Symmetry properties and exact solutions of the time fractional Kolmogorov-Petrovskii-Piskunov equation

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In this paper, the time fractional Kolmogorov-Petrovskii-Piskunov (TFKPP) equation is analyzed by means of Lie symmetry approach. The TFKPP is reduced to ordinary differential equation of fractional order via the attained point symmetries. Moreover, the simplest equation method is used in construct the exact solutions of underlying equation with recently introduced conformable fractional derivative.

Keywords: Time fractional Kolmogorov-Petrovskii-Piskunov equation; Lie symmetry analysis; Erdélyi-Kober fractional derivative; Riemann-Liouville derivative; conformable fractional derivative; simplest equation method.

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1. Introduction

The fractional calculus (FC) began to wind up exceptionally famous in a few parts of science and engineering. Numerous important event, that is, acoustics, anomalous diffusion, chemistry, control processing, electro-magnetics, and viscoelasticity have been expressed by FC. It is known that a systematic method for extracting the analytical solution of both ordinary differential equations (ODEs) and partial differential equations (PDEs) was first proposed by the Norwegian mathematician Sophus Lie in the early 19th century. The fundamental overview of this strategy is the estimation of variable changes that can leave differential condition unchanged. Therefore, a vital role in the field of FC is to attain the Lie symmetries and the solutions of the equations with the FC derivatives. There have been some properties of the fractional sense that could not be found in classical sense, owing to this we feel motivated to establish the symmetries of TFKPP equation. This equation has the following generalized form [1-4]

$$\begin{aligned}&\partial_t^\alpha u = u_{xx} + \lambda u + \mu u^2 + \gamma u^3,\ \lambda+\mu+\gamma=0,\\ &\psi^2 = \mu^2 - 4\lambda\gamma \geq 0,\end{aligned} \tag{1}$$

where $\partial_t^\alpha u := \mathcal{D}_t^\alpha u$ stands for Riemann-Liouville of order α, expressed as [5]

$$\mathcal{D}_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int\limits_0^t (t-\xi)^{n-\alpha-1}u(x,\xi)d\xi, & n-1<\alpha<n\\ \frac{\partial^n u}{\partial t^n}, & \alpha = n\in\mathbb{N}\end{cases} \tag{2}$$

The TFKPP Eq. (1), has a large application and includes as particular cases the time fractional Fitzhugh-Nagumo equation ($\lambda=-c,\ \mu=c+1,\ \gamma=-1,\ 0<c<1$), which is used in population genetics, the time fractional Newell-Whitehead equation ($\lambda=1,\ \mu=0,\ \gamma=-1$). Recently, the homotopy perturbation method and homotopy analysis method have utilized to consider the TFKPP equation by Gepreel [1] and Hariharan [2], respectly with $\lambda=\mu=0$ and $\gamma=-2$.

In FC, there are large amount of differential derivatives were defined *e.g.* [6-9]. In the calculus, the chain rule is a useful and an applicable. It is also hold for conformable fractional derivatives.

As far as we know, every proposed fractional derivative has some disadvantages. Therefore, Khalil *et al.*, [9], proposed a new definitions:

Definition 1.1. Surmise that $f : [a,b]\times(0,\infty)\to\mathbb{R}$, then the conformable fractional derivative of f is given by

$$\begin{aligned}{}_t\mathbb{T}_\alpha(f)(x,t) = \lim_{\epsilon\to 0}\frac{f\left(x,t+\epsilon t^{1-\alpha}\right)-f(x,t)}{\epsilon},\\ \alpha\in(0,1],\end{aligned} \tag{3}$$

for all $t>0$.

Theorem 1.1 [9] *Suppose that* $a,b\in\mathbb{R}$ *and* $\alpha\in(0,1]$, *then*

(i) ${}_t\mathbb{T}_\alpha(au+bv) = a\,{}_t\mathbb{T}_\alpha(u) + b\,{}_t\mathbb{T}_\alpha(v)$,

(ii) ${}_t\mathbb{T}_\alpha(t^\lambda) = \lambda t^{\lambda-\alpha},\ \ \lambda\in\mathbb{R}$,

(iii) ${}_t\mathbb{T}_\alpha(uv) = u\,{}_t\mathbb{T}_\alpha(v) + v\,{}_t\mathbb{T}_\alpha(u)$,

(iv) ${}_t\mathbb{T}_\alpha\left(\frac{u}{v}\right) = \frac{u\,{}_t\mathbb{T}_\alpha(v) - v\,{}_t\mathbb{T}_\alpha(u)}{v^2}$,

(v) ${}_t\mathbb{T}_\alpha(u)(t) = t^{1-\alpha}u'(t),\ \ u\in C^1$.

