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A $SU(5) \times Z_2$ kink solution and its local stability

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A non-abelian kink inducing asymptotically the breaking pattern $SU(5)\times Z_2 \to SU(4)\times U(1)/Z_4$ is obtained. We consider a fourth order Higgs potential in a $1+1$ theory where the scalar field is in the adjoint representation of $SU(5)$. The perturbative stability of the kink is also evaluated. A Schrödinger-like equation for the excitations along each $SU(5)$ generator is determined, and in none of the cases negative eigenvalues compromising the stability of solution are found. In particular, several bounded scalar states are determined, being one of them the translational zero mode of the flat space $SU(5)\times Z_2$ kink.

Keywords: $SU(5)$ kink; local stability.

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1. Introduction

In theories with a simple scalar field and a Z_2 invariant self-interaction potential, it is possible to find stable topological kinks interpolating asymptotically between the minima of the system. These solutions are interesting in the framework of the gravitational theories with extra dimensions because the kink or domain wall interpolates between two Anti de Sitter spacetimes and induces the standard gravitational interaction on the four-dimensional sector of the warped structure [1–4]. Such scenarios are referred to as brane-worlds, and it is surmised that the standard model fields should be localized on the topological defect [5–7].

As a first approximation to a scenario where the symmetry group of the standard model can be recovered inside the domain wall, in absence of gravity but in presence of a non-abelian symmetry, kink solutions have been obtained in several opportunities [8–10]. In Ref. [8] three kink solutions for a $SO(10)$ theory inducing asymptotically the breaking symmetry $SO(10)\to SU(5)$ were determined; subsequently, in Ref. [10] the local stability of these scenarios was evaluated finding in two of them tachyonic Pöschl-Teller modes in the spectrum of scalar perturbations. Other models in terms of E_6 group were discussed in Ref. [9].

Among the non-abelian solutions we highlights the one where the kink interpolates between $SU(3)\times SU(2)\times U(1)$ vacuum expectation values (vev) of a $SU(5)\times Z_2$ invariant potential [11–14]. Since this issue is the focus of this paper, let us review in detail this scenario.

Consider the bosonic sector of the $SU(5)$ model in $(1{+}1)$ dimensions

$$L = -\mathrm{Tr}(\partial_m \mathbf{\Phi}\partial^m \mathbf{\Phi}) - V(\mathbf{\Phi}), \tag{1}$$

$$V(\mathbf{\Phi}) = -\mu^2 \mathrm{Tr}(\mathbf{\Phi}^2) + h(\mathrm{Tr}(\mathbf{\Phi}^2))^2 + \lambda \mathrm{Tr}(\mathbf{\Phi}^4) + V_0 \tag{2}$$

with $\mathbf{\Phi}$ a scalar field transforming in the adjoint representation of the symmetry group

$$\mathbf{\Phi} \to \mathbf{U}\mathbf{\Phi}\mathbf{U}^\dagger, \quad \mathbf{U} = \exp(i\alpha_j \mathbf{T}_j),$$
$$\mathrm{Tr}(\mathbf{T}_{j_1}\mathbf{T}_{j_2}) = \frac{1}{2}\delta_{j_1 j_2} \tag{3}$$

where $\mathbf{T}_j$, $j=1,\ldots,24$, are traceless hermitian generators of $SU(5)$. In the potential (2), μ, h and λ are the parameters of the theory and V_0 is a constant to adjust conveniently the minimum to zero.

It is well know that there are two possible non-trivial form for the minimum of (2) [15]:

$$< \mathbf{\Phi}_{\mathrm{A}} > \sim \mathrm{diag}(2,2,2,-3,-3), \qquad \lambda > 0, \tag{4}$$

and

$$< \mathbf{\Phi}_{\mathrm{B}} > \sim \mathrm{diag}(1,1,1,1,-4), \qquad \lambda < 0; \tag{5}$$

which lead to the breaking patterns

$$SU(5) \to SU(3)\times SU(2)\times U(1) \tag{6}$$

and

$$SU(5) \to SU(4)\times U(1), \tag{7}$$

respectively.

