Analytical solution of the time fractional diffusion equation and fractional convection-diffusion equation

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In this paper, we obtain analytical solutions for the time-fractional diffusion and time-fractional convection-diffusion equations. These equations are obtained from the standard equations by replacing the time derivative with a fractional derivative of order α. Fractional operators of type Liouville-Caputo, Atangana-Baleanu-Caputo, fractional conformable derivative in Liouville-Caputo sense, and Atangana-Koca-Caputo were used to model diffusion and convection-diffusion equation. The Laplace and Fourier transforms were applied to obtain analytical solutions for the fractional order diffusion and convection-diffusion equations. The solutions obtained can be useful to understand the modeling of anomalous diffusion, subdiffusive systems and super-diffusive systems, transport processes, random walk and wave propagation phenomenon.

Keywords: Fractional calculus; Mittag-Leffler kernel; fractional conformable derivative; diffusion equation.

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1. Introduction

Recent studies in science and engineering demonstrated that the dynamics of many systems may be described more accurately by means of differential equations of non-integer order. The diffusion equation is a partial differential equation that portrays density dynamics in a material subject to diffusion [1,2]. The convection-diffusion equation explains the flow of heat, particles, oil reservoir simulations, transport of mass and energy, global weather production, or other physical quantities in conditions where there are both diffusion and convection or advection [3-5]. Fractional diffusion equations are largely used in describing abnormal slowly-diffusion phenomenon, and fractional diffusion equations are always used in describing abnormal convection phenomenon. Time-fractional diffusion is derived by considering continuous time random walk problems, which are in general non-Markovian processes.

Several definitions, related to fractional order-derivatives have been used in the literature. These definitions include, Riemann-Liouville, Liouville-Caputo, conformable derivatives, Caputo-Fabrizio, Atangana-Baleanu and Atangana-Koca, to mention a few [6]. The choice of fractional differentiation is motivated by the fact that the interaction with the medium is not local but global. The fractional operators can be a useful way to include memory in a dynamical process. A dynamical process that is modelled through fractional order derivatives carries information about its present as well as past states.

In this paper, we consider the time-fractional diffusion and convection-diffusion equations, obtained from the standard equations by replacing the time derivative with fractional derivatives of type Liouville-Caputo, Atangana-Baleanu-Caputo, fractional conformable derivative in Liouville-Caputo sense and Atangana-Koca-Caputo of order α, with $0<\alpha\leq 1$.

The following fractional diffusion equation is considered

$$({}_0\mathcal{D}_t^{\alpha}u)(x,t)=\mu\frac{\partial^2}{\partial x^2}u(x,t),$$
$$t>0,\ \ x\in\mathbb{R},\ \ \mu\in\mathbb{R}^+,\ \ 0<\alpha\leq 1, \tag{1}$$
$$u(x,0)=\psi(x), \tag{2}$$

where μ is the diffusion coefficient.

The fractional convection-diffusion equation considered is [4]

$$({}_0\mathcal{D}_t^{\alpha}u)(x,t)=-\epsilon\eta\frac{\partial}{\partial x}u(x,t)+\mu\frac{\partial^2}{\partial x^2}u(x,t)+\frac{Q(x,t)}{c\rho},$$
$$t>0,\ \ 0<\alpha\leq 1, \tag{3}$$
$$u(x,0)=\psi(x),\ \ x\in\mathbb{R},\ \ \mu\in\mathbb{R}^+, \tag{4}$$

where $\mu=\lambda/c\rho$ is the diffusion coefficient, ϵ is the porosity, η is the velocity, λ is the thermal conductivity, c is the specific heat, ρ is the mass density, and $Q(x,t)$ is the source term.

2. Basic Tools

The Liouville-Caputo (C) fractional operator of order α is defined as [7]

$$ {}_a^C\mathcal{D}_t^\alpha u(t,x) = \begin{cases} \frac{d^n}{dt^n}u(x,t), & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)}\int\limits_a^t (t-z)^{n-\alpha-1}\frac{\partial^n}{\partial z^n}u(x,z)dz, & n-1<\alpha<n\in\mathbb{N}. \end{cases} \quad (5) $$

where ${}_0^C\mathcal{D}_t^\alpha$ is the Liouville-Caputo fractional operator of order α with respect to t.

