Generalities on finite element discretization for fractional pressure diffusion equation in the fractal continuum

H.D. Sánchez-Chávez, C.A. López-Ortiz and L. Flores-Cano
Departamento de Física, Universidad Tecnológica de la Mixteca
Km. 2.5 Carretera a Acatlima, Huajuapan de León, Oaxaca, 69000, México.
e-mail: hchavez@mixteco.utm.mx

Received 27 September 2018; accepted 19 October 2018

In this study we explore the application of the novel Fractional Calculus in Fractal Continuum (FCFC), together with the Finite Element Method (FEM), in order to analize explicitly how these differential operators act in the process of discretizing the generalized fractional pressure diffusion equation for a three-dimensional anisotropic continuous fractal flow. The Master Finite Element Equation (MFEE) for arbitrary interpolation functions is obtained. As an example, MFEE for the case of a generic linear tetrahedron in $\mathbb{R}^3$ is shown. Analytic solution for the spatial variables is determined over a canonical tetrahedral finite element in global coordinates.

Keywords: Finite element; fractional calculus in fractal continuum; anisotropic continuous fractal flow; fractional pressure diffusion equation; continuum mechanics.

PACS: 47.56.+r; 47.53.+n; 02.30.Jr.

DOI: https://doi.org/10.31349/RevMexFis.65.251

1. Introduction

Origins of Fractional Order Calculus (FOC) back in time to the end of XVII century in the famous question of L'Hospital to Leibnitz; "What if n be $1/2$?" (question obviously inspired in the very known notation invented by Leibnitz for derivatives), Leibnitz's response to L'Hospital was; "It will lead to a paradox, from this apparent paradox, one day useful consequences will be drawn" [1]. This was the point in time line in which seed of FOC had been planted. Due to the Liouville's works together with those of Riemann, the current definitions of the differential and integral fractional operators of Riemann-Liouville were published in the 1800's, in the same period the definition of the fractional integral of Gründwald-Letnikov also emerges. In the twentieth century, the definitions of the fractional operators of Weyl, Riesz and Caputo arise. The operators mentioned above are among many more definitions of fractional operators, the ones that are currently used or common the most [2, 3]. In the last fifty years, many works based on this fractional calculus operators have been published, to name; Kilbas *et al*. [3], Miller and Ross [4], Oldham and Spanier [5] and Samko *et al*. [6] in the rigorous mathematical context and some others like Strichartz [7] and Kigami [8] have been started to solve partial differential equations on mathematical fractal sets. Recently, important studies related to the application of FOC have been reported, for example; Gómez *et al*. [9] in the modeling of electrical circuits; Coronel *et al.* [13] stuying fractional behavior of BFT and CK oscillators; Atangana and Gómez in the study of the fundamental differences between power law, exponential decay, Mittag-Leffler law and their possible applications to real problems [10]; Atangana [11] in the application of the semigroup principle to the analysis of fractional derivatives of evolutions equations; Morales *et al.* [12] in the discussion of generalized Cauchy problems in diffusion wave processes. Authors like Herrmann [14] and West *et al*. [15] had focussed the FOC to some engineering applications. On the other hand, many researchers have reported findings based on Mandelbrot's ideas for fractal characterization of natural systems [16]; for example, from biological systems [17, 18], computer simulation [19], geological sciences [20–22], folded and crumpled of thin matter [23–25] to fluid flow [27,34], but from the point of view of physics, there was not a proposal on fractality and fractional calculus in the continuum until continuum-type equations for fractal media were proposed by Tarasov [26], that essentially links the fractal dimension of a fractal set with the order of the derivative (or integral). The works in the same line are [26–33, 35, 37]. In [33, 34] the explicit proposal of the FCFC is done.

In the present work, we used the results published in [33, 34] about the fractional calculus operators in the fractal continuum in order to discretize the pressure diffusion equation. Section 2 is devoted to resume important definitions of FCFC together with the pressure transient equation for fractal continuum flow, also derivation of master finite element equation is included in this section. Section 3 includes the discussion of our results and potential uses. We wrote our conclusions in Sec. 4 and finally, details of calculations are shown in Appendix.

2. Basic Theory and Formula Derivation

2.1. Fractional calculus in fractal continuum

The FCFC of authors of [33,34], is built on the basis of Tarazov's aproximation to the continuum physics and mechanics [26, 27], and it basically consist in the transformation of a problem of a intrinsically discontinuous medium (fractal) onto a problem in a continuous space (Euclidean) in which this fractal is embedded [30], dealing in the process with linear, superficial and volume fractional infinitesimal coefficients, this coefficients are written in terms of fractal dimensionalities proper of the medium and are supported by a specific metric well defined as we can see in [34] and its function is to vinculate the Euclidean differential elements with fractals ones, they rewrite the concept of Hausdorff derivative given in [32] in terms of an ordinary derivative multiplied by a power law function of the variable x as:

$$\frac{d^H}{dx^\zeta}f = \left(\frac{x}{l_0}+1\right)^{1-\zeta}\frac{d}{dx}f = \frac{{l_0}^{\zeta-1}}{c_1}\frac{d}{dx}f = \frac{d}{d^\zeta x}f$$

where the function $c_1 = c_1(x,\zeta)$ is defined as the Density Of States (DOS) in the fractal continuum along R^1 [33,34]. The DOS describes in this case, how permitted states of particles are closely packed in the x axis. The expression $dx_D = c_1 dx$ represents the number of states (permitted places) between x and $x+dx$ [34]. Now, Hausdorff's partial derivative is defined as:

$$\nabla_k^H = \left(\frac{x_k}{l_k}+1\right)^{1-\zeta_k}\frac{\partial}{\partial x_k} \quad \text{where} \quad \zeta_k = D - d_k \tag{1}$$

and definition of fractional Laplacian is:

$$\nabla_i^H\nabla_i^H\psi = \sum_i^3\left(\chi^{(i)}\right)^2\left[\frac{\partial^2\psi}{\partial x_i^2}+\frac{1-\zeta_i}{x_i+l_i}\left(\frac{\partial\psi}{\partial x_i}\right)\right] \tag{2}$$

where:

$$\chi^{(i)} = \frac{l_i^{\zeta_i-1}}{c_1^{(i)}(x_i)} = \left(\frac{x_i}{l_i}+1\right)^{1-\zeta_i} \tag{3}$$