More than that, the chain rule is valid for conformable fractional derivatives, shown by Abdeljawad [10].

Theorem 1.2. *Surmise that* $f:(0,\infty)\to\mathbb{R}$ *is a real differentiable,* $\alpha-$*differentiable function. Assume that* g *is a function defined in the range of* f *and also differentiable; then, one has the following rule:*

$$ {}_t\mathbb{T}_\alpha(fog)(t)=t^{1-\alpha}g'(t)f'(g(t)). \quad (4)$$

There are many investigation about conformable fractional derivatives [11-14] and also some physical interpretations of this newly introduced fractional derivative are described in [15].

The organization of the manuscript is given below: In Sec. 2, we provide some preliminaries. Section 3, is devoted to the description of Lie symmetry analysis of TFKPP Eq. (1). General similarity forms and symmetry reductions are established. In Sec. 4, exact solutions to the TFKPP equation with conformable fractional derivative are investigated. Finally, the last section is devoted to conclusions.

2. Lie symmetry analysis of fractional partial differential equations

Here, some description for solving fractional partial differential equations (FPDEs) via Lie symmetry analysis will be provided. Surmise that FPDE having as in [16-26]

$$\partial_t^\alpha u=F(x,t,u,u_x,u_{xx}),\qquad 0<\alpha<1. \quad (5)$$

If (5) is invariant under a one parameter Lie group of point transformations

$$\bar t=\bar t(x,t,u;\epsilon),\quad \bar x=\bar x(x,t,u;\epsilon),\quad \bar u=\bar u(x,t,u;\epsilon), \quad (6)$$

the vector field of an evolution type of equation is as follows:

$$V=\xi^t(x,t,u)\frac{\partial}{\partial t}+\xi^x(x,t,u)\frac{\partial}{\partial x}+\phi(x,t,u)\frac{\partial}{\partial u}, \quad (7)$$

where the coefficients $\xi^t,\ \xi^x$ and ϕ of the vector field are to be determined. When V satisfy the Lie symmetry condition, the vector field (7) generates a symmetry of (5),

$$pr^{(\alpha,2)}V(\Delta)|_{\Delta=0}=0,\ \ \Delta=\partial_t^\alpha u-F.$$

Thus the extension operator take the form

$$pr^{(\alpha,2)}V=V+\phi_\alpha^0\partial_{\partial_t^\alpha u}+\phi^x\partial_{u_x}+\phi^{xx}\partial_{u_{xx}},$$

where

$$\begin{aligned}
\phi^x&=\mathcal{D}_x(\phi)-u_x\mathcal{D}_x(\xi^x)-u_t\mathcal{D}_x(\xi^t),\\
\phi^{xx}&=\mathcal{D}_x(\phi^x)-u_{xt}\mathcal{D}_x(\xi^t)-u_{xx}\mathcal{D}_x(\xi^x),\\
\phi_\alpha^0&=\mathcal{D}_t^\alpha(\phi)+\xi^x\mathcal{D}_t^\alpha(u_x)-\mathcal{D}_t^\alpha(\xi^x u_x)\\
&\quad+\mathcal{D}_t^\alpha(\mathcal{D}_t(\xi^t)u)-\mathcal{D}_t^{\alpha+1}(\xi^t u)+\xi^t\mathcal{D}_t^{\alpha+1}(u).
\end{aligned}$$