Atangana and Baleanu considered the generalized Mittag-Leffer function as the kernel of differentiation. This kernel is non-singular and nonlocal and preserves the benefits of the above fractional operators. Replaced the exponential kernel with the generalized Mittag-Leffler function, we obtain the fractional operator of type Atangana-Baleanu in Liouville-Caputo sense (ABC) of order α defined as follows [8]

$$ \left({}_a^{ABC}\mathcal{D}_t^{(n+\alpha)}u\right)(x,t) = \frac{1}{g(\alpha)}\int\limits_a^t E_\alpha\big(-g(\alpha)(t-z)^\alpha\big)\frac{\partial^{n+1}u}{\partial z^{n+1}}(x,z)dz, \;\; n-1<\alpha<n\in\mathbb{N}. \quad (6) $$

where $n\in\mathbb{N}$ and $g(\alpha)$ is a normalization function that depends of α, which satisfies that, $g(0)=g(1)=1$.

Let $0<\alpha\leq 1$ and $n\in\mathbb{N}$, the Laplace transforms of the Liouville-Caputo and Atangana-Baleanu-Caputo fractional operators are given by

$$ \mathcal{L}\left[{}_0^C\mathcal{D}_t^{(n+\alpha)}u(x,t)\right](x,s) = \frac{1}{s^{n-\alpha}}\Big(s^n\mathcal{L}\left[u\left(x,t\right)\right] - s^{n-1}u\left(x,0\right) - \ldots - u^{(n-1)}\left(x,0\right)\Big). \quad (7) $$

$$ \mathcal{L}\left[{}_0^{ABC}\mathcal{D}_t^{(n+\alpha)}u(x,t)\right](x,s) = \frac{1}{g(\alpha)}\,\frac{1}{s^{1-\alpha}}\,\frac{s^{n+1}\mathcal{L}\left[u\left(x,t\right)\right] - s^n u\left(x,0\right) - s^{n-1}\dot{u}\left(x,0\right)\ldots - u^{(n)}\left(x,0\right)}{s+g\left(\alpha\right)}. \quad (8) $$

Khalil in [9] gives a new definition of derivative called "conformable derivative". Let $f:[a,\infty)\longrightarrow\mathbb{R}$. The conformable derivative of $f(t)$ is given by

$$ {}_a\mathcal{D}_t^\alpha f(t) = \lim_{\epsilon\to 0}\frac{f\Big(t+\epsilon t^{1-\alpha}\Big) - f(t)}{\epsilon}, \quad (9) $$

for all $t>0$, $\alpha\in(0,1)$. If $f(t)$ is α-differentiable in some $(0,a)$, $a>0$, and $\lim\limits_{\epsilon\to 0^+} f^{(\alpha)}(t)$ exists, then define $f^\alpha(0) = \lim\limits_{\epsilon\to 0^+} f^{(\alpha)}(t)$.

The left conformable integral is given by

$$ {}_aI_t^\alpha f(t) = \int\limits_a^t \frac{f(x)}{(x-a)^{1-\alpha}}dx, \;\; x\geq a, \;\; 0<\alpha\leq 1, \quad (10) $$

Iterating n-times the integral (10) and replacing the integer n, for $\beta\in\mathbb{C}$, with $\mathrm{Re}(\beta)>0$, we define the following fractional conformable integral

$$ {}_a^\beta I_t^\alpha f(t) = \frac{1}{\Gamma(\beta)}\int\limits_a^t \left(\frac{(t-a)^\alpha - (x-a)^\alpha}{\alpha}\right)^{\beta-1} \times \frac{f(x)}{(x-a)^{1-\alpha}}\,dx. \quad (11) $$

Considering the definition given by Eq. (11) we get the left fractional conformable derivative in the Liouville-Caputo sense. Let $\mathrm{Re}(\beta)\geq 0$, $n=[\mathrm{Re}(\beta)]+1$, $f\in C^n_{\alpha,a}([a,b])$, $(f\in C^n_{\alpha,b}([a,b]))$. Then the left fractional conformable derivative in the Liouville-Caputo sense is given by [10]

$$ {}_a^{c\;\beta}\mathcal{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\beta)}\int\limits_a^t \Big(\frac{(t-a)^\alpha - (x-a)^\alpha}{\alpha}\Big)^{n-\beta-1} \times \frac{{}_a\mathcal{D}_x^\alpha f(x)}{(x-a)^{1-\alpha}}\,dx = {}^{n-\beta}_{\;\;a} I_t^\alpha\Big({}_a^n\mathcal{D}_t^\alpha f(t)\Big), \quad (12) $$