this Hausdorff Laplacian turns to ordinary Laplacian when $\zeta_i = \alpha_i = 1$. Other vector operators with significant relevance for this work are $\overrightarrow{\nabla}^H$, $\overrightarrow{\nabla}^H\psi$ and $\overrightarrow{\nabla}^H\cdot\overrightarrow{\Psi}$ where $\overrightarrow{\Psi} = (\psi_1,\psi_2,\psi_3)$ represents any vector field in the fractal flow, which are defined as:

$$\overrightarrow{\nabla}^H = \overrightarrow{e}_1\chi^{(1)}\frac{\partial}{\partial x_1}+\overrightarrow{e}_2\chi^{(2)}\frac{\partial}{\partial x_2}+\overrightarrow{e}_3\chi^{(3)}\frac{\partial}{\partial x_3} \tag{4}$$

$$\overrightarrow{\nabla}^H\psi = \left(\nabla_1^H\psi\right)\overrightarrow{e}_1+\left(\nabla_2^H\psi\right)\overrightarrow{e}_2+\left(\nabla_3^H\psi\right)\overrightarrow{e}_3 \tag{5}$$

$$\overrightarrow{\nabla}^H\cdot\overrightarrow{\Psi} = \sum_i^3\nabla_i^H\psi_i \tag{6}$$

respectively, where $\overrightarrow{e}_i$ are the base vectors, $\psi(x_i)$ scalar function and symbol "·" is the usual scalar product. Accordingly with [34] in the 3D case, the DOS, is defined analogous to dx_D for one-dimension by the expression:

$$dV_D = c_3(x_i,D)\,dV = c_3(x_i,D)\,dxdydz \tag{7}$$

where c_3 is part of the fractal metric defined in [34]. A useful and clarifying definition of c_3 is done in [30]. More definitions of operators of FCFC can be consulted in [33, 34], we have included just those ones we are going to employ in the next sections.

2.2. Pressure transient equation for fractal continuum flow

In order to get the transient pressure equation for fractal continuum flow, as in the classical case, it is necessary to relate the generalized Darcy equation:

$$u_i = -\frac{K_{ij}^{(c)}}{\mu_c}\nabla_i^H(p-h_g) \tag{8}$$

with equation for slightly compressible liquids:

$$\frac{\partial\rho_c}{\partial t} = c\rho_c\frac{\partial p}{\partial t} \tag{9}$$

and continuity equation:

$$\frac{\partial\rho_c}{\partial t} = -\overrightarrow{\nabla}^H\cdot\rho_c\overrightarrow{u} \tag{10}$$

then, susbtituing (8) and (9) into (10) the result reads:

$$c\mu_c\frac{\partial p}{\partial t} = \overrightarrow{\nabla}^H\cdot\left(K_{ii}^{(c)}\overrightarrow{\nabla}^H(p-h_g)\right) \tag{11}$$

where is assume that characteristic tensor property of the fractal continuum flow $K_{ij}^{(c)} = 0$ for $i \neq j$ [34]. Equation (11) is the well known pressure diffusion equation for the case of an anisotropic three-dimensional fractal continuum flow as is referred in [34], h_g from expression (11) represents the gravitational head defined as:

$$h_g = p_0 - g\zeta_z\rho_0 l_3\left(\frac{x_3}{l_3}+1\right)^{\zeta_z} \tag{12}$$

and c is the coefficient of fractal continuum compressibility [34].

2.3. Formula derivation

Using Eqs. (5) and (6) to rewrite (11) we obtain the partial differential equation:

$$c\mu_c\frac{\partial\phi}{\partial t}=\chi^{(x)}\frac{\partial}{\partial x}\left(K_{11}^{(c)}\chi^{(x)}\frac{\partial\phi}{\partial x}\right)+\chi^{(y)}\frac{\partial}{\partial y}\left(K_{22}^{(c)}\chi^{(y)}\frac{\partial\phi}{\partial y}\right)+\chi^{(z)}\frac{\partial}{\partial z}\left(K_{33}^{(c)}\chi^{(z)}\frac{\partial\phi}{\partial z}\right) \tag{13}$$

where:

$$\phi = p(x_i,t) - h_g(x_i) \tag{14}$$

with $p(x_i,t) = N^T d$ and h_g given by (12) [35], multiplying (13) by $A = c\mu_c\left(\chi^{(x)}\chi^{(y)}\chi^{(z)}\right)^{-1}$ and rearranging terms, we get:

$$\frac{\partial\phi}{\partial t}-\frac{1}{A}\left\{\frac{\partial}{\partial x}\left(K_{11}^{(c)}A_x\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{22}^{(c)}A_y\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_{33}^{(c)}A_z\frac{\partial\phi}{\partial z}\right)\right\}=0$$

where $A_i = (\chi^i/\chi^j\chi^k)$. In order to use Galerkin's method, we first develop an appropriate weak form, as is usual in FEM [35]. We can assume that V is the volume of an arbitrary finite element then, multiplying by the weighting functions N_i, integrating over all the volume and taking into account (7), the Galerkin weighted residual is:

$$\iiint\limits_V \frac{\partial\phi}{\partial t} N_i \mathrm{d}V_D - \iiint\limits_V \left\{ \frac{\partial}{\partial x}\left(K_{11}^{(c)} A_x \frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{22}^{(c)} A_y \frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_{33}^{(c)} A_z \frac{\partial\phi}{\partial z}\right) \right\} \frac{c_3}{A} N_i \mathrm{d}V_3 = 0$$

writing $\gamma = (c_3/A)$ and carrying out an integration by parts process, leads to:

$$\iiint\limits_V \frac{\partial\phi}{\partial t} N_i \mathrm{d}V_D - \sum_{j=1}^{3} \iint\limits_{S_n} A_j K_{jj}^{(c)} \frac{\partial\phi}{\partial x_j} N_i n_j \gamma \mathrm{d}S_2 + \sum_{j=1}^{3} \iiint\limits_V A_j K_{jj}^{(c)} \frac{\partial\phi}{\partial x_j} \frac{\partial N_i}{\partial x_j} \gamma \mathrm{d}V_3 = 0.$$

applying the surface natural boundary condition:

$$k\frac{\partial\phi}{\partial n} \equiv \left(k_x \frac{\partial\phi}{\partial x} n_x + k_y \frac{\partial\phi}{\partial y} n_y + k_z \frac{\partial\phi}{\partial z} n_z\right) = \alpha\phi + \beta$$

with $k_i = K_{ii} A_i$, α and β are known parameters along the boundary [35]. Taking into account that general solution over an element has the form:

$$\phi(x,y,z,t) = \left(N_1(x,y,z)\, N_2(x,y,z) \cdots N_n(x,y,z)\right) \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix} = \mathbf{N^T d}$$

we get:

$$\iiint\limits_V N_i \mathbf{N}^T \mathrm{d}V_D \;\; \dot{\mathbf{d}} - \iint\limits_{S_n} \alpha N_i \mathbf{N}^T \gamma \mathrm{d}S_2 \;\; \mathbf{d} + \iint\limits_{S_n} \alpha h_g N_i \gamma \mathrm{d}S_2 - \iint\limits_{S_n} \beta N_i \gamma \mathrm{d}S_2$$
$$+ \sum_{j=1}^{3} \iiint\limits_V K_{jj}^{(c)} A_j \frac{\partial N_i}{\partial x_j} \frac{\partial \mathbf{N}^T}{\partial x_j} \gamma \mathrm{d}V_3 \;\; \mathbf{d} - \iiint\limits_V K_{33}^{(c)} A_3 \frac{\partial h_g}{\partial x_3} \frac{\partial N_i}{\partial x_3} \gamma \mathrm{d}V_3 = 0 \tag{15}$$

Taking into consideration expression (12) and arranging terms, (15) turns to:

$$\iiint\limits_V N_i \mathbf{N}^T \mathrm{d}V_D \;\; \dot{\mathbf{d}} - \iint\limits_{S_n} \alpha N_i \mathbf{N}^T \gamma \mathrm{d}S_2 \;\; \mathbf{d} + \iint\limits_{S_n} \alpha h_g N_i \gamma \mathrm{d}S_2 - \iint\limits_{S_n} \beta N_i \gamma \mathrm{d}S_2$$
$$+ \sum_{j=1}^{3} \iiint\limits_V K_{jj}^{(c)} A_j \frac{\partial N_i}{\partial x_j} \frac{\partial \mathbf{N}^T}{\partial x_j} \gamma \mathrm{d}V_3 \;\; \mathbf{d} - \iiint\limits_V K_{33}^{(c)} g \zeta_z^2 \ell_z^{1-\zeta_z} (z+\ell_z)^{\zeta_z - 1} \frac{\partial N_i}{\partial x_3} \gamma \mathrm{d}V_3 = 0. \tag{16}$$

the three terms inside the second integral of volume of (16), can be expressed in matrix form as follows:

$$g_1 \mathbf{B}_x \mathbf{B}_x^T + g_2 \mathbf{B}_y \mathbf{B}_y^T + g_3 \mathbf{B}_z \mathbf{B}_z^T = \begin{pmatrix} \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{pmatrix} \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix} \begin{pmatrix} \mathbf{B}_x^T \\ \mathbf{B}_y^T \\ \mathbf{B}_z^T \end{pmatrix} \equiv \mathbf{BCB}^T$$

where:

$$\mathbf{B}^T = \begin{pmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} & \frac{\partial N_4}{\partial z} \end{pmatrix}; \qquad \mathbf{C} = \begin{pmatrix} A_1 K_{11}^{(c)} \gamma & 0 & 0 \\ 0 & A_2 K_{22}^{(c)} \gamma & 0 \\ 0 & 0 & A_3 K_{33}^{(c)} \gamma \end{pmatrix}$$

and:

$$q = K_{33}^{(c)} g \zeta_z^2 \ell_z^{\zeta_z - 1} c_1^{(x)} c_1^{(y)}$$

therefore, the finite element equations are:

$$\iiint\limits_V \mathbf{NN}^T \mathrm{d}V_D \;\; \dot{\mathbf{d}} + \iiint\limits_V \mathbf{BCB}^T \mathrm{d}V \;\; \mathbf{d} - \iint\limits_{S_n} \alpha \mathbf{NN}^T \gamma \mathrm{d}S_2 \;\; \mathbf{d}$$
$$= \iiint\limits_V q \frac{\partial \mathbf{N}}{\partial x_3} \mathrm{d}V + \iint\limits_{S_n} \beta \mathbf{N} \gamma \mathrm{d}S_2 - \iint\limits_{S_n} \alpha h_g \mathbf{N} \gamma \mathrm{d}S_2$$

that has the typical form:

$$M\dot{\mathbf{d}} + (K_k + K_\alpha)\,\mathbf{d} = r_q + r_\beta + r_\alpha \tag{17}$$

where:

$$M = \iiint\limits_V \mathbf{N}\mathbf{N}^T \mathrm{d}V_D \qquad K_k = \iiint\limits_V \mathbf{B}\mathbf{C}\mathbf{B}^T \mathrm{d}V \qquad K_\alpha = \iint\limits_{S_n} \alpha\mathbf{N}\mathbf{N}^T\gamma \mathrm{d}S_2$$

$$r_q = \iiint\limits_V q\frac{\partial \mathbf{N}}{\partial x_3}\mathrm{d}V \qquad r_\beta = \iint\limits_{S_n} \beta\mathbf{N}\gamma\mathrm{d}S_2 \quad \text{and} \quad r_\alpha = \iint\limits_{S_n} \alpha h_g \mathbf{N}\gamma\mathrm{d}S_2 \tag{18}$$