The condition of invariance

$$\xi^t(x,t,u)|_{t=0}=0,$$

is inevitable for the (6), due to the (2).
The α^{th} extended infinitesimal is presented as:

$$\begin{aligned}
\phi_\alpha^0&=\mathcal{D}_t^\alpha(\phi)+\xi^x\mathcal{D}_t^\alpha(u_x)-\mathcal{D}_t^\alpha(\xi^x u_x)+\mathcal{D}_t^\alpha(\mathcal{D}_t(\xi^t)u)\\
&\quad-\mathcal{D}_t^{\alpha+1}(\xi^t u)+\xi^t\mathcal{D}_t^{\alpha+1}(u),
\end{aligned} \quad (8)$$

where $\mathcal{D}_t^\alpha$ exhibits the total fractional derivative operator. The fractional generalized Leibnitz rule is expressed as

$$\mathcal{D}_t^\alpha\big[u(t)v(t)\big]=\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathcal{D}_t^{\alpha-n}u(t)\mathcal{D}_t^n v(t),\ \ \alpha>0, \quad (9)$$

here

$$\binom{\alpha}{n}=\frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}.$$

Therefore using (9) one can represent (8) as

$$\begin{aligned}
\phi_\alpha^0&=\mathcal{D}_t^\alpha(\phi)-\alpha\mathcal{D}_t(\xi^t)\frac{\partial^\alpha u}{\partial t^\alpha}-\sum_{n=1}^{\infty}\binom{\alpha}{n}\mathcal{D}_t^n(\xi^x)\mathcal{D}_t^{\alpha-n}(u_x)\\
&\quad-\sum_{n=1}^{\infty}\binom{\alpha}{n+1}\mathcal{D}_t^{n+1}(\xi^t)\mathcal{D}_t^{\alpha-n}(u).
\end{aligned} \quad (10)$$

Using chain rule

$$\begin{aligned}
\frac{d^m f(g(t))}{dt^m}&=\sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\\
&\quad\times\frac{1}{k!}[-g(t)]^r\frac{d^m}{dt^m}[g(t)^{k-r}]\frac{d^k f(g)}{dg^k},
\end{aligned}$$

and setting $f(t)=1$, one can get

$$\begin{aligned}
\mathcal{D}_t^\alpha(\phi)&=\frac{\partial^\alpha\phi}{\partial t^\alpha}+\phi_u\frac{\partial^\alpha u}{\partial t^\alpha}-u\frac{\partial^\alpha\phi_u}{\partial t^\alpha}\\
&\quad+\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\phi_u}{\partial t^n}\mathcal{D}_t^{\alpha-n}(u)+\vartheta,
\end{aligned}$$

where

$$\begin{aligned}
\vartheta&=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\\
&\quad\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}[u^{k-r}]\frac{\partial^{n-m+k}\phi}{\partial t^{n-m}\partial u^k}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\phi_\alpha^0&=\frac{\partial^\alpha\phi}{\partial t^\alpha}+\big(\phi_u-\alpha\mathcal{D}_t(\xi^t)\big)\frac{\partial^\alpha u}{\partial t^\alpha}-u\frac{\partial^\alpha\phi_u}{\partial t^\alpha}+\vartheta\\
&\quad+\sum_{n=1}^{\infty}\Big[\binom{\alpha}{n}\frac{\partial^\alpha\phi_u}{\partial t^\alpha}-\binom{\alpha}{n+1}\mathcal{D}_t^{n+1}(\xi^t)\Big]\mathcal{D}_t^{\alpha-n}(u)\\
&\quad-\sum_{n=1}^{\infty}\binom{\alpha}{n}\mathcal{D}_t^n(\xi^x)\mathcal{D}_t^{\alpha-n}(u_x).
\end{aligned}$$

3. Symmetry representation of TFKPP equation

In view of the Lie theory, we have:

$$\phi_\alpha^0 = \phi^{xx} + \lambda\phi + 2\mu\phi u + 3\gamma\phi u^2. \tag{11}$$

Substituting (10) into (11), the determining equations for Eq. (1) is attained,consequently, we have

$$\xi^t = 4tc_3, \qquad \xi^x = c_1 + 2\alpha x c_3,$$
$$\phi = c_2 u + (3\alpha - 2)uc_3 + \mathcal{C}(x,t),$$

where c_1, c_2 and c_3 are constants and $\mathcal{C}(x,t)$ is a solution of Eq. (1). Therefore, the algebra g of Eq. (1) can be written as

$$V_1 = \frac{\partial}{\partial x},\ V_2 = u\frac{\partial}{\partial u},$$
$$V_3 = 4t\frac{\partial}{\partial t} + 2\alpha x\frac{\partial}{\partial x} + (3\alpha - 2)u\frac{\partial}{\partial u},$$
$$V_4 = \mathcal{C}(x,t)\frac{\partial}{\partial u}.$$