Due to the absence of cubic terms in (2) the Z_2 symmetry is included in the model and kink solutions are expected, such that

$$\mathbf{\Phi}(z=-\infty) = -\mathbf{U}\mathbf{\Phi}(z=+\infty)\mathbf{U}^\dagger, \tag{8}$$

where U is an element of $SU(5)$ and $\mathbf{\Phi}$ depends only on the coordinate z. In fact, for $h=-3\lambda/20$ and $\lambda>0$, the symmetry breaking pattern

$$SU(5)\times Z_2 \to \frac{SU(3)\times SU(2)\times U(1)}{Z_3\times Z_2} \tag{9}$$

can be induced asymptotically by the following non-abelian kink

$$\mathbf{\Phi}_{\rm A} = \frac{\sqrt{5}}{2}\frac{\mu}{\sqrt{\lambda}}\Big[\text{diag}(1,-1,0,1,-1) + \frac{1}{5}\tanh\Big(\frac{\mu z}{\sqrt{2}}\Big)\text{diag}(-1,-1,4,-1,-1)\Big]. \tag{10}$$

which, at $z=\pm\infty$ goes to

$$\mathbf{\Phi}_{\rm A}(z=+\infty) = \frac{1}{\sqrt{5}}\frac{\mu}{\sqrt{\lambda}}\text{diag}(2,-3,2,2,-3) \tag{11}$$

$$\mathbf{\Phi}_{\rm A}(z=-\infty) = \frac{1}{\sqrt{5}}\frac{\mu}{\sqrt{\lambda}}\text{diag}(3,-2,-2,3,-2). \tag{12}$$

Notice that (12) is compatible with the constraint (8) and that, from trace properties, the vacuums (4) and (11) are equivalent for the scalar potential. On the other hand, in accordance with (11) and (12), $SU(3)\times SU(2)\times U(1)$ is embedded asymptotically in $SU(5)$ in different ways. Moreover, inside the kink, $\mathbf{\Phi}_{\rm A}(z=0)\sim$ diag$(1,-1,0,1,-1)$, the unbroken group is

$$\frac{SU(2)\times SU(2)\times U(1)\times U(1)}{Z_2\times Z_2}. \tag{13}$$

This solution, as well as its perturbative stability, were determined in [11]; subsequently, (10) was recovered as a particular case of a kink in $SU(N)\times Z_2$ [12], the extension to curved spacetime in five dimensions was found in [13, 14] where (10) induces the symmetry group of the standard model at the boundary of an AdS$_5$ warped spacetime.

With respect to the vev (5), curiously, up to now, a kink solution for the model (1) has not been reported; however, a self-gravitating kink inducing unbroken group (7) was obtained in [13]. Considering the absence of a flat kink for (5), our proposal for this paper is to find this solution and evaluate its stability under small perturbations.

2. Kink solution

Let us consider a kink solution, $\mathbf{\Phi}_{\rm B}$, for (1) in correspondence, at infinity, with the symmetry breaking pattern

$$SU(5)\times Z_2 \to \frac{SU(4)\times U(1)}{Z_4}. \tag{14}$$

For this case, it is convenient to write the non-abelian field in terms of diagonal generators of $SU(5)$

$$\mathbf{\Phi} = \phi_1\mathbf{T}_{21} + \phi_2\mathbf{T}_{22} + \phi_3\mathbf{T}_{23} + \phi_4\mathbf{T}_{24} \tag{15}$$

where

$$\mathbf{T}_{21} = \frac{1}{2}\text{diag}(1,-1,0,0,0), \tag{16}$$

$$\mathbf{T}_{22} = \frac{1}{2\sqrt{3}}\text{diag}(1,1,-2,0,0), \tag{17}$$

$$\mathbf{T}_{23} = \frac{1}{2}\text{diag}(0,0,0,1,-1), \tag{18}$$

$$\mathbf{T}_{24} = \frac{1}{2\sqrt{15}}\text{diag}(2,2,2,-3,-3). \tag{19}$$

The unbroken group (14) is induced by a kink when (15) satisfy the boundary conditions

$$\mathbf{\Phi}(z=+\infty) = v\Big(\mathbf{T}_{23} + \sqrt{\frac{3}{5}}\mathbf{T}_{24}\Big) = <\mathbf{\Phi}_{\rm B}>, \tag{20}$$

$$\mathbf{\Phi}(z=-\infty) = v\Big(\mathbf{T}_{23} - \sqrt{\frac{3}{5}}\mathbf{T}_{24}\Big) = -\mathbf{U}<\mathbf{\Phi}_{\rm B}>\mathbf{U}^{\dagger}. \tag{21}$$