The Atangana-Koca fractional derivative in Liouville-Caputo sense (AKC) is given by [11,12]

$$ \left({}_a^{AKC}\mathcal{D}_t^\alpha u\right)(x,t) = \frac{1}{g(\alpha)} \times \int\limits_a^t E_{\alpha,\beta}^{\gamma,q}\big(-g(\alpha)(t-z)^\alpha\big)\frac{\partial u}{\partial z}(x,z)dz, \quad (13) $$

where $g(\alpha)$ is a normalization function as in the previous cases.

Let $0<\alpha\leq 1$, the Laplace transform of the Atangana-Koca fractional-order derivative is given as

$$ \mathcal{L}\Big\{{}_0^{AKC}\mathcal{D}_t^\alpha u(x,t)\Big\}(x,s) = \frac{1}{g(\alpha)\big(1-g(\alpha)\big)^q} \times \Big(s^{-n\alpha}\mathcal{L}\left[u\left(x,t\right)\right] - s^{-n\alpha-1}u\left(x,0\right)\Big). \quad (14) $$

Given a function $u(x)\in L_1(\mathbb{R})$, the Fourier transform is given by

$$\widehat{u}(k) = \left(\mathcal{F}_x u(x)\right)(k) := \int_{-\infty}^{\infty} e^{ikx}\, u(x)dx, \tag{15}$$

and the inverse Fourier transform of $u(x)$ is given by

$$\mathcal{F}_k^{-1}\left(\mathcal{F}_x u(k)\right)(x) := \frac{1}{2\pi} \times \int_{-\infty}^{\infty} e^{-ikx}\, \left(\mathcal{F}_x u(x)\right)(k)dk. \tag{16}$$

3. Fractional diffusion equations

In this paper, we solved the diffusion and convection-diffusion equation considering fractional operators of type Liouville-Caputo, Atangana-Baleanu-Caputo, fractional conformable derivative in Liouville-Caputo sense and Atangana-Koca-Caputo.

Diffusion Equation.

In the Liouville-Caputo sense we have the following diffusion equation

$$({}_0^C\mathcal{D}_t^{\alpha} u)(x,t) = \mu\frac{\partial^2}{\partial x^2}u(x,t),$$
$$t>0,\ \ x\in\mathbb{R},\ \ \mu\in\mathbb{R}^+,\ \ 0<\alpha\leq 1, \tag{17}$$
$$u(x,0)=\psi(x), \tag{18}$$

where μ is the diffusion coefficient.

Solution. Applying the Laplace transform to Eq. (17) and taking the condition (18) we get

$$s^{\alpha}\left(\mathcal{L}_t u\right)(x,s) - s^{\alpha-1}\psi(x) = \mu\frac{\partial^2}{\partial x^2}\left(\mathcal{L}_t u\right)(x,s). \tag{19}$$

Applying the Fourier transform in the left hand of the Eq. (19) we have

$$\mathcal{F}_x\Big\{s^{\alpha}\left(\mathcal{L}_t u\right)(x,s) - s^{\alpha-1}\psi(x)\Big\}(k,s) = s^{\alpha}\,\widehat{u}(k,s) - s^{\alpha-1}\Psi(k), \tag{20}$$

and for the right hand of the Eq. (19) we have

$$\mathcal{F}_x\Big\{\mu\frac{\partial^2}{\partial x^2}(\mathcal{L}_t u)\Big\}(k,s) = \mu\frac{\partial^2}{\partial x^2}(\mathcal{F}_x\mathcal{L}_t u)(k,s) = \mu(-ik)^2\widehat{u}(k,s). \tag{21}$$

Equating Eqs. (20) and (21) the following explicit relation is deduced for $\widehat{u}(k,s)$

$$\widehat{u}(k,s) = \frac{s^{\alpha-1}\Psi(k)}{s^{\alpha}+\mu k^2}. \tag{22}$$

Now, applying the inverse Laplace and inverse Fourier transforms to Eq. (22) we have