For V_3, one can write

$$\frac{dt}{4t} = \frac{dx}{2\alpha x} = \frac{du}{(3\alpha - 2)u},$$

and this give

$$\zeta = xt^{\frac{-\alpha}{2}},\ \ u(x,t) = t^{\frac{3\alpha-2}{4}}\mathcal{F}(\zeta). \tag{12}$$

Theorem 3.1. *The transformation* (12) *reduces* (1) *to the following:*

$$\left(\mathcal{P}_{\frac{2}{\alpha}}^{-\frac{\alpha}{4}+\frac{1}{2},\alpha}\mathcal{F}\right)(\zeta) = \mathcal{F}'' + \lambda\mathcal{F} + \mu\mathcal{F}^2 + \gamma\mathcal{F}^3, \tag{13}$$

with the Erdélyi-Kober (EK) fractional differential operator $\mathcal{P}_\beta^{\tau,\alpha}$ *defined by*

$$\left(\mathcal{P}_\beta^{\tau,\alpha}\mathcal{F}\right) := \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\beta}\zeta\frac{d}{d\zeta}\right)\left(\mathcal{K}_\beta^{\tau+\alpha,n-\alpha}\mathcal{F}\right)(\zeta),$$
$$n = \begin{cases} [\alpha]+1, & \alpha \notin \mathbb{N} \\ \alpha, & \alpha \in \mathbb{N} \end{cases}$$

where

$$\left(\mathcal{K}_\beta^{\tau,\alpha}\mathcal{F}\right)(\zeta) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int\limits_1^\infty (u-1)^{\alpha-1}u^{-(\tau+\alpha)}\mathcal{F}(\zeta u^{\frac{1}{\beta}})du, \\ \mathcal{F}(\zeta),\ \ \alpha = 0, \end{cases}$$

is the EK fractional integral operator.

Proof: *Let* $n-1 < \alpha < n,\ \ n = 1,2,3,....$ *By means of Reimann-Liouville, one reaches*

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\Bigg[\frac{1}{\Gamma(n-\alpha)} \times \int\limits_0^t (t-s)^{n-\alpha-1}s^{\frac{3\alpha-2}{4}}\mathcal{F}\left(xs^{\frac{-\alpha}{2}}\right)ds\Bigg]. \tag{14}$$

Letting $\varrho = t/s$, one can get $ds = -(t/\varrho^2)d\varrho$, therefore (14) can be written as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{\alpha}{4}-\frac{1}{2}}\left(\mathcal{K}_{\frac{2}{\alpha}}^{\frac{3\alpha+2}{4},n-\alpha}\mathcal{F}\right)(\zeta)\right].$$

Taking into account the relation $(\zeta = xt^{-\alpha/2})$, we can obtain

$$t\frac{\partial}{\partial t}\phi(\zeta) = t\frac{\partial\zeta}{\partial t}\frac{d\phi(\zeta)}{d\zeta} = -\frac{\alpha}{2}\zeta\frac{d\phi(\zeta)}{d\zeta}.$$

Therefore one can get

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{\alpha}{4}-\frac{1}{2}}\left(\mathcal{K}_{\frac{2}{\alpha}}^{\frac{3\alpha+2}{4},n-\alpha}\mathcal{F}\right)(\zeta)\right] \\
&= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{\alpha}{4}-\frac{1}{2}}\left(\mathcal{K}_{\frac{2}{\alpha}}^{\frac{3\alpha+2}{4},n-\alpha}\mathcal{F}\right)(\zeta)\right)\right] \\
&= \frac{\partial^{n-1}}{\partial t^{n-1}}\Bigg[t^{n-\frac{\alpha}{4}-\frac{3}{2}}\left(n - \frac{\alpha}{4} - \frac{1}{2} - \frac{\alpha}{2}\zeta\frac{d}{d\zeta}\right) \\
&\times\left(\mathcal{K}_{\frac{2}{\alpha}}^{\frac{3\alpha+2}{4},n-\alpha}\mathcal{F}\right)(\zeta)\Bigg] \\
&= \ldots = t^{-\frac{\alpha}{4}-\frac{1}{2}}\prod_{j=0}^{n-1}\left(-\frac{\alpha}{4} + \frac{1}{2} + j - \frac{\alpha}{2}\zeta\frac{d}{d\zeta}\right) \\
&\times\left(\mathcal{K}_{\frac{2}{\alpha}}^{\frac{3\alpha+2}{4},n-\alpha}\mathcal{F}\right)(\zeta) \\
&= t^{-\frac{\alpha}{4}-\frac{1}{2}}\left(\mathcal{P}_{\frac{2}{\alpha}}^{-\frac{\alpha}{4}+\frac{1}{2},\alpha}\mathcal{F}\right)(\zeta).
\end{aligned}$$

