\ -\partial_1 & \partial_4 & 0 & 0 \end{pmatrix}, \qquad (13)$$

where the directional derivatives ∂_n are

$$\partial_1 = \frac{1}{\sqrt{2}}(\partial_x + i\partial_y), \quad \partial_2 = \frac{1}{\sqrt{2}}(\partial_x - i\partial_y),$$

$$\partial_3 = \frac{1}{\sqrt{2}}(\partial_z + \partial_t), \quad \partial_4 = \frac{1}{\sqrt{2}}(\partial_z - \partial_t). \qquad (14)$$

The associated matrices $\sigma^{ab} = (1/2)(\gamma^a\gamma^b - \gamma^b\gamma^a)$ were given in [8]; here we repeat them in the appendix for the sake of completeness. From their explicit form, it follows that for the electromagnetic tensor f_{ab}, in particular, and for *any* antisymmetric tensor, $f_{ab} = -f_{ba}$, in general,

$$f_{ab}\sigma^{ab} = 2\begin{pmatrix} f_{12}+f_{34} & -2f_{42} & 0 & 0 \\ 2f_{31} & -f_{12}-f_{34} & 0 & 0 \\ 0 & 0 & f_{12}-f_{34} & -2f_{32} \\ 0 & 0 & 2f_{41} & -f_{12}+f_{34} \end{pmatrix}. \tag{15}$$

Comparing with (5), we see that the cartesian components of the RS vector **F** are

$$\begin{aligned} &F_x + iF_y = 2f_{31}, \\ &F_x - iF_y = -2f_{42}, \quad F_z = f_{12}+f_{34}, \\ &F^*_x + iF^*_y = -2f_{41}, \\ &F^*_x - iF^*_y = 2f_{32}, \quad F^*_z = -f_{12}+f_{34}. \end{aligned} \tag{16}$$

3.1. General coordinates system

In a general system of coordinates, the directional derivatives ∂_n must be replaced by the covariant directional derivative ∇_n. The Dirac equation in a general coordinates system can thus be written as

$$\gamma^n\nabla_n\psi + im\psi = 0 \tag{17}$$

in terms of the covariant derivative

$$\nabla_n = \partial_n\psi + \Gamma_n, \tag{18}$$

where Γ_n are the Fock coefficients [9, 10]. In a tetradial representation, they are defined as

$$\Gamma_n = -\frac{1}{4}\Gamma_{abn}\sigma^{ab},$$

where $\Gamma^a{}_{bc}$ are the Ricci rotation coefficients given by

$$de^a = e^b \wedge \Gamma^a{}_b,$$

with $\Gamma^a{}_b = \Gamma^a{}_{bc}e^c$, and $\Gamma_{abc} = -\Gamma_{bac}$ (see, *e.g.* Refs. [6, 7]). In the null-tetrad formalism, their forms follow from (15) (see also [8]).

For a rank two tensor, in particular, we have

$$\nabla_c f_{ab} = \partial_c f_{ab} - \Gamma^n{}_{ac}f_{nb} - \Gamma^n{}_{bc}f_{an}. \tag{19}$$

Applying this formula to our matrix $\mathbb{F}$, one finds after some lengthy but straightforward algebra that

$$\nabla_n\mathbb{F} = \partial_n\mathbb{F} + \mathbf{\Gamma}_n\,\mathbb{F} - \mathbb{F}\,\mathbf{\Gamma}_n\ , \tag{20}$$

a formula that can also be checked by direct substitution.

Accordingly, the inhomogeneous Maxwell equations take the form

$$\gamma^n\nabla_n\mathbb{F} = 4\pi\not{J}, \tag{21}$$

valid in general.

4. Concluding remark

The above analysis clarifies the role of the Riemann-Silberstein vector in the context of a spinorial approach to classical electromagnetism. Given the full covariance of all the formulas obtained in this paper, the present formulation can be applied in future publications to problems in general relativy involving electromagnetism and Dirac fields.

Appendix

A.

The matrices σ^{ab} associated to the Dirac matrices (12) are [8]

$$\begin{aligned} \sigma^{12}+\sigma^{34} &= 2\begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \\ \sigma^{31} &= 2\begin{pmatrix} \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \\ \sigma^{42} &= 2\begin{pmatrix} \begin{matrix} 0 & -1 \\ 0 & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \\ -\sigma^{12}+\sigma^{34} &= 2\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \\ \sigma^{32} &= 2\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 0 & -1 \\ 0 & 0 \end{matrix} \end{pmatrix} \\ \sigma^{41} &= 2\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \end{pmatrix}. \end{aligned} \tag{A.1}$$

1. L. Silberstein, *Annalen der Physik* **327** (1907) 579.
2. I. Bialynicki-Birula and Z. Bialynicka-Birula, *Phys. Rev. A* **67** (2003) 062114.
3. I. Bialynicki-Birula and Z. Bialynicka-Birula, *J. Phys. A: Math. Theor*. **46** (2013) 053001.
4. I. V. Belkovich and B. L. Kogan, *Progress Electromag. Research B*, **69** (2016) 103.
5. E. T. Newman and R. Penrose, *J. Math. Phys*. **3** (1962) 566.
6. J. Plebanski, *J. Math. Phys.* **16** (1975) 2395.
7. F. J. Ernst, *J. Math. Phys.* **19** (1978) 489.
8. S. Hacyan, *Rev. Mex. Fis.* **38** (1992) 59.
9. V. Fock, *Comptes Rendus Acad. Sci.* **189** (1929) 25.
10. T.C Chapman and D.J. Leiter, *Am. J. Phys.* **44** (1976) 858.