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{\alpha,1}\big(-\mu k^2 t^{\alpha}\big)\,\Psi(k)e^{-ikx}dk. \tag{23}$$

In the Atangana-Baleanu-Caputo sense we have the following diffusion equation

$$({}_0^{ABC}\mathcal{D}_t^{\alpha} u)(x,t) = \mu\frac{\partial^2}{\partial x^2}u(x,t),$$
$$t>0,\ \ x\in\mathbb{R},\ \ \mu\in\mathbb{R}^+,\ \ 0<\alpha\leq 1, \tag{24}$$
$$u(x,0)=\psi(x), \tag{25}$$

where μ is the diffusion coefficient.

Solution. Applying the Laplace transform to Eq. (24) and taking the condition (25) we get

$$\frac{s^{\alpha}\left(\mathcal{L}_t u\right)(x,s) - s^{\alpha-1}\psi(x)}{s+g(\alpha)} = \mu\frac{\partial^2}{\partial x^2}\left(\mathcal{L}_t u\right)(x,s). \tag{26}$$

Applying the Fourier transform to Eq. (26) and simplifying, we have the following relation for $\widehat{u}(k,s)$

$$\widehat{u}(k,s) = \frac{s^{\alpha-1}\Psi(k)}{s^{\alpha}+\mu k^2\left(s^{\alpha}+g(\alpha)\right)}, \tag{27}$$

and applying the inverse Fourier transform to Eq. (27) we have

$$\tilde{u}\left(x,s\right) = \left(\mathcal{F}_k^{-1}(\widehat{u})\right)(x,s) = \frac{s^{\alpha-1}}{2\pi} \times \int_{-\infty}^{\infty}\frac{\Psi(k)}{s^{\alpha}+\mu k^2\left(s^{\alpha}+g(\alpha)\right)}e^{-ikx}dk. \tag{28}$$

Finally, applying the inverse Laplace transform to the above equation we get

$$u(x,t) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{st}s^{\alpha-1}ds \times \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi(k)}{s^{\alpha}+\mu k^2\left(s^{\alpha}+g(\alpha)\right)}e^{-ikx}dk. \tag{29}$$

Considering the fractional conformable derivative in the Liouville-Caputo sense we have the following diffusion equation

$$({}_0^{c\ \beta}\mathcal{D}_t^{\alpha} u)(x,t) = \mu\frac{\partial^2}{\partial x^2}u(x,t),$$
$$t>0,\ \ x\in\mathbb{R},\ \ \mu\in\mathbb{R}^+,\ \ 0<\alpha\leq 1, \tag{30}$$
$$u(x,0)=\psi(x), \tag{31}$$

where μ is the diffusion coefficient.

Solution. Applying the Laplace transform to Eq. (30) and taking the condition (31) we get

$$\frac{\Gamma(1-\alpha\beta)}{\alpha^{-\beta}\Gamma(1-\beta)}\Big(s^{\alpha\beta}\left(\mathcal{L}_t u\right)(x,s)-s^{\alpha\beta-1}\psi(x)\Big)$$
$$=\mu\frac{\partial^2}{\partial x^2}\left(\mathcal{L}_t u\right)(x,s). \quad (32)$$

Applying the Fourier transform to Eq. (32) and simplifying, we have

$$\widehat{u}(k,s)=\frac{s^{\alpha\beta-1}\Psi(k)}{s^{\alpha\beta}+\mu k^2\dfrac{\Gamma(1-\beta)}{\alpha^{\beta}\Gamma(1-\alpha\beta)}}. \quad (33)$$

Now applying the inverse Laplace and inverse Fourier transforms to Eq. (33) we have

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{\alpha\beta,1}\left(-\mu k^2\frac{\Gamma(1-\beta)}{\Gamma(1-\alpha\beta)}t^{\alpha\beta}\right)$$
$$\times\Psi(k)e^{-ikx}dk. \quad (34)$$

In the case when $\alpha=1$ the expression (34) matches the solution obtained in the Eq. (23) in the Liouville-Caputo sense.