This completes the proof.

Also, for the symmetry of $V_1 + V_2 + V_3$, one can write

$$\frac{dt}{4t} = \frac{dx}{2\alpha x + 1} = \frac{du}{(3\alpha - 1)u},$$

which yields

$$\zeta = \frac{2\alpha x + 1}{2\alpha}t^{\frac{-\alpha}{2}},\ \ u(x,t) = t^{\frac{3\alpha-1}{4}}\mathcal{F}(\zeta). \tag{15}$$

Theorem 3.2. *The transformation* (15) *reduces* (1) *to the following nonlinear ordinary differential equation of fractional order:*

$$\left(\mathcal{P}_{\frac{2}{\alpha}}^{\frac{3-\alpha}{4},\alpha}\mathcal{F}\right)(\zeta) = \mathcal{F}'' + \lambda\mathcal{F} + \mu\mathcal{F}^2 + \gamma\mathcal{F}^3. \tag{16}$$

Proof: Similar to the proof of previous theorem.

4. Exact Solutions of TFKPP equation

Symmetry analysis of differential equations gives many information about geometric properties of various differential equations. For example, it is possible to extract vector fields, infinitesimals, conservation laws and reductions of differential equations. Reduction procedure of differential equations allows us to reduce dimension of these equations by one less. In two dimensional partial differential equations (PDEs), reduction procedure gives an ordinary differential equation (ODE). So, solving this ODE concludes exact solution of original PDE. However, in FPDEs with Riemann-Liouville fractional derivatives we get ODEs with the EK derivatives which there is not a systematic method to find their exact solution. Therefore, after reduction of TFKPP equation with the Riemann-Liouville fractional derivative we obtain Eqs. (13) and (16) which it is not possible to find analytical solutions. However, we can obtain exact solution of Eq. (1) with $\partial_t^\alpha u := {}_t\mathbb{T}_\alpha(u)$. In this section, we investigate the exact solutions of TFKPP equation with conformable fractional derivative.

4.1. Simplest equation method and its applications to time fractional differential equations

This approach was proposed in [27,28]. The steps for the approach is stated as follows:

Let the TFDE is given by

$$P\left(u,\ {}_t\mathbb{T}_\alpha(u), u_x, u_{xx}, ...\right) = 0.\ \ 0 < \alpha \le 1, \tag{17}$$

Then the modified version of simplest equation method procedure have the following steps:

Step 1: We utilize the following

$$u(x,t) = \Theta(\xi),\ \ \xi = A\left(x - \nu\frac{t^\alpha}{\alpha}\right), \tag{18}$$

where A and ν are nonzero constants to be determined later.

Consequently we attain with parameters A and ν the following

$$P\left(\Theta, -A\nu\Theta', A\Theta', A^2\Theta'', ...\right) = 0. \tag{19}$$

Step 2: Suppose that Eq. (19) possesses

$$\Theta(\xi) = \sum_{i=0}^{N} a_i[z(\xi)]^i, \tag{20}$$

where $a_i,\ i = 0, 1, ..., N$, are constants to be determined later. The positive value of N in (20), which the pole order for the general solution of Eq. (19), can be determined by substituting $\Theta(\xi) = \xi^{-m},\ (m > 0)$.

