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New reflections on higher dimensional linearized gravity

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We make a number of remarks on linearized gravity with cosmological constant in any dimension, which, we argue, can be useful in a quantum gravity framework. For this purpose we assume that the background space-time metric corresponds to the de Sitter or anti-de Sitter space. Moreover, *via* the graviton mass and the cosmological constant correspondence, we make some interesting observations, putting special attention on the possible scenario of a graviton-tachyon connection. We compare our proposed formalism with the Novello and Neves approach.

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1. Introduction

It is known that there are a number of works relating tachyons with M-theory [1] (see also Ref. 2 and references therein), including the brane and anti-brane systems [3], closed-string tachyon condensation [4], tachyonic instability and tachyon condensation in the $E(8)$ heterotic string [5], among many others. Part of the motivation of these developments emerges because it was discovered that the ground state of the bosonic string is tachyonic [6] and that the spectrum in AdS/CFT [7] can contain a tachyonic structure.

On the other hand, it is also known that the $(5+5)$-signature and the $(1+9)$-signature are common to both type IIA strings and type IIB strings. In fact, versions of M-theory lead to type IIA string theories in space-time of signatures $(0+10)$, $(1+9)$, $(2+8)$, $(4+6)$ and $(5+5)$, and to type IIB string theories of signatures $(1+9)$, $(3+7)$ and $(5+5)$ [8]. It is worth mentioning that some of these theories are linked by duality transformations. So, one wonders whether tachyons may also be related to the various signatures. In particular, here we are interested to see the possible relation of tachyons with a space-time of $(4+6)$-dimensions. Part of the motivation in the $(4+6)$-signature arises from the observation that $(4+6) = (1+4) + (3+2)$. This means that the world of $(4+6)$-dimensions can be splitted into a de Sitter world of $(1+4)$-dimensions and an anti-de Sitter world of $(3+2)$-dimensions. Moreover, looking the $(4+6)$-world from the perspective of $(3+7)$-dimensions obtained by compactifying-uncompactifying prescription such that $4 \to 3$ and $6 \to 7$, one can associate with the 3 and 7 dimensions of the $(3+7)$-world with a S^3 and S^7, respectively, which are two of the parallelizable spheres; the other it is S^1. As it is known these spheres are related to both Hopf maps and division algebras (see Ref. 9 and references therein).

In this work, we develop a formalism that allows us to address the $(4+6)$-dimensional world *via* linearized gravity. In this case, one starts assuming the Einstein field equations with cosmological constant Λ in $(4+6)$-dimensions and develops the formalism considering a linearized metric in such equations. We note that the result is deeply related to the cosmological constant $\Lambda \lessgtr 0$ sign. In fact, one should remember that in $(1+4)$-dimensions, Λ is positive, while in $(3+2)$-dimensions, Λ is negative. At the level of linearized gravity, one searches for the possibility of associating these two different signs of Λ with tachyons. This leads us to propose a unified tachyonic framework in $(4+6)$-dimensions which includes these two separate cases of Λ. Moreover, we argue that our formalism may admit a possible connection with the increasing interesting proposal of duality in linearized gravity (see Refs. [10-12] and references therein).

In order to achieve our goal, we first introduce, in a simple context, the tachyon theory. Secondly, in a novel form we develop the de Sitter and anti-de Sitter space-times formalism, clarifying the meaning of the main constraints. Moreover, much work allows us to describe a new formalism for higher dimensional linearized gravity. Our approach is focused on the space-time signature in any dimension and in particular in $(4+6)$-dimensions.

A further motivation of our approach may emerge from the recent direct detections of gravitational waves [13-15]. According to this detection the upper bound of the graviton mass is $m_g \leq 1.3 \times 10^{-58}$ kg [15]. Since in our computations the mass and the cosmological constant are proportional, such an upper bound must also be reflected in the cosmological constant value.