Considering the Atangana-Koca fractional-order derivative in the Liouville-Caputo sense we have the following diffusion equation

$$({}_0^{AKC}\mathcal{D}_t^{\alpha}u)(x,t)=\mu\frac{\partial^2}{\partial x^2}u(x,t),$$
$$t>0,\ \ x\in\mathbb{R},\ \ \mu\in\mathbb{R}^+,\ \ 0<\alpha\leq 1, \quad (35)$$
$$u(x,0)=\psi(x), \quad (36)$$

where μ is the diffusion coefficient.

Solution. Applying the Laplace transform to Eq. (35) and taking the condition (36) we get

$$\frac{1}{a}\Big(s^{-n\alpha}\mathcal{L}\left[u\left(x,t\right)\right]-s^{-n\alpha-1}u\left(x,0\right)\Big)$$
$$=\mu\frac{\partial^2}{\partial x^2}\left(\mathcal{L}_t u\right)(x,s), \quad (37)$$

where $a=g(\alpha)\big(1-g(\alpha)\big)^{\alpha}$.

Applying the Fourier transform to Eq. (37) and simplifying, we have

$$\widehat{u}(k,s)=\frac{s^{-n\alpha-1}\Psi(k)}{s^{-n\alpha}+a\mu k^2}. \quad (38)$$

Now applying the inverse Laplace and inverse Fourier transforms to Eq. (38) we have

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[1-E_{n\alpha,1}\left(-\frac{t^{n\alpha}}{a\mu k^2}\right)\right]$$
$$\times\Psi(k)e^{-ikx}dk. \quad (39)$$

Convection-Diffusion Equation

In the Liouville-Caputo sense we have the following convection-diffusion equation

$$({}_0^{C}\mathcal{D}_t^{\alpha}u)(x,t)=-\epsilon\eta\frac{\partial}{\partial x}u(x,t)+\mu\frac{\partial^2}{\partial x^2}u(x,t)$$
$$+\frac{Q(x,t)}{c\rho},\ \ t>0,\ \ 0<\alpha\leq 1, \quad (40)$$
$$u(x,0)=\psi(x),\ \ x\in\mathbb{R},\ \ \mu\in\mathbb{R}^+, \quad (41)$$

where $\mu=\lambda/c\rho$ is the diffusion equation.

Solution. Applying the Laplace transform to Eq. (40) and taking the condition (41) we get

$$s^{\alpha}\left(\mathcal{L}_t u\right)(x,s)-s^{\alpha-1}\psi(x)=-\epsilon\eta\frac{\partial}{\partial x}\left(\mathcal{L}_t u\right)(x,s)$$
$$+\mu\frac{\partial^2}{\partial x^2}\left(\mathcal{L}_t u\right)(x,s)+\frac{Q(x,s)}{c\rho}. \quad (42)$$

Applying the Fourier transform to Eq. (42) and simplifying, we have the following relation for $\widehat{u}(k,s)$

$$\widehat{u}(k,s)=\frac{s^{\alpha-1}\Psi(k)}{s^{\alpha}+\big(\mu k^2-\epsilon\eta ik\big)}+\frac{1}{c\rho}\frac{Q(k,s)}{s^{\alpha}+\big(\mu k^2-\epsilon\eta ik\big)}. \quad (43)$$

Applying the inverse Laplace transform and the inverse Fourier transforms to Eq. (43) we get

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{\alpha,1}\big(-\big(\mu k^2-\epsilon\eta ik\big)t^{\alpha}\big)\ \Psi(k)e^{-ikx}dk$$
$$+\frac{1}{2\pi c\rho}\int_{-\infty}^{\infty}e^{-ikx}dk\frac{1}{2\pi i}$$
$$\times\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{Q(k,s)}{s^{\alpha}+\big(\mu k^2-\epsilon\eta ik\big)}e^{st}ds. \quad (44)$$

In the Atangana-Baleanu-Caputo sense we have the following convection-diffusion equation

$$({}_0^{ABC}\mathcal{D}_t^{\alpha}u)(x,t)=-\epsilon\eta\frac{\partial}{\partial x}u(x,t)+\mu\frac{\partial^2}{\partial x^2}u(x,t)$$
$$+\frac{Q(x,t)}{c\rho},\quad t>0,\ 0<\alpha\leq 1, \quad (45)$$
$$u(x,0)=\psi(x),\ \ x\in\mathbb{R},\ \ \mu\in\mathbb{R}^+, \quad (46)$$

where $\mu=\lambda/c\rho$ is the diffusion equation.