In the present paper, we use the Bernoulli and Riccati equations which their solutions can be expressed by elementary functions. For the Bernoulli equation:[i]

$$\frac{dz}{d\xi} = az(\xi) + b[z(\xi)]^k,\ k \in \mathbb{N}\backslash\{1\},$$

we use the solutions

$$z(\xi) = \sqrt[k-1]{\frac{a\exp\left[a(k-1)(\xi+\xi_0)\right]}{1 - b\exp\left[a(k-1)(\xi+\xi_0)\right]}},$$

for the case $a > 0,\ b < 0$ and

$$z(\xi) = \sqrt[k-1]{-\frac{a\exp\left[a(k-1)(\xi+\xi_0)\right]}{1 + b\exp\left[a(k-1)(\xi+\xi_0)\right]}},$$

for the case $a < 0,\ b > 0$ and ξ_0 is a constant of integration. For the Riccati equation

$$\frac{dz}{d\xi} = a + b[z(\xi)]^2,$$

which admits the following exact solutions:

$$z(\xi) = -\frac{\sqrt{-ab}}{b}\tanh\left[\sqrt{-ab}\xi - \frac{\epsilon\ln(\xi_0)}{2}\right],\ \xi_0 > 0,\ \epsilon = \pm 1,$$

when $ab < 0$ and

$$z(\xi) = \frac{\sqrt{ab}}{b}\tan\left[\sqrt{ab}\xi + \xi_0\right],\ \xi_0 = Const.,$$

when $ab > 0$.

Step 3: Plugging (20) into (19) and equating the coefficients of z^i to zero, one can obtain an algebraic system in $A,\ \nu$ and $a_i,\ i = 0, ..., N$.

4.2. Application to the TFKPP equation

The transformation

$$u(x,t) = \Theta(\xi),\ \ \xi = A\left(x - \nu\frac{t^\alpha}{\alpha}\right), \tag{21}$$

changes Eq. (1) with $\partial_t^\alpha u = {}_t\mathbb{T}_\alpha(u)$ to:

$$A^2\Theta'' + \nu A\Theta' + \lambda\Theta + \mu\Theta^2 + \gamma\Theta^3 = 0. \tag{22}$$

We suppose that Eq. (22) has solution of the form (20). Balancing the highest order derivative terms with nonlinear terms in Eq. (22), we get $N = 1$, and hence

$$\Theta(\xi) = a_0 + a_1 z(\xi),\ a_1 \neq 0. \tag{23}$$

Substituting (23) along with (21) into Eq. (22) and then vanishing the coefficients of z^i, one can get some algebraic equations about a_0, a_1, A and ν, which solving them by *Maple*, concludes:

- Case 1:

$$a_0 = 0,\ a_1 = \frac{b\left(\mu^2 \mp |\mu|\psi\right)}{2a\mu\gamma},$$

$$A = \frac{\sqrt{-\mu^2 + 2\lambda\gamma \pm |\mu|\psi}}{2a\sqrt{\gamma}},$$

$$\nu = \frac{\mu^2 - 6\lambda\gamma \mp |\mu|\psi}{2\sqrt{\gamma\left(-\mu^2 + 2\lambda\gamma \pm |\mu|\psi\right)}},$$

where $\psi = \sqrt{\mu^2 - 4\lambda\gamma}$. In this case, the exact solutions of Eq. (22) are:

$$\Theta(\xi) = \frac{b(-\mu^2 \pm |\mu|\psi)\exp\left[a(\xi+\xi_0)\right]}{2\gamma\mu\left(-1+b\exp\left[a(\xi+\xi_0)\right]\right)},\ \ a>0,\ b<0,$$
$$\Theta(\xi) = \frac{b(-\mu^2 \pm |\mu|\psi)\exp\left[a(\xi+\xi_0)\right]}{2\gamma\mu\left(1+b\exp\left[a(\xi+\xi_0)\right]\right)},\ \ a<0,\ b>0,$$

and using the substitution in (18) we get the final solutions:

$$u(x,t) = \frac{b(-\mu^2 \pm |\mu|\psi \exp\left[\eta(x,t)\right])}{2\gamma\mu\left(-1+b\exp\left[\eta(x,t)\right]\right)},\ \ a>0,\ b<0,$$
$$u(x,t) = \frac{b(-\mu^2 \pm |\mu|\psi)\exp\left[\eta(x,t)\right]}{2\gamma\mu\left(1+b\exp\left[\eta(x,t)\right]\right)},\ \ a<0,\ b>0,$$

where

$$\eta(x,t) = \frac{2x\sqrt{\gamma(-\mu^2+2\gamma\lambda \pm |\mu|\psi)}}{4\gamma} + \frac{(-\mu^2+6\gamma\lambda \pm |\mu|\psi)t^\alpha + 4\xi_0\gamma a\alpha}{4\gamma\alpha}.$$