(17) represents a system of first order ordinary differential equations, also is the MFEE of (11) for general weighting functions N_i in $\mathbb{R}^3$ [37, 38]. In the case that $\mathbf{N}$ is conformed by the conventional interpolation functions for arbitrary linear tetrahedron in $\mathbb{R}^3$ as is referred in [36, 37], it would be simple to see that (18) includes information of the tetrahedral coordinates N_i (or $(r, s, t, 1-r-s-t)$), such expression would be written as:

$$M = \iiint\limits_V \mathbf{N}\mathbf{N}^T \mathrm{d}V_D = \iiint\limits_V \mathbf{N}\mathbf{N}^T c_3 dxdydz = \int\limits_0^1 \int\limits_0^{1-r} \int\limits_0^{1-r-s} \mathbf{N}\mathbf{N}^T c_3 \mathbf{J} drdsdt \tag{19}$$

where $\mathbf{J}$ represents the Jacobian transformation matrix between both reference frames [38] and explicit value of c_3 is given by:

$$c_3\,(r,s,t) = \left[l_x\left(\frac{\sum_{i=1}^4 N_i x_i}{l_x} + 1\right)\right]^{\zeta_x - 1}\left[l_y\left(\frac{\sum_{i=1}^4 N_i y_i}{l_y} + 1\right)\right]^{\zeta_y - 1}\left[l_z\left(\frac{\sum_{i=1}^4 N_i z_i}{l_z} + 1\right)\right]^{\zeta_z - 1} \tag{20}$$

according with [33]. Analog expressions can be arise for the remaining terms of (18). Term M is the coefficient of time derivatives of the nodal variables. From equation (17), ith-equation is written as

$$(M)_{ij}\dot{d}_j + \left((K_k)_{ij} + (K_\alpha)_{ij}\right) d_j = (r_q)_i + (r_\beta)_i + (r_\alpha)_i \tag{21}$$

which, in this work, we solved analitically for the spatial variables of the particular case of a canonical tetrahedron in the Euclidean reference frame (vertices $(-\ell_x, -\ell_y, -\ell_z)$, $(1-\ell_x, -\ell_y, -\ell_z)$, $(-\ell_x, 1-\ell_y, -\ell_z)$ and $(-\ell_x, -\ell_y, 1-\ell_z)$). For this case, master finite element equation is:

$$\begin{aligned}
&\left(\frac{1}{6V}\right)^2 \Bigg[6V_{0i}6V_{0j}\theta + (6V_{0i}a_j + 6V_{0j}a_i)\left(\frac{\zeta_x}{\zeta_x+\zeta_y+\zeta_z+1} - \ell_x\right)\theta \\
&\quad + (6V_{0i}b_j + 6V_{0j}b_i)\left(\frac{\zeta_y}{\zeta_x+\zeta_y+\zeta_z+1} - \ell_y\right)\theta + (6V_{0i}c_j + 6V_{0j}c_i)\left(\frac{\zeta_z}{\zeta_x+\zeta_y+\zeta_z+1} - \ell_z\right)\theta \\
&\quad + (a_i b_j + a_j b_i)\left(\frac{\zeta_x\zeta_y}{(\zeta_x+\zeta_y+\zeta_z+2)(\zeta_x+\zeta_y+\zeta_z+1)} - \ell_y\frac{\zeta_x}{\zeta_x+\zeta_y+\zeta_z+1} - \ell_x\frac{\zeta_y}{\zeta_x+\zeta_y+\zeta_z+1} + \ell_x\ell_y\right)\theta \\
&\quad + (a_i c_j + a_j c_i)\left(\frac{\zeta_x\zeta_z}{(\zeta_x+\zeta_y+\zeta_z+2)(\zeta_x+\zeta_y+\zeta_z+1)} - \ell_z\frac{\zeta_x}{\zeta_x+\zeta_y+\zeta_z+1} - \ell_x\frac{\zeta_z}{\zeta_x+\zeta_y+\zeta_z+1} + \ell_x\ell_z\right)\theta \\
&\quad + (b_i c_j + b_j c_i)\left(\frac{\zeta_y\zeta_z}{(\zeta_x+\zeta_y+\zeta_z+2)(\zeta_x+\zeta_y+\zeta_z+1)} - \ell_z\frac{\zeta_y}{\zeta_x+\zeta_y+\zeta_z+1} - \ell_y\frac{\zeta_z}{\zeta_x+\zeta_y+\zeta_z+1} + \ell_y\ell_z\right)\theta \\
&\quad + a_i a_j\left(\frac{\zeta_x(\zeta_x+1)}{(\zeta_x+\zeta_y+\zeta_z+2)(\zeta_x+\zeta_y+\zeta_z+1)} - 2\ell_x\frac{\zeta_x}{\zeta_x+\zeta_y+\zeta_z+1} + \ell_x^2\right)\theta \\
&\quad + b_i b_j\left(\frac{\zeta_y(\zeta_y+1)}{(\zeta_x+\zeta_y+\zeta_z+2)(\zeta_x+\zeta_y+\zeta_z+1)} - 2\ell_y\frac{\zeta_y}{\zeta_x+\zeta_y+\zeta_z+1} + \ell_y^2\right)\theta \\
&\quad + c_i c_j\left(\frac{\zeta_z(\zeta_z+1)}{(\zeta_x+\zeta_y+\zeta_z+2)(\zeta_x+\zeta_y+\zeta_z+1)} - 2\ell_z\frac{\zeta_z}{\zeta_x+\zeta_y+\zeta_z+1} + \ell_z^2\right)\theta\Bigg]\dot{d}_j \\
&\quad + \Bigg(a_i a_j\frac{\Gamma(2-\zeta_x)\Gamma(\zeta_y)\Gamma(\zeta_z)}{\Gamma(3-\zeta_x+\zeta_y+\zeta_z)}\ell_x^{2(\zeta_x-1)} + b_i b_j\frac{\Gamma(\zeta_x)\Gamma(2-\zeta_y)\Gamma(\zeta_z)}{\Gamma(3+\zeta_x-\zeta_y+\zeta_z)}\ell_y^{2(\zeta_y-1)} \\
&\quad + c_i c_j\frac{\Gamma(\zeta_x)\Gamma(\zeta_y)\Gamma(2-\zeta_z)}{\Gamma(3+\zeta_x+\zeta_y-\zeta_z)}\ell_z^{2(\zeta_z-1)} + K_\alpha\Bigg)d_j = 2K_{33}^{(c)} g\zeta_z^2\ell_z^{\zeta_z-1}\frac{\Gamma(\zeta_x)\Gamma(\zeta_y)}{\Gamma(\zeta_x+\zeta_y+2)}c_i + r_\beta + r_\alpha.
\end{aligned} \tag{22}$$

details of calculations that we made can be read in Appendix.