Solution. Applying the Laplace transform to Eq. (45) and taking the condition (46) we get

$$\frac{s^{\alpha}\left(\mathcal{L}_t u\right)(x,s)-s^{\alpha-1}\psi(x)}{s+g(\alpha)}=-\epsilon\eta\frac{\partial}{\partial x}\left(\mathcal{L}_t u\right)(x,s)$$
$$+\mu\frac{\partial^2}{\partial x^2}\left(\mathcal{L}_t u\right)(x,s)+\frac{Q(x,s)}{c\rho}. \quad (47)$$

Applying the Fourier transform to Eq. (47) and simplifying, we have the following relation for $\widehat{u}(k,s)$

$$\widehat{u}(k,s)=\frac{s^{\alpha-1}\Psi(k)}{s^{\alpha}+\big(\mu k^2-\epsilon\eta ik\big)\,(s+g(\alpha))}+\frac{1}{c\rho}\frac{\big(s+g(\alpha)\big)Q(k,s)}{s^{\alpha}+\big(\mu k^2-\epsilon\eta ik\big)\,(s+g(\alpha))}. \tag{48}$$

Finally, applying the inverse Fourier transform and the inverse Laplace transform to Eq. (48) we get

$$u(x,t)=\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}e^{st}s^{\alpha-1}ds\frac{1}{2\pi}\times\int_{-\infty}^{\infty}\frac{\Psi(k)}{s^{\alpha}+\big(\mu k^2-\epsilon\eta ik\big)\,(s+g(\alpha))}e^{-ikx}dk+\frac{1}{2\pi ic\rho}\int_{\epsilon-i\infty}^{\epsilon+i\infty}e^{st}(s-g(\alpha))ds\;\frac{1}{2\pi}\times\int_{-\infty}^{\infty}\frac{Q(k,s)}{s^{\alpha}+\big(\mu k^2-\epsilon\eta ik\big)\,(s+g(\alpha))}e^{-ikx}dk. \tag{49}$$

Considering the fractional conformable derivative in the Liouville-Caputo sense we have the following convection-diffusion equation

$$({}_{0}^{c\,\beta}\mathcal{D}_t^{\alpha}u)(x,t)=-\epsilon\eta\frac{\partial}{\partial x}u(x,t)+\mu\frac{\partial^2}{\partial x^2}u(x,t)+\frac{Q(x,t)}{c\rho},\;\;t>0\;\;0<\alpha\leq 1, \tag{50}$$

$$u(x,0)=\psi(x),\;\;x\in\mathbb{R},\;\;\mu\in\mathbb{R}^{+}, \tag{51}$$

where μ is the diffusion coefficient.

Solution. Applying the Laplace transform to Eq. (50) and taking the condition (51) we get

$$\frac{\Gamma(1-\alpha\beta)}{\alpha^{-\beta}\Gamma(1-\beta)}\Big(s^{\alpha\beta}\left(\mathcal{L}_t u\right)(x,s)-s^{\alpha\beta-1}\psi(x)\Big)=-\epsilon\eta\frac{\partial}{\partial x}\left(\mathcal{L}_t u\right)(x,s)+\mu\frac{\partial^2}{\partial x^2}\left(\mathcal{L}_t u\right)(x,s)+\frac{Q(x,s)}{c\rho}. \tag{52}$$

Applying the Fourier transform to Eq. (42) and simplifying, we have

$$\widehat{u}(k,s)=\frac{s^{\alpha\beta-1}\Psi(k)}{s^{\alpha\beta}+(\mu k^2-\epsilon\eta ik)\frac{\Gamma(1-\beta)}{\alpha^{\beta}\Gamma(1-\alpha\beta)}}+\frac{\Gamma(1-\beta)}{c\rho\alpha^{\beta}\Gamma(1-\alpha\beta)}\times\frac{Q(k,s)}{s^{\alpha\beta}+(\mu k^2-\epsilon\eta ik)\frac{\Gamma(1-\beta)}{\alpha^{\beta}\Gamma(1-\alpha\beta)}}. \tag{53}$$