• Case 2:

$$a_0 = \frac{\psi-\mu}{2\gamma},\ a_1 = \frac{b(3\mu^2-12\lambda\gamma-\mu\psi)}{\gamma a(3\psi-\mu)},$$
$$A = \frac{\sqrt{2}\psi}{2a\sqrt{-\gamma}},\ \nu = \frac{\sqrt{2}\mu}{2\sqrt{-\gamma}}.$$

In this case, the exact solutions of Eq. (22) are:

$$\Theta(\xi) = \frac{-2\mu^2+2\mu\psi+6\lambda\gamma+b\left(6\lambda\gamma-\mu^2-\mu\psi\right)\exp\left[a(\xi+\xi_0)\right]}{\gamma(3\psi-\mu)\left(-1+b\exp\left[a(\xi+\xi_0)\right]\right)},\qquad a>0, b<0,$$
$$\Theta(\xi) = \frac{2\mu^2+2\mu\psi-6\lambda\gamma+b\left(6\lambda\gamma-\mu^2-\mu\psi\right)\exp\left[a(\xi+\xi_0)\right]}{\gamma(3\psi-\mu)\left(1+b\exp\left[a(\xi+\xi_0)\right]\right)},\qquad a<0, b>0,$$

or equivalently

$$u(x,t) = \frac{-2\mu^2+2\mu\psi+6\lambda\gamma+b\left(6\lambda\gamma-\mu^2-\mu\psi\right)\exp\left[\eta(x,t)\right]}{\gamma(3\psi-\mu)\left(-1+b\exp\left[\eta(x,t)\right]\right)},\qquad a>0,\ b<0,$$
$$u(x,t) = \frac{2\mu^2+2\mu\psi-6\lambda\gamma+b\left(6\lambda\gamma-\mu^2-\mu\psi\right)\exp\left[\eta(x,t)\right]}{\gamma(3\psi-\mu)\left(1+b\exp\left[\eta(x,t)\right]\right)},\qquad a<0,\ b>0,$$

where

$$\eta(x,t) = \xi_0 a + \frac{x\psi\sqrt{-2\gamma}}{2\gamma} - \frac{\mu\psi t^\alpha}{2\gamma\alpha}.$$

• Case 3:

$$a_0 = \frac{\psi-\mu}{2\gamma},\quad a_1 = \frac{2b(\mu^2-3\lambda\gamma-\mu\psi)}{\gamma a(3\psi-\mu)},$$
$$A = \frac{\sqrt{2\lambda\gamma-\mu^2+\mu\psi}}{2a\sqrt{\gamma}},\quad \nu = \frac{6\lambda\gamma-\mu^2+\mu\psi}{2\sqrt{\gamma\left(2\lambda\gamma-\mu^2+\mu\psi\right)}}.$$

In this case, we can obtain

$$\Theta(\xi) = \frac{2\left(-\mu^2+3\lambda\gamma+\mu\psi\right)}{\gamma(3\psi-\mu)\left(-1+b\exp\left[a(\xi+\xi_0)\right]\right)},$$
$$a>0,\ b<0,$$
$$\Theta(\xi) = \frac{2\left(\mu^2-3\lambda\gamma-\mu\psi\right)}{\gamma(3\psi-\mu)\left(1+b\exp\left[a(\xi+\xi_0)\right]\right)},$$
$$a<0,\ b>0,$$

and using the substitution in (18) we have

$$u(x,t) = \frac{2\left(-\mu^2+3\lambda\gamma+\mu\psi\right)}{\gamma(3\psi-\mu)\left(-1+b\exp\left[\eta(x,t)\right]\right)},$$
$$a>0,\ b<0,$$
$$u(x,t) = \frac{2\left(\mu^2-3\lambda\gamma-\mu\psi\right)}{\gamma(3\psi-\mu)\left(1+b\exp\left[\eta(x,t)\right]\right)},$$
$$a<0,\ b>0, \tag{24}$$

where

$$\eta(x,t) = \xi_0 a + \frac{x\sqrt{-\mu^2+\mu\psi+2\gamma\lambda}}{2\sqrt{\gamma}} - \frac{-\mu^2+\mu\psi+6\gamma\lambda}{4\gamma\alpha}t^\alpha.$$