Technically, this work is structured as follows. In Sec. 2, we make a simple introduction of tachyon theory. In Sec. 3, we discuss a possible formalism for the de Sitter and anti-de Sitter space-times. In Sec. 4, we develop the most general formalism of higher dimensional linearized gravity with cosmological constant. In Sec. 5, we establish a novel approach for considering the constraints that determine the de Sitter and anti-de Sitter space. In Sec. 6, we associate the concept of tachyons with higher dimensional linearized gravity. In Sec. 7, we develop linearized gravity with cosmological con-

stant in $(4+6)$-dimensions. We add an Appedix A in attempt to further clarify the negative mass squared term-tachyon association. Finally, in Sec. 8, we make some final remarks.

2. Special relativity and the signature of the space-time

Let us start considering the well known time dilatation formula

$$dt = \frac{d\tau}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{1}$$

Here, τ is the proper time, $v^2 \equiv (dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2$ is the velocity of the object and c denotes the speed of light. Of course, the expression (1) makes sense over the real numbers only if one assumes $v < c$. It is straightforward to see that (1) leads to the line element

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{2}$$

In tensorial notation one may write (2) as

$$ds^2 = -c^2 d\tau^2 = \eta^{(+)}_{\mu\nu} dx^\mu dx^\nu, \tag{3}$$

where the indices μ, ν take values in the set $\{1, 2, 3, 4\}$, $x^1 = ct$, $x^2 = x$, $x^3 = y$ and $x^4 = z$. Moreover, $\eta^{(+)}_{\mu\nu}$ denotes the flat Minkowski metric with associated signature $(-1, +1, +1, +1)$. Usually, one says that such a signature represents a world of $(1 + 3)$-dimensions.

If one now defines the linear momentum

$$p^\mu = m_0^{(+)} \frac{dx^\mu}{d\tau}, \tag{4}$$

with $m_0^{(+)} \neq 0$ a constant, one sees that (3) implies

$$p^\mu p^\nu \eta^{(+)}_{\mu\nu} + m_0^{(+)2} c^2 = 0. \tag{5}$$

Of course, $m_0^{(+)}$ plays the role of the rest mass of the object. This is because setting $p^i = 0$, with $i \in \{2, 3, 4\}$, in the rest frame and defining $E = cp^1$, the constraint (5) leads to the famous formula $E = \pm m_0 c^2$.

Let us follow similar steps, but instead of starting with the expression (1), one now assumes the formula

$$d\lambda = \frac{d\xi}{\sqrt{\frac{u^2}{c^2} - 1}}, \tag{6}$$

where $u^2 = (dw/d\xi)^2 + (d\rho/d\xi)^2 + (d\zeta/d\xi)^2$. Note that in this case one has the condition $u > c$. Here, in order to emphasize the differences between (1) and (6), we are using a different notation. Indeed, the notation used in (1) and (6) is introduced in order to establish an eventual connection with $(4 + 6)$-dimensions. From (6) one obtains

$$ds^2 = -c^2 d\xi^2 = +c^2 d\lambda^2 - dw^2 - d\rho^2 - d\zeta^2. \tag{7}$$

In tensorial notation, one may write (7) as

$$ds^2 = -c^2 d\xi^2 = \eta^{(-)}_{\mu\nu} dy^\mu dy^\nu, \tag{8}$$

where $y^1 = c\lambda$, $y^2 = w$, $y^3 = \rho$ and $y^4 = \zeta$. Moreover, $\eta^{(-)}_{\mu\nu}$ denotes the flat Minkowski metric with associated signature $(+1, -1, -1, -1)$. One says that this signature represents a world of $(3 + 1)$-dimensions.

If one now defines the linear momentum

$$\mathcal{P}^\mu = m_0^{(-)} \frac{dy^\mu}{d\xi}, \tag{9}$$

with $m_0^{(-)} \neq 0$ a constant, one sees that (9) implies

$$\mathcal{P}^\mu \mathcal{P}^\nu \eta^{(-)}_{\mu\nu} + m_0^{(-)2} c^2 = 0. \tag{10}$$

Since, $u > c$ one observes that in this case the constraint (10) corresponds to a tachyon system with mass $m_0^{(-)}$.