Finally, applying the inverse Laplace transform and the inverse Fourier transform to Eq. (53), we get

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{\alpha\beta,1}\left(-(\mu k^2-\epsilon\eta ik)\frac{\Gamma(1-\beta)}{\alpha^{\beta}\Gamma(1-\alpha\beta)}t^{\alpha\beta}\right)\times\Psi(k)e^{-ikx}dk+\frac{\Gamma(1-\beta)}{c\rho\alpha^{\beta}\Gamma(1-\alpha\beta)}\;\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}dk\times\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{Q(k,s)}{s^{\alpha\beta}+(\mu k^2-\epsilon\eta ik)\frac{\Gamma(1-\beta)}{\alpha^{\beta}\Gamma(1-\alpha\beta)}}e^{st}ds. \tag{54}$$

In the case when $\alpha=1$ the expression (54) matches the solution obtained in the Eq. (44) in the Liouville-Caputo sense.

Considering the Atangana-Koca fractional-order derivative in the Liouville-Caputo sense we have the following convection-diffusion equation

$$({}_{0}^{AKC}\mathcal{D}_t^{\alpha}u)(x,t)=-\epsilon\eta\frac{\partial}{\partial x}u(x,t)+\mu\frac{\partial^2}{\partial x^2}u(x,t)+\frac{Q(x,t)}{c\rho},\;\;t>0,\;\;0<\alpha\leq 1, \tag{55}$$

$$u(x,0)=\psi(x),\;\;x\in\mathbb{R},\;\;\mu\in\mathbb{R}^{+}, \tag{56}$$

where μ is the diffusion coefficient.

Solution. Applying the Laplace transform to Eq. (55) and taking the condition (56) we get

$$\frac{1}{b}\Big(s^{-n\alpha}\mathcal{L}\left[u\left(x,t\right)\right]-s^{-n\alpha-1}u\left(x,0\right)\Big)=-\epsilon\eta\frac{\partial}{\partial x}\left(\mathcal{L}_t u\right)(x,s)+\mu\frac{\partial^2}{\partial x^2}\left(\mathcal{L}_t u\right)(x,s)+\frac{Q(x,s)}{c\rho}, \tag{57}$$

where $b=g(\alpha)\big(1-g(\alpha)\big)^{\alpha}$.

Applying the Fourier transform to Eq. (57) and simplifying, we have

$$\widehat{u}(k,s)=\frac{s^{-n\alpha-1}\Psi(k)}{s^{-n\alpha}+b\mu k^2-\epsilon\eta bik}+\frac{b}{c\rho}\;\frac{Q(k,s)}{s^{-n\alpha}+b\mu k^2-\epsilon\eta bik}. \tag{58}$$

Applying the inverse Laplace transform and the inverse Fourier transform to Eq. (58) we get

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[1-E_{n\alpha,1}\left(-\frac{t^{n\alpha}}{b\mu k^2-\epsilon\eta bik}\right)\right]\times\Psi(k)e^{-ikx}dk+\frac{b}{2\pi c\rho}\int_{\epsilon-i\infty}^{\epsilon+i\infty}e^{-ikx}dk\;\;\frac{1}{2\pi i}\times\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{Q(k,s)}{s^{-n\alpha}+b\mu k^2-\epsilon\eta bik}e^{st}ds. \tag{59}$$