3. Discussion

Actually, problems dealing with transport phenomena are very important in science and engineering, particularly, in the study of porous media there is a great research activity both theoretical and experimental [27, 33, 34, 40–43, 45–54]. On the other hand, since researchers began to apply fractional calculus in order to solve diverse engineering problems, many authors have made important contributions as we have referred before because of that, importance of modelling this type of systems lies in the successful forecast of the behavior that have quantities like flows, speeds, amounts of matter, pressure drops, etc. In real systems, the difficulty is that big because the medium in question is characterized by very complex geometric shapes, turning the modelling in a strong mathematical challenge, for that reason, the FCFC has special significance [34]. In that sense, we can notice that differential equation (22) contains the geometry information associated with the fractal medium under study through the corresponding fractal dimensions, ζ_i, D, d_i, cut off lower limits ℓ_i and transformation function c_3, [33, 34]. In the present case, we have employed the FCFC in the discretization process of the three-dimensional pressure diffusion equation for the anisotropic continuum fractal flow published in [34], it can be written in computer codes in any programming language and be of great interest in the field of computer simulation.

The discretization process of the parabolic equation (11) was written in (18) for general form functions N_i, rewritten for arbitrary linear tetrahedron in (19) and solved analytically for spatial variables over the canonical tetrahedron in (22). We have shown explicitly the process to be followed with other types of finite elements. We also mention that the integral formulas that have been obtained analytically for the spatial case are general in the sense that they were solved for non-particular fractional parameter values, such parameters will depend on the geometry of the system to be simulated.

The fractional transient-pressure equation for flow in a porous medium has been solved analytically in [34], its solution corresponds to the specific case of radial contribution in a cylindrical symmetry domain with isotropic porosity. This type of results are helpful, for example, in the oil industry (well production analysis) or in the characterization of aquifers. From the point of view of software tools, it is useful to have numerical procedures for the solution of this type of equations moreover, in the computational field, one can aspire to solve more complex cases like anisotropic one. In the present work, we have focused on the application of FEM for the most generic resolution of such pressure equation.

The results, by themselves, are already of significance for the computational implementation and allow the more accurate calculation of the integrals that appear in the matrix elements of the formulation, reducing computational complexity and also clarifies the panorama of the applicability of such method in this case of relative novelty.

4. Conclusions

We employ the FCFC defined by means of fractional operators (1), (2) and (6) of [33, 34] that relate a discontinuous system with a continuous one through the transformation function defined by (7) in order to get the MFEE for the transient-pressure equation in a three-dimensional continuum fractal flow. Explicit form of coefficient c_3 for the geometry of a linear tetrahedron is given in (20). We have solved analitically the integral formulas for the spatial variables of (17) for the case of a canonical tetrahedron anchored in vertices $(-\ell_x, -\ell_y, -\ell_z)$, $(1-\ell_x, -\ell_y, -\ell_z)$, $(-\ell_x, 1-\ell_y, -\ell_z)$ and $(-\ell_x, -\ell_y, 1-\ell_z)$ using a very similar process to the one carried out in the literature of mathematical methods to obtain the Dirichlet's integral formula [55, 56]. We also mention that the results we obtained in this work can be linked to real field data that allow the development of adequate computer simulations. As a continuity of the present work, in a future publication, we will report a robust implementation that allows to see graphically the contrast that has the inclusion of the intrinsic geometry of the medium in the modeling of real application pressure diffusion problems, in contrast with the usual Euclideans aproximations implemented in commercial simulations softwares that not include fractional and fractal features.

Appendix

A.

In this section, we include the details of calculations done in order to solve each volume term of (17). Let's start with:

$$M = \iiint_V \mathbf{N}\mathbf{N}^T \mathrm{d}V_D$$

where:

$$\mathbf{N}\mathbf{N}^T = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix} \begin{pmatrix} N_1 & N_2 & N_3 & N_4 \end{pmatrix} = \begin{pmatrix} N_1N_1 & N_1N_2 & N_1N_3 & N_1N_4 \\ N_2N_1 & N_2N_2 & N_2N_3 & N_2N_4 \\ N_3N_1 & N_3N_2 & N_3N_3 & N_3N_4 \\ N_4N_1 & N_4N_2 & N_4N_3 & N_4N_4 \end{pmatrix} \tag{A.1}$$

and term M_{ij} is:

$$M_{ij} = \left(\frac{1}{6V}\right)^2 \int\limits_W [6V_{0i}6V_{0j} + (6V_{0i}a_j + 6V_{0j}a_i)x + (6V_{0i}b_j + 6V_{0j}b_i)y + (6V_{0i}c_j + 6V_{0j}c_i)z$$
$$+(a_ib_j + a_jb_i)xy + (a_ic_j + a_jc_i)xz + (b_ic_j + b_jc_i)yz + a_ia_jx^2 + b_ib_jy^2 + c_ic_jz^2\big]\, c_3(x_i, D)\mathrm{d}V_3 \qquad \text{(A.2)}$$

letting $u_x = (x+\ell_x)$, $u_y = (y+\ell_y)$ y $u_z = (z+\ell_z)$ the transformation function $c_3(u_i, D)\mathrm{d}V_3$ turns into $u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x$, working with the tetrahedron mentioned in previous sections, we get the next ten integrals whose procedure solution and solutions are shown:

1. $$6V_{0i}6V_{0j}\int\limits_0^1\int\limits_0^{1-u_x}\int\limits_0^{1-u_x-u_y} u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x,$$

2. $$(6V_{0i}a_j + 6V_{0j}a_i)\int\limits_0^1\int\limits_0^{1-u_x}\int_0^{1-u_x-u_y} (u_x-\ell_x)u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x,$$

3. $$(6V_{0i}ab_j + 6V_{0j}b_i)\int\limits_0^1\int\limits_0^{1-u_x}\int_0^{1-u_x-u_y} (u_y-\ell_y)u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x,$$

4. $$(6V_{0i}c_j + 6V_{0j}c_i)\int\limits_0^1\int\limits_0^{1-u_x}\int\limits_0^{1-u_x-u_y} (u_z-\ell_z)u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x,$$

5. $$(a_ib_j + a_jb_i)\int\limits_0^1\int\limits_0^{1-u_x}\int\limits_0^{1-u_x-u_y} (u_x-\ell_x)(u_y-\ell_y)u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x,$$

6. $$(a_ic_j + a_jc_i)\int\limits_0^1\int\limits_0^{1-u_x}\int\limits_0^{1-u_x-u_y} (u_x-\ell_x)(u_z-\ell_z)u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x,$$

7. $$(b_ic_j + b_jc_i)\int\limits_0^1\int\limits_0^{1-u_x}\int\limits_0^{1-u_x-u_y} (u_y-\ell_y)(u_z-\ell_z)u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x,$$

8. $$(a_ia_j)\int\limits_0^1\int\limits_0^{1-u_x}\int\limits_0^{1-u_x-u_y} (u_x-\ell_x)^2u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x,$$

9. $$(b_ib_j)\int\limits_0^1\int\limits_0^{1-u_x}\int\limits_0^{1-u_x-u_y} (u_y-\ell_y)^2u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x,$$

10. $$(c_ic_j)\int\limits_0^1\int\limits_0^{1-u_x}\int\limits_0^{1-u_x-u_y} (u_z-\ell_z)^2u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x.$$

Solution of integral 1.