The equations of motion for the coefficients of (15) are given by

$$\phi_1'' = -\left[\mu^2 - \left(h+\frac{2\lambda}{5}\right)\phi_4^2 - \left(h+\frac{\lambda}{2}\right)(\phi_1^2+\phi_2^2) - h\phi_3^2\right]\phi_1 + \frac{2\lambda}{\sqrt{5}}\phi_1\phi_2\phi_4, \tag{22}$$

$$\phi_2'' = -\left[\mu^2 - \left(h+\frac{2\lambda}{5}\right)\phi_4^2 - \left(h+\frac{\lambda}{2}\right)(\phi_1^2+\phi_2^2) - h\phi_3^2\right]\phi_2 + \frac{\lambda}{\sqrt{5}}\phi_4(\phi_1^2-\phi_2^2), \tag{23}$$

$$\phi_3'' = -\left[\mu^2 - \left(h+\frac{9\lambda}{10}\right)\phi_4^2 - \left(h+\frac{\lambda}{2}\right)\phi_3^2 - h(\phi_1^2+\phi_2^2)\right]\phi_3, \tag{24}$$

$$\phi_4'' = -\left[\mu^2 - \left(h+\frac{7\lambda}{30}\right)\phi_4^2 - \left(h+\frac{2\lambda}{5}\right)(\phi_1^2+\phi_2^2) - \left(h+\frac{9\lambda}{10}\right)\phi_3^2\right]\phi_4 + \frac{\lambda}{\sqrt{5}}\phi_2\left(\phi_1^2 - \frac{\phi_2^2}{3}\right). \tag{25}$$

where prime means derivative with respect to z. Suggested by (20, 21) we choose conveniently $\phi_1=\phi_2=0$; thus, we obtain a pair of coupled equations for ϕ_3 and ϕ_4, namely

$$\phi_3'' = \left[-\mu^2 + \left(h+\frac{9\lambda}{10}\right)\phi_4^2 + \left(h+\frac{\lambda}{2}\right)\phi_3^2\right]\phi_3, \tag{26}$$

$$\phi_4'' = \left[-\mu^2 + \left(h+\frac{7\lambda}{30}\right)\phi_4^2 + \left(h+\frac{9\lambda}{10}\right)\phi_3^2\right]\phi_4, \tag{27}$$

which can be decouple for $\lambda=-10h/9$, $h>0$. Now, for the remaining equations we require

$$\phi_3(z=\pm\infty) = v, \qquad \phi_4(z=\pm\infty) = \pm\sqrt{\frac{3}{5}}v; \tag{28}$$

thus, we find the solution

$$\mathbf{\Phi}_{\rm B} = v\left[\mathbf{T}_{23} + \sqrt{\frac{3}{5}}\tanh\left(\frac{\mu z}{\sqrt{2}}\right)\mathbf{T}_{24}\right], \qquad v = \frac{3\mu}{2\sqrt{h}} \tag{29}$$

such that at $z = \pm\infty$, the unbroken group (14) is embedded in $SU(5)$ in different ways. On the other hand, in the core of the kink, $z = 0$, the remaining symmetry group is

$$\frac{SU(3) \times U(1) \times U(1)}{Z_3}. \tag{30}$$

3. Perturbative stability

Now, in order to study the perturbative stability of (29) let us consider, in the energy of the system, small deviations from kink solution

$$E = \int dz \left[\mathrm{Tr}(\partial_z \mathbf{\Phi}_{\mathrm{B}} + \epsilon\, \partial_z \mathbf{\Psi})^2 + V(\mathbf{\Phi}_{\mathrm{B}} + \epsilon \mathbf{\Psi})\right],$$
$$\epsilon \ll 1, \tag{31}$$

where $\mathbf{\Psi} = \psi_j \mathbf{T}_j$. Thus, to second order in ϵ (the term proportional to ϵ is zero via the equation of motion of $\mathbf{\Phi}_{\mathrm{B}}$), we find that