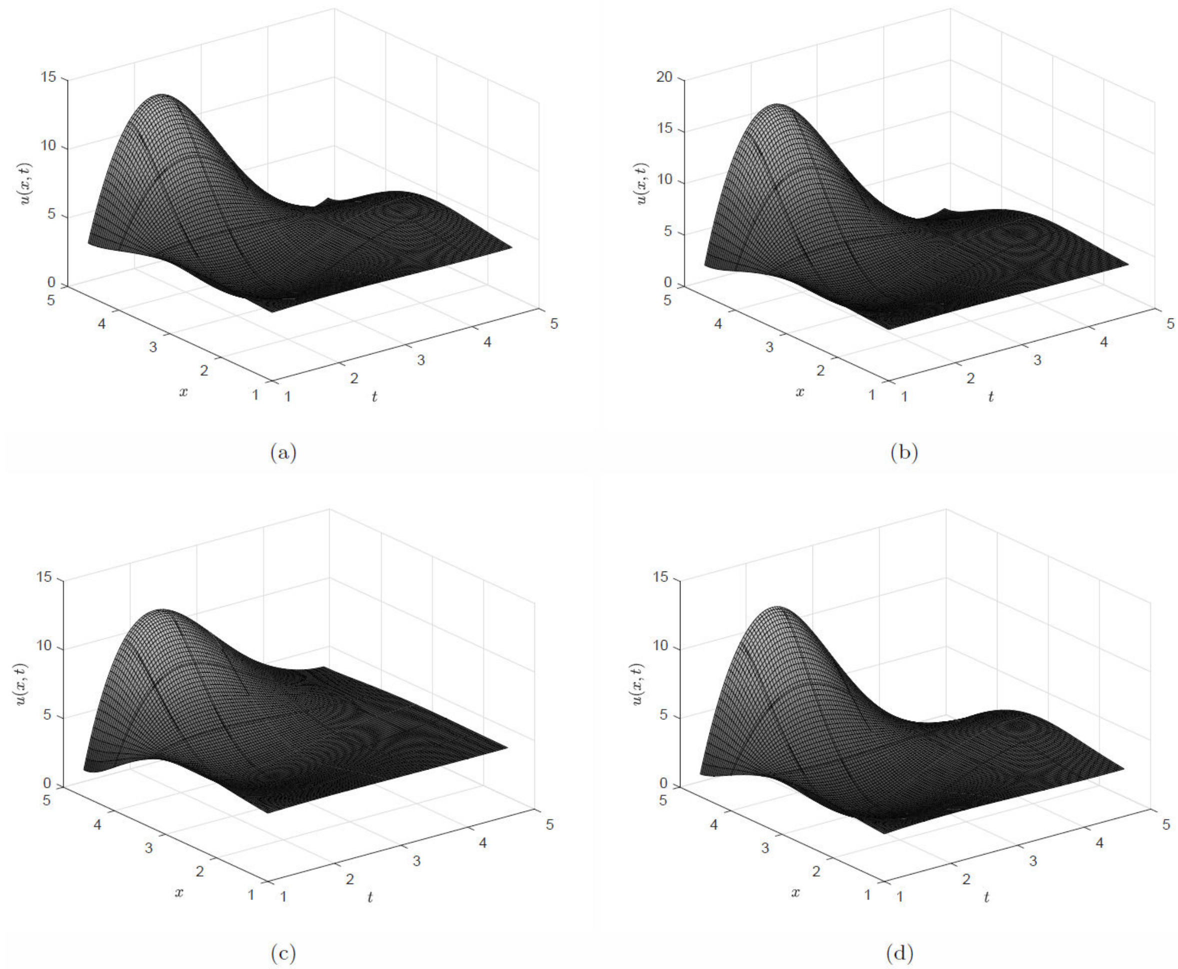

FIGURE 1. Numerical solutions of Eqs. (44), (49), (54) and (59). In (a) Eq. (44); in (b) Eq. (49); in (c) Eq. (54) and (d) (59), we consider $\alpha = 0.85$ for the cases (a), (b), (d) and for (c), we consider $\alpha = 0.92$-$\beta = 0.83$ for the fractional conformable derivative in the Liouville-Caputo sense.

4. Illustrative examples

Figures 1(a-d) show numerical simulations of the Eqs. (44), (49), (54) and (59) for $\alpha = 0.85$ and $\alpha = 0.92$-$\beta = 0.83$ for the fractional conformable derivative in the Liouville-Caputo sense, these values were chosen arbitrarily.

5. Conclusion

In this work we applied fractional-order derivatives of type Liouville-Caputo, Atangana-Baleanu, fractional conformable derivative and Atangana-Koca to obtain analytical solutions for the diffusion and convection-diffusion equation. The fractional equations were solved using the Laplace and Fourier transform. The anomalous diffusion concept is naturally obtained from diffusion equations using the fractional calculus approach. Our results indicate that the kernel involved in the fractional derivative and the fractional-order α has an important influence on the concentration. When memory effects described by the fractional order α are incorporated using fractional time derivatives, the crossover dynamics is richer. The alternative solutions obtained in this paper provide a new theoretical perspective of the diffusion and convection-diffusion phenomena.

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