1. $$\int\limits_0^1\int\limits_0^{1-u_x}\int\limits_0^{1-u_x-u_y} u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x = \int\limits_0^1\int\limits_0^{1-u_x} u_x^{\zeta_x-1}u_y^{\zeta_y-1}\left(\frac{u_z^{\zeta_z}}{\zeta_z}\right)_0^{1-u_x-u_y}\mathrm{d}u_y\mathrm{d}u_x$$
$$= \frac{1}{\zeta_z}\int\limits_0^1\int\limits_0^{1-u_x} u_x^{\zeta_x-1}u_y^{\zeta_y-1}(1-u_x-u_y)^{\zeta_z}\mathrm{d}u_y\mathrm{d}u_x$$

setting $u_y = (1-u_x)t \rightarrow du_y = (1-u_x)dt$, changing the integration limits to $t(u_y = 0) = 0$ and $t(u_y = 1-u_x) = 1$, and rearranging integrals we get:

$$= \frac{1}{\zeta_z}\left(\int_0^1 u_x^{\zeta_x-1}(1-u_x)^{\zeta_y+\zeta_z}\mathrm{d}u_x\right)\left(\int_0^1 t^{\zeta_y-1}(1-t)^{\zeta_z}\mathrm{d}t\right)$$

using definition of Beta Function:

$$\frac{1}{\zeta_z}\frac{\Gamma(\zeta_x)\Gamma(\zeta_y+\zeta_z+1)}{\Gamma(\zeta_x+\zeta_y+\zeta_z+1)}\frac{\Gamma(\zeta_y)\Gamma(\zeta_z+1)}{\Gamma(\zeta_y+\zeta_z+1)} = \frac{1}{\zeta_z}\frac{\Gamma(\zeta_x)\Gamma(\zeta_y)\Gamma(\zeta_z+1)}{\Gamma(\zeta_x+\zeta_y+\zeta_z+1)}$$

from properties $\Gamma(\beta+1) = \beta\Gamma(\beta)$ of Gamma function we have:

$$\int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y} u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x = \frac{\Gamma(\zeta_x)\Gamma(\zeta_y)\Gamma(\zeta_z)}{\Gamma(\zeta_x+\zeta_y+\zeta_z+1)} = \theta \tag{A.3}$$

carrying out same procedures for the remaining integrals:

2. $$\int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y} (u_x-\ell_x)u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x = \left(\frac{\zeta_x}{\zeta_x+\zeta_y+\zeta_z+1}-\ell_x\right)\theta$$

3. $$\int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y} (u_y-\ell_y)u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x = \left(\frac{\zeta_y}{\zeta_x+\zeta_y+\zeta_z+1}-\ell_y\right)\theta$$

4. $$\int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y} (u_y-\ell_y)u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x = \left(\frac{\zeta_z}{\zeta_x+\zeta_y+\zeta_z+1}-\ell_z\right)\theta$$

5. $$\int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y} (u_x-\ell_x)(u_y-\ell_y)u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x$$

$$= \left(\frac{\zeta_x\zeta_y}{(\zeta_x+\zeta_y+\zeta_z+2)(\zeta_x+\zeta_y+\zeta_z+1)} - \ell_y\frac{\zeta_x}{\zeta_x+\zeta_y+\zeta_z+1} - \ell_x\frac{\zeta_y}{\zeta_x+\zeta_y+\zeta_z+1} + \ell_x\ell_y\right)\theta$$

6. $$\int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y} (u_x-\ell_x)(u_z-\ell_z)u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x$$

$$= \left(\frac{\zeta_x\zeta_z}{(\zeta_x+\zeta_y+\zeta_z+2)(\zeta_x+\zeta_y+\zeta_z+1)} - \ell_z\frac{\zeta_x}{\zeta_x+\zeta_y+\zeta_z+1} - \ell_x\frac{\zeta_z}{\zeta_x+\zeta_y+\zeta_z+1} + \ell_x\ell_z\right)\theta$$

7. $$\int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y} (u_y-\ell_y)(u_z-\ell_z)u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x$$

$$= \left(\frac{\zeta_y\zeta_z}{(\zeta_x+\zeta_y+\zeta_z+2)(\zeta_x+\zeta_y+\zeta_z+1)} - \ell_z\frac{\zeta_y}{\zeta_x+\zeta_y+\zeta_z+1} - \ell_y\frac{\zeta_z}{\zeta_x+\zeta_y+\zeta_z+1} + \ell_y\ell_z\right)\theta$$

8. $$\int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y} (u_x-\ell_x)^2u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x$$

$$= \left(\frac{\zeta_x(\zeta_x+1)}{(\zeta_x+\zeta_y+\zeta_z+2)(\zeta_x+\zeta_y+\zeta_z+1)} - 2\ell_x\frac{\zeta_x}{\zeta_x+\zeta_y+\zeta_z+1} + \ell_x^2\right)\theta$$

$$\mathbf{9}.\quad \int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y}(u_y-\ell_y)^2u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x$$
$$=\left(\frac{\zeta_y(\zeta_y+1)}{(\zeta_x+\zeta_y+\zeta_z+2)(\zeta_x+\zeta_y+\zeta_z+1)}-2\ell_y\frac{\zeta_y}{\zeta_x+\zeta_y+\zeta_z+1}+\ell_y^2\right)\theta$$

$$\mathbf{10}.\quad \int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y}(u_z-\ell_z)^2u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x$$
$$=\left(\frac{\zeta_z(\zeta_z+1)}{(\zeta_x+\zeta_y+\zeta_z+2)(\zeta_x+\zeta_y+\zeta_z+1)}-2\ell_z\frac{\zeta_z}{\zeta_x+\zeta_y+\zeta_z+1}+\ell_z^2\right)\theta$$