Now, for the case of ordinary matter, if one wants to quantize, one starts promoting p^μ as an operator identifying $\hat{p}^\mu = -i\partial^\mu$. Thus, at the quantum level (5) becomes

$$(-\partial^\mu \partial^\nu \eta^{(+)}_{\mu\nu} + m_0^{(+)2})\varphi = 0. \tag{11}$$

It is important to mention that here we are using a coordinate representation for φ in the sense that $\varphi(x^\mu) =< x^\mu|\varphi >$.

By defining the d'Alembert operator $\Box^{(+)^2} = \eta^{(+)}_{\mu\nu}\partial^\mu\partial^\nu$ one notes that last equation reads

$$(\Box^{(+)^2} - m_0^{(+)2})\varphi = 0. \tag{12}$$

Analogously, in the constraint (10) one promotes the momentum $\mathcal{P}^\mu$ as an operator $\hat{\mathcal{P}}^\mu = -i\partial^\mu$ and using $\eta^{(+)}_{\mu\nu} = -\eta^{(-)}_{\mu\nu}$, the expression (10) yields

$$(\Box^{(+)^2} + m_0^{(-)2})\varphi = 0. \tag{13}$$

The last two expressions are Klein-Gordon type equations for ordinary matter and tachyonic systems, respectively. In fact, these two equations will play an important role in the analysis in Sec. 6, concerning linearized gravity with positive and negative cosmological constant.

3. Clarifying de Sitter and anti-de Sitter space-time

Let us start with the constraint

$$x^i x^j \eta^{(+)}_{ij} + (x^{d+1})^2 = r_0^2, \tag{14}$$

where $\eta^{(+)}_{ij} = diag(-1, 1, ..., 1)$ is the Minkowski metric and the i index goes from 1 to d. and r_0^2 is a positive constant. The line element is given by

$$ds^2 \equiv dx^A dx^B \eta_{AB} = dx^i dx^j \eta^{(+)}_{ij} + (dx^{d+1})^2. \tag{15}$$

It is not difficult to see that the corresponding Christoffel symbols and the Riemann tensor are given by

$$\Gamma^{i}_{kl} = \frac{g_{kl}x^{i}}{r_0^2} \tag{16}$$

and

$$R_{ijkl} = \frac{1}{r_0^2}(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{17}$$

respectively.

Here, the metric g_{ij} is given by

$$g_{ij} = \eta^{(+)}_{ij} + \frac{x_i x_j}{(r_0^2 - x^r x^s \eta^{(+)}_{rs})}. \tag{18}$$

It is worth mentioning that one can even consider a flat metric $\eta_{ij} = diag(-1,-1,....,1,1)$, with t-times and s-space coordinates and analogue developments leads to the formulas (14)-(18).

Of course, the line element associated with the metric (18) is

$$ds^2 \equiv \left(\eta^{(+)}_{ij} + \frac{x_i x_j}{\left(r_0^2 - x^r x^s \eta^{(+)}_{rs}\right)}\right) dx^i dx^j, \tag{19}$$

which in spherical coordinates becomes

$$ds^2 \equiv -\left(1 - \frac{r^2}{r_0^2}\right) dt^2 + \frac{dr^2}{(1-\frac{r^2}{r_0^2})} + r^2 d\Omega^{d-2}. \tag{20}$$

Here, one is assuming that $x^m x^n \eta^{(+)}_{mn} = -x^1 x^1 + r^2$, where $r^2 = x^a x^b \delta_{ab}$, with a, b running from 2 to d. Moreover, $d\Omega^{d-2}$ is a volume element in $d-2$ dimensions. The expression (20) is, of course, very useful when one considers black-holes or cosmological models in the de Sitter space (or anti-de Sitter space).

In the anti-de Sitter case, instead of starting with the formula (14) one considers the constraint is $x^i x^j \eta^{(+)}_{ij} - (x^{d+1})^2 = -r_0^2$. This constraint will play an important role in Sec. 5.

4. Linearized gravity with cosmological constant in any dimension

Although in the literature there are similar computations [16], the discussion of this section seems to be new, in sense that it is extended to any background metric in higher dimensions. Usually, one starts linearized gravity by writing the metric of the space-time $g_{\mu\nu} = g_{\mu\nu}(x^\alpha)$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{21}$$

where $\eta_{\mu\nu} = diag(-1,-1,....1,1)$ is the Minkowski metric, with t-times and s-space coordinates, and $h_{\mu\nu}$ is a small perturbation. Therefore, the general idea is to keep only with the first order terms in $h_{\mu\nu}$, in the Einstein field equations.