$$E = E[\mathbf{\Phi}_{\mathrm{B}}] + \epsilon^2 \int dz\, \psi_{j_1} \left(-\delta_{j_1 j_2}\partial_z^2 + V_{j_1 j_2}\right)\psi_{j_2} + \mathcal{O}(\epsilon^3) \tag{32}$$

where

$$\begin{aligned} V_{j_1 j_2}(\mathbf{\Phi}_{\mathrm{B}}) = &-\frac{1}{2}\mu^2 \delta_{j_1 j_2} + h\mathrm{Tr}(\mathbf{\Phi}_{\mathrm{B}}^2)\delta_{j_1 j_2} \\ &+ 4h\mathrm{Tr}(\mathbf{\Phi}_{\mathrm{B}}\mathbf{T}_{j_1})\mathrm{Tr}(\mathbf{\Phi}_{\mathrm{B}}\mathbf{T}_{j_2}) \\ &- \frac{40}{9}h\mathrm{Tr}(\mathbf{\Phi}_{\mathrm{B}}^2\mathbf{T}_{j_1}\mathbf{T}_{j_2}) \\ &- \frac{20}{9}h\mathrm{Tr}(\mathbf{\Phi}_{\mathrm{B}}\mathbf{T}_{j_1}\mathbf{\Phi}_{\mathrm{B}}\mathbf{T}_{j_2}) \end{aligned} \tag{33}$$

which is diagonal and, hence, the double sum in (32) can be written as follows

$$\left(-\frac{1}{2}\delta_{j_1 j_2}\partial_z^2 + V_{j_1 j_2}\right)\psi_{j_2} = m_{j_1 j_2}^2 \psi_{j_2}^2, \tag{34}$$

and the stability problem consists in determining the eigenvalue associated with each generator of $SU(5)$. Fortunately, for several generators we obtain the same eigenvalues equation and, after group them, only five non trivial cases need to be checked. For the trivial cases, $j = 19, 20$, whose generators are broken everywhere, a vanishing potential is obtained.

For the remaining five cases it is convenient to consider $\xi = \mu z/\sqrt{2}$, to make dimensionless the equations of motions. For the set of generators labelled by $j = 1, \ldots, 6$, $21, 22$, basis for $SU(3)$ in (30), we get

$$\left[-\frac{1}{2}\partial_\xi^2 + \frac{1}{4}\left(5 + 3\tanh^2\xi\right)\right]\psi_j = 2\frac{m_j^2}{\mu^2}\psi_j, \tag{35}$$

which can be rewritten in terms of a Pöschl-Teller potential, whose bound states are well known [16]. Thus, the spectrum of scalar states is determined for all j by a pair of normalizable eigenfunctions

$$m_0^2 = \frac{4+\sqrt{7}}{8}\mu^2, \quad \psi_0 \sim \cosh^{(1-\sqrt{7})/2}(\xi) \tag{36}$$

$$m_1^2 = \frac{3\sqrt{7}}{8}\mu^2, \quad \psi_1 \sim \cosh^{(1-\sqrt{7})/2}(\xi)\sinh(\xi) \tag{37}$$

and a set of free states from $m^2 > 3\sqrt{7}\mu^2/8$.

In the cases identified with $j_+ = 7, \ldots, 12$ and $j_- = 13, \ldots, 18$, broken generators with respect to (30) but unbroken with respect to (14) at $\xi \to \pm\infty$, we have

$$\left[-\frac{1}{2}\,\partial_\xi^2 + \tanh\xi\,(\tanh\xi \pm 1)\right]\psi_{j_\pm} = 2\frac{m_{j_\pm}^2}{\mu^2}\psi_{j_\pm}. \tag{38}$$

In this case the Schrödinger-like equation is associated to a non-conventional potential (see Fig. 1) which exhibits a negative well in $-\infty < \xi \le 0$ for ψ_+. For ψ_- the potential has an equivalent profile, but in the region $\infty > \xi \ge 0$. However, the eigenvalues of the equation are positive since (38) can be factorized as follows [17]

$$(-\partial_\xi + \beta_\pm)(\partial_\xi + \beta_\pm)\psi_{j_\pm} = 4\frac{m_{j_\pm}^2}{\mu^2}\psi_{j_\pm}, \tag{39}$$

where

$$\beta_\pm = \mp 2\frac{1 + 3\cosh 2\xi \mp \sinh 2\xi + 2(\tanh\xi \mp 1)\xi}{4 + e^{\pm 2\xi} + e^{\mp 2\xi}(3 \pm 4\xi)}. \tag{40}$$

In addition, the zero mode can be obtained, $\psi_0 \sim 1 \mp \tanh\xi$, which is out of the spectrum of eigenfunctions because it is not normalizable, as expected in concordance with Fig. 1.