For the term:

$$K_k=\int_W \mathbf{BCB}^T\mathrm{d}V_3$$

where:

$$\mathbf{BCB}^T=\begin{pmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\\a_4&b_4&c_4\end{pmatrix}\begin{pmatrix}g_1&0&0\\0&g_2&0\\0&0&g_3\end{pmatrix}\begin{pmatrix}a_1&a_2&a_3&a_4\\b_1&b_2&b_3&b_4\\c_1&c_2&c_3&c_4\end{pmatrix}$$

with general form:

$$(K_k)_{ij}=\int_W(a_ia_jg_1+b_ib_jg_2+c_ic_jg_3)\,\mathrm{d}V_3 \tag{A.4}$$

gives the next three follow integrals, whose solutions can be obtained in the analog manner we showed before:

$$1.\quad a_ia_j\int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y}\ell_x^{2(\zeta_x-1)}u_x^{1-\zeta_x}u_y^{\zeta_y-1}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x=\frac{\Gamma(2-\zeta_x)\Gamma(\zeta_y)\Gamma(\zeta_z)}{\Gamma(3-\zeta_x+\zeta_y+\zeta_z)}\ell_x^{2(\zeta_x-1)}$$

$$2.\quad b_ib_j\int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y}\ell_y^{2(\zeta_y-1)}u_x^{\zeta_x-1}u_y^{1-\zeta_y}u_z^{\zeta_z-1}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x=\frac{\Gamma(\zeta_x)\Gamma(2-\zeta_y)\Gamma(\zeta_z)}{\Gamma(3+\zeta_x-\zeta_y+\zeta_z)}\ell_y^{2(\zeta_y-1)}$$

$$3.\quad c_ic_j\int_0^1\int_0^{1-u_x}\int_0^{1-u_x-u_y}\ell_z^{2(\zeta_z-1)}u_x^{\zeta_x-1}u_y^{\zeta_y-1}u_z^{1-\zeta_z}\mathrm{d}u_z\mathrm{d}u_y\mathrm{d}u_x=\frac{\Gamma(\zeta_x)\Gamma(\zeta_y)\Gamma(2-\zeta_z)}{\Gamma(3+\zeta_x+\zeta_y-\zeta_z)}\ell_z^{2(\zeta_z-1)}.$$

the general term remains as:

$$(K_k)_{ij}=a_ia_j\frac{\Gamma(2-\zeta_x)\Gamma(\zeta_y)\Gamma(\zeta_z)}{\Gamma(3-\zeta_x+\zeta_y+\zeta_z)}\ell_x^{2(\zeta_x-1)}+b_ib_j\frac{\Gamma(\zeta_x)\Gamma(2-\zeta_y)\Gamma(\zeta_z)}{\Gamma(3+\zeta_x-\zeta_y+\zeta_z)}\ell_y^{2(\zeta_y-1)}$$
$$+c_ic_j\frac{\Gamma(\zeta_x)\Gamma(\zeta_y)\Gamma(2-\zeta_z)}{\Gamma(3+\zeta_x+\zeta_y-\zeta_z)}\ell_z^{2(\zeta_z-1)}. \tag{A.5}$$

finally for the integral:

$$r_q=\int_W q\frac{\partial N_i}{\partial x_3}\mathrm{d}V_3 \tag{A.6}$$

where:

$$q\frac{\partial N_i}{\partial x_3}=K_{33}^{(c)}g\zeta_z^2\ell_z^{\zeta_z-1}c_1^{(x)}c_1^{(y)}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}\qquad\Rightarrow\qquad r_q=2K_{33}^{(c)}g\zeta_z^2\ell_z^{\zeta_z-1}\frac{\Gamma(\zeta_x)\Gamma(\zeta_y)}{\Gamma(\zeta_x+\zeta_y+2)}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix} \tag{A.7}$$

Acknowledgments

The authors are grateful for the constructive comments of the anonymous referees who helped to improve this article. Also the authors thank to M. G. Jiménez-Escamilla for clarifying discussions. This work was supported by the PROMEP UTMIX-PTC-062 grant.