Also, in the use of Riccati equation, substituting (23) along with (21) into Eq. (22) and then vanishing the coefficients of z^i, we can obtain some algebraic equations about

a_0, a_1, A and ν, that solving them by Computer algebra technique , concludes:

- Case 1:

$$a_0 = -\frac{\mu}{2\gamma}, \quad a_1 = \pm\frac{ib\psi}{2\gamma\sqrt{ab}},$$
$$A = \pm\frac{\sqrt{2}\psi}{4\sqrt{\gamma ab}}, \quad \nu = \frac{i\sqrt{2}\mu}{2\sqrt{\gamma}}.$$

In this case, the exact solutions of Eq. (22) are:

$$\Theta(\xi) = \frac{-\mu \pm i\psi\tan\left[\sqrt{ab}\xi + \xi_0\right]}{2\gamma}, \quad ab > 0,$$
$$\Theta(\xi) = \frac{-\mu \pm \psi\tanh\left[\sqrt{-ab}\xi - \frac{\epsilon\ln(\xi_0)}{2}\right]}{2\gamma}, \quad ab < 0,$$

and using the substitution in (18) we get the following final solutions:

$$u(x,t) = -\frac{1}{2\gamma}\left(\mu + i\psi\tan\left[\frac{-\sqrt{2\gamma}\psi x\alpha \mp 4\gamma\xi_0\alpha + i\mu\psi t^\alpha}{4\gamma\alpha}\right]\right),$$

when $ab > 0$ and

$$u(x,t) = -\frac{1}{2\gamma}\left(\mu + \psi\tanh\left[\frac{-i\sqrt{2\gamma}\psi x\alpha \pm 2\gamma\epsilon\ln(\xi_0)\alpha - \mu\psi t^\alpha}{4\gamma\alpha}\right]\right),$$

when $ab < 0$.

- Case 2:

$$a_0 = -\frac{\mu}{2\gamma}, \; a_1 = \mp\frac{ib\psi}{2\gamma\sqrt{ab}},$$
$$A = \pm\frac{\sqrt{2}\psi}{4\sqrt{\gamma ab}}, \; \nu = -\frac{i\sqrt{2}\mu}{2\sqrt{\gamma}},$$

Exact solutions of Eq. (22) extracted from this case are:

$$\Theta(\xi) = \frac{-\mu \mp i\psi\tan\left[\sqrt{ab}\xi + \xi_0\right]}{2\gamma}, \; ab > 0,$$
$$\Theta(\xi) = \frac{-\mu \mp \psi\tanh\left[\sqrt{-ab}\xi - \frac{\epsilon\ln(\xi_0)}{2}\right]}{2\gamma}, \; ab < 0,$$

or equivalently

$$u(x,t) = -\frac{1}{2\gamma}\left(\mu + i\psi\tan\left[\frac{\sqrt{2\gamma}\psi x\alpha \pm 4\gamma\xi_0\alpha + i\mu\psi t^\alpha}{4\gamma\alpha}\right]\right),$$

when $ab > 0$ and

$$u(x,t) = -\frac{1}{2\gamma}\left(\mu + \psi\tanh\left[\frac{i\sqrt{2\gamma}\psi x\alpha \mp 2\gamma\epsilon\ln(\xi_0)\alpha - \mu\psi t^\alpha}{4\gamma\alpha}\right]\right),$$

when $ab < 0$.

5. Conclusion

In this study, the Lie group analysis method was successfully applied to investigate the reduction and symmetry properties of the TFKPP equation. Moreover, we have arrived to some exact solutions of the conformable TFKPP equation, thanks to the application of simplest equation method. The results of this study undoubtedly offer helpful information about the TFKPP equation.

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i. In this paper, the case $k = 2$ has been used to find solutions.

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