Here, we shall replace the Minkowski metric $\eta_{\mu\nu}$ by a general background metric denoted by $g^{(0)}_{\mu\nu}$. At the end we shall associate $g^{(0)}_{\mu\nu}$ with the de Sitter or anti-de Sitter space. So, the analogue of (21) becomes

$$g_{\mu\nu} = g^{(0)}{}_{\mu\nu} + h_{\mu\nu}. \tag{22}$$

The inverse of $g_{\mu\nu}$ is

$$g^{\mu\nu} = g^{(0)\mu\nu} - h^{\mu\nu}. \tag{23}$$

Here, (23) is the inverse metric of (22) at first order in $h_{\mu\nu}$. Also, the metric $g^{(0)}{}_{\mu\nu}$ is used to raise and lower indices. Therefore, neglecting the terms of second order in $h_{\mu\nu}$ one finds that the Christoffel symbols can be written as

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{(0)\lambda}{}_{\mu\nu} + \Sigma^{\lambda}_{\mu\nu,} \tag{24}$$

where $\Gamma^{(0)\lambda}{}_{\mu\nu}$ are the Christoffel symbols associated with $g^{(0)}{}_{\mu\nu}$ and $\Sigma^{\lambda}_{\mu\nu}$ is given by

$$\Sigma^{\lambda}_{\mu\nu} \equiv \frac{1}{2} g^{(0)\lambda\alpha}(\mathcal{D}_\nu h_{\alpha\mu} + \mathcal{D}_\mu h_{\nu\alpha} - \mathcal{D}_\alpha h_{\mu\nu}). \tag{25}$$

Here, the symbol $\mathcal{D}_\mu$ denotes covariant derivative with respect the metric $g^{(0)}{}_{\mu\nu}$.

Similarly, one obtains that at first order in $h_{\mu\nu}$, the Riemann tensor becomes

$$R^{\mu}{}_{\nu\alpha\beta} = R^{(0)\mu}{}_{\nu\alpha\beta} + \mathcal{D}_\alpha \Sigma^{\mu}{}_{\nu\beta} - \mathcal{D}_\beta \Sigma^{\mu}{}_{\nu\alpha}, \tag{26}$$

which can be rewritten as

$$R_{\mu\nu\alpha\beta} = R^{(0)}_{\mu\nu\alpha\beta} + \mathcal{D}_\alpha \Sigma_{\nu\beta\mu} - \mathcal{D}_\beta \Sigma_{\nu\alpha\mu} + R^{(0)\sigma}{}_{\nu\alpha\beta} h_{\sigma\mu}, \tag{27}$$

where

$$\Sigma_{\mu\nu\alpha} \equiv \frac{1}{2}(\mathcal{D}_\nu h_{\alpha\mu} + \mathcal{D}_\mu h_{\alpha\nu} - \mathcal{D}_\alpha h_{\mu\nu}). \tag{28}$$

Then, using the definition (28), the Riemann tensor becomes

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}(\mathcal{D}_\alpha \mathcal{D}_\beta h_{\mu\nu} - \mathcal{D}_\beta \mathcal{D}_\alpha h_{\mu\nu} + \mathcal{D}_\alpha \mathcal{D}_\nu h_{\beta\mu} - \mathcal{D}_\beta \mathcal{D}_\nu h_{\alpha\mu} + \mathcal{D}_\beta \mathcal{D}_\mu h_{\nu\alpha} - \mathcal{D}_\alpha \mathcal{D}_\mu h_{\nu\beta}) + R^{(0)}_{\mu\nu\alpha\beta} + R^{(0)\sigma}_{\nu\alpha\beta} h_{\sigma\mu}. \tag{29}$$

Note that in this case the covariant derivatives $\mathcal{D}_\mu$ do not commute as is the case of the ordinate partial derivatives ∂_μ in a Minkowski space-time background.