Finally, in the scalar field directions $\mathbf{T}_{23}$ and $\mathbf{T}_{24}$, we find

$$\left(-\frac{1}{2}\partial_\xi^2 + 2\right)\psi_{23} = 2\frac{m_{23}^2}{\mu^2}\psi_{23} \tag{41}$$

$$\left(-\frac{1}{2}\partial_\xi^2 + 3\tanh^2\xi - 1\right)\psi_{24} = 2\frac{m_{24}^2}{\mu^2}\psi_{24}, \tag{42}$$

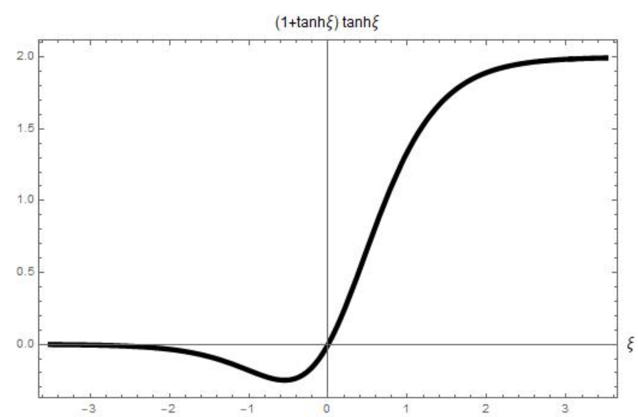

FIGURE 1. Plot of the potential for the scalar perturbations ψ_{j_+} along the broken generators $j_+ = 7, \ldots, 12$. The potential profile for ψ_{j_-} with $j_- = 13, \ldots, 18$ is a mirror image of the shown one.

with eigenvalues bounded as $m_{23}^2 > \mu^2$ and $m_{24}^2 \geq 0$. In the last case, two localized states are found

$$m_0^2 = 0\,, \quad \psi_0 \sim \cosh^{-2}(\xi) \tag{43}$$

$$m_1^2 = \frac{3}{4}\mu^2\,, \quad \psi_1 \sim \cosh^{-2}(\xi)\sinh(\xi), \tag{44}$$

being the first one, (43), the translational mode of the flat space $SU(5) \times Z_2$ kink [11].

Since in all cases the eigenvalues are positive, the perturbations do not induce instability on (29) and hence the non-abelian flat domain wall $\mathbf{\Phi}_{\rm B}$ is a locally stable solution of (1, 2).

4. Ending comments

We have derived a flat $SU(5) \times Z_2$ kink interpolating asymptotically between Minkowskian vacuums with the symmetry breaking pattern $SU(5) \times Z_2 \to SU(4) \times U(1)/Z_4$ and with unbroken group $SU(3) \times U(1) \times U(1)/Z_3$ in the core of the wall.

With regard to the spectrum of scalar fluctuations, we do not find tachyonic modes compromising the local stability of the non-abelian wall. In particular, we find the translational zero mode and several Pöschl-Teller confined scalar states along $SU(3)$ basis.

Non-abelian kinks as brane worlds is the next issue that we would like to study. The main problem is that we need to find a scenario where the symmetry group of the standard model corresponds to the unbroken symmetry inside the kink. In our opinion, solutions similar to (10) and (29) are a first approximation to this open problem.

As commented in the introduction, another option has been already explored in [8, 9] where the symmetry of the theory is determined by the $SO(10)$ group. In this case, the unbroken symmetry for finite z is achieved used the clash-of-symmetries mechanism. Thus, $SU(3) \times SU(2) \times U(1) \times U(1)$ may be obtained in the core of a $SO(10)$ wall, which is almost the symmetry expected for a more realistic model.

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