---

1. B. Ross, *Historia Mathematica*, (Academic Press, Inc., New Heaven Conn, 1977), **Vol. 4**, pp. 75-89.
2. A.L. Ozores, *Pensamiento Matemático*, **4** (2014) 77.
3. A.A. Kilbas, H.M. Srivastava, & J.J. Trujillo, *Theory and Applications of Fractional Differential Equations* (North-Holland Mathematics Studies 204, Netherlands, 2006), pp. 69-132.
4. K.S. Miller, & B. Ross, *An introduction to the fractional calculus and fractional differential equations* (Wiley, USA, 1993). pp. 44-121.
5. K.B. Oldham, & J. Spanier, *The fractional calculus* (Academic Press, Inc., USA, 1974), pp. 46-159.
6. S.G. Samko, A.A.Kilbas, & O.I. Marichev, *Fractional integrals and derivatives: theory and applications* (Gordon and Breach Science Publishers, Amsterdan, 1993), pp. 28-172.
7. R.S. Strichartz, *Differential equations on fractals: a tutorial* (Princeton University Press, New Jersey, 2006), pp. 39-89.
8. J. Kigami, *Analysis on fractals* (Cambridge University Press, UK, 2001), pp. 107-152 .
9. J.F. Gomez-Aguilar, H. Yepez-Martinez, R.F. Escobar-Jiménez, C.M. Astorga-Zaragoza, & J. Reyes-Reyes, *Applied Mathematical Modelling*, **40** (2016) 9079.
10. A. Atangana, *Physica A: Statistical Mechanics and its Applications* **505** (2018) 688; J.F. Gómez-Aguilar, R.F. Escobar-Jiménez, M.G. López-López, & V.M. Alvarado-Martínez, *Journal of Electromagnetic Waves and Applications* **30** (2016) 1937.
11. V.F. Morales-Delgado, M.A. Taneco-Hernández, & J.F. Gómez-Aguilar, *EPJ Plus* **132** (2017) 47.
12. A. Atangana, & J.F. Gómez-Aguilar, *EPJ Plus* **133** (2018) 1.
13. A. Coronel-Escamilla *et al*., *Entropy* **19** (2017) 55.
14. R. Herrmann, *Fractional Calculus: an introduction for physicists* 2nd ed. (World Scientific, Singapore, 2014), pp. 43-66.
15. B. West, M. Bologna, & P. Grigolini, *Physics of fractal operators* (Springer Science & Business Media, USA, 2003), pp. 37-117.
16. B.B. Mandelbrot, *The fractal geometry of nature* (WH freeman, NY 1997), pp. 15-244.
17. P.J. Bentley, *Genetic Programming and Evolvable Machines* **5** (2004) 71.
18. P.J. Bentley, *International Conference on Evolvable Systems* (Springer, Berlin, Heidelberg, 2003), 81.
19. P.J. Bentley, *J. Royal Soc. Interface*, **6** (2009) 451.
20. M. Hayakawa, T. Ito, & N. Smirnova, *Geophys. Res. Lett*, **26** (1999) 2797.
21. D.L. Turcotte, *Fractals and chaos in geology and geophysics*, 2nd. ed., (Cambridge university press 1997), pp. 56-78.
22. V.P. Dimri, *Dimri, V.P. (2005). Fractals in geophysics and seismology: an introduction. In Fractal Behaviour of the Earth System (pp. 1-22). Springer, Berlin, Heidelberg* (Springer, Berlin, Heidelberg, 2005)), pp. 23-30.
23. A.S. Balankin, A.H. Rangel, G.G. Pérez, F.G. Martinez, H.D. Sánchez-Chávez, & C.L. Martínez-González, *Phys. Rev. E* **87** (2013) 052806.
24. A.S. Balankin, D. S. Ochoa, I.A. Miguel, J.P. Ortiz, & M.A. Martínez-Cruz, *Phys. Rev. E* **81** (2010) 061126.
25. H.D. Sáchez-Chávez, & L. Flores-Cano, *Rev. Mex. Fis*. **64** 415.
26. V.E. Tarasov, *Phys. Lett. A* **336** (2005) 167.
27. V.E. Tarasov, *Ann. Phys* **318** (2005) 286.
28. V.E. Tarasov, *J. Math. Phys* **55** (2014) 083510.
29. M. Ostoja-Starzewski, J. Li, H. Joumaa, & P.N. Demmie, *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik: Applied Mathematics and Mechanics* **94** 373.
30. J. Li, & M. Ostoja-Starzewski, in *Proceedings of Proc. Royal Soc. Lond. A* **465** (2014) 2521.
31. A.S. Balankin, J. Bory-Reyes, & M. Shapiro, *Physica A: Statistical Mechanics and its Applications* **444** (2016) 345.
32. W. Chen, Chaos, *Solitons Fractals* **28** (2006) 923.
33. A.S. Balankin, & B.E. Elizarraraz, *Phys. Rev. E* **85** (2012) 025302.
34. A.S. Balankin, & B.E. Elizarraraz, *Phys. Rev. E* **85** (2012) 056314.
35. M.A. Bhatti, *Fundamental Finite Element Analysis and Aplications* (Wiley, New Jersey, 2005), pp. 545-549.
36. K.J. Bathe, *Finite element procedures* (Prentice Hall, New Jersey, 2006), pp. 339-456.
37. M. Kaltenbacher, *Numerical simulation of mechatronic sensors and actuators: finite elements for computational multiphysics* 3rd ed., (Springer-Verlag Berlin, Heidelberg, 2015), pp. 27-28.
38. T.R. Chandrupatla, A.D. Belegundu, T. Ramesh, & C. Ray, *Introduction to finite elements in engineering* **Vol. 2**, (Prentice Hall, New Jersey, 2002), pp. 275-279.
39. B. Yu, *Appl. Mech. Rev*, **61** (2008) 050801.
40. B. Xiao, X. Zhang, W. Wang, G. Long, H. Chen, H. Kang, & W. Ren, *Fractals*, **26** (2018) 1840015.
41. M.J. Blunt, *Current opinion in colloid, & interface science*, **6** (2001) 197.
42. R. Lenormand, *Proc. R. Soc. Lond. A*, **423** (1989) 159.
43. D. Poon, *Annual Technical Meeting*, Petroleum Society of Canada, (1995).
44. Q. Zheng, J. Fan, X. Li, & C. Xu, *Chemical Engineering Science*, **189** (2018), 260.

45. P.M. Adler, *Journal of hydrology*, **187** (2018) 195.
46. H.W. Park, J. Choe, & J.M. Kang, *Energy Sources*, **22** (2000) 881.
47. A.S. Balankin, A.K. Golmankhaneh, J. Patiño-Ortiz, & M. Patiño-Ortiz, *Phys. Lett. A*, **382** (2018), 1534.
48. C. Chen, & R. Raghavan, *Journal of Petroleum Science and Engineering*, **128** (2015) 194.
49. A. Carpinteri, & A. Sapora, *Applied Mathematics and Mechanics*, **90** (2010) 203.
50. Y. Jin, X. Li, M. Zhao, X. Liu, & H. Li, *International Journal of Heat and Mass Transfer*, **108** (2017) 1078.
51. K. Razminia, A. Razminia, & D. Baleanu, *Applied Mathematical Modelling*, **39** (2015) 86.
52. G. Alaimo, & M. Zingales, *Communications in Nonlinear Science and Numerical Simulation*, **22** (2015) 889.
53. W. Wei, J. Cai, J. Xiao, Q. Meng, B. Xiao, & Q. Han, *Fuel* **234** (2018) 1373.
54. T. Miao, Z. Long, A. Chen, & B. Yu, *International Communications in Heat and Mass Transfer*, **88** (2017) 194.
55. M.R. Spiegel, *Advanced mathematics: for engineers and scientists*. (The McGraw-Hill Companies, Inc., NY, 1971), pp. 212-220.
56. G.B. Arfken, & H.J. Weber, *Mathematical methods for physicists* 3rd. ed., (Harcourt Academic Press, Harcourt, 2001), pp. 487-490.