One can show that the term $\mathcal{D}_\alpha \mathcal{D}_\beta h_{\mu\nu} - \mathcal{D}_\beta \mathcal{D}_\alpha h_{\mu\nu}$ leads to

$$\mathcal{D}_\alpha \mathcal{D}_\beta h_{\mu\nu} - \mathcal{D}_\beta \mathcal{D}_\alpha h_{\mu\nu} = -h_{\lambda\mu} R^{(0)\lambda}{}_{\nu\alpha\beta} - h_{\lambda\nu} R^{(0)\lambda}{}_{\mu\alpha\beta}. \tag{30}$$

Then using (29), (30) and properties of the Riemann tensor, one can rewrite $R_{\mu\nu\alpha\beta}$ as

$$R_{\mu\nu\alpha\beta} = R^{(0)}_{\mu\nu\alpha\beta} + \frac{1}{2}(h_{\lambda\alpha}R^{(0)\lambda}{}_{\mu\beta\nu} - h_{\lambda\beta}R^{(0)\lambda}{}_{\mu\alpha\nu} - h_{\lambda\nu}R^{(0)\lambda}_{\mu\alpha\beta} + \mathcal{D}_\nu\mathcal{D}_\alpha h_{\mu\beta} - \mathcal{D}_\nu\mathcal{D}_\beta h_{\mu\alpha} + \mathcal{D}_\beta\mathcal{D}_\mu h_{\nu\alpha} - \mathcal{D}_\alpha\mathcal{D}_\mu h_{\nu\beta}). \quad (31)$$

Multiplying (31) by $g^{\mu\nu}$, as given in (23), leads to the Ricci tensor

$$R_{\mu\nu} = R^{(0)}{}_{\mu\nu} + \frac{1}{2}(h_{\lambda\nu}R^{(0)\lambda}{}_{\mu} + h_{\lambda\mu}R^{(0)\lambda}{}_{\nu}) - h^{\alpha\beta}R^{(0)}_{\alpha\mu\beta\nu} + \frac{1}{2}(\mathcal{D}_\mu\mathcal{D}^\alpha h_{\alpha\nu} + \mathcal{D}_\nu\mathcal{D}^\alpha h_{\alpha\mu} - \mathcal{D}_\mu\mathcal{D}_\nu h - \mathcal{D}^\alpha\mathcal{D}_\alpha h_{\mu\nu}). \quad (32)$$

Thus, the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$ becomes

$$R = R^{(0)} + \mathcal{D}^\alpha\mathcal{D}^\beta h_{\alpha\beta} - \mathcal{D}^\alpha\mathcal{D}_\alpha h - h^{\alpha\beta}R^{(0)}{}_{\alpha\beta}. \quad (33)$$

Now one can use (32) and (33) in the Einstein gravitational field equations with a cosmological constant

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \quad (34)$$

When one sets $g^{(0)}_{\mu\nu}$ as a de Sitter (or anti-de Sitter) background one obtains

$$\mathcal{D}_\mu\mathcal{D}_\nu h + \mathcal{D}^\alpha\mathcal{D}_\alpha h_{\mu\nu} - \mathcal{D}_\mu\mathcal{D}^\alpha h_{\alpha\nu} - \mathcal{D}_\nu\mathcal{D}^\alpha h_{\alpha\mu} + g^{(0)}{}_{\mu\nu}(\mathcal{D}^\alpha\mathcal{D}^\beta h_{\alpha\beta} - \mathcal{D}^\alpha\mathcal{D}_\alpha h) - \frac{2}{r_0^2}h_{\mu\nu} - \frac{(d-3)}{2r_0^2}hg^{(0)}{}_{\mu\nu} = 0. \quad (35)$$

As it is commonly done, in linearized gravity in four dimensions, one shall define $\bar{h}_{\mu\nu} = h_{\mu\nu} - (1/2)g^{(0)}{}_{\mu\nu}$. Therefore, substituting this expression for $\bar{h}_{\mu\nu}$ in (35), fixing the Lorentz gauge $\mathcal{D}^\nu\bar{h}_{\mu\nu} = 0$ and assuming the trace $\bar{h} = 0$, one finally gets

$$\Box^2\bar{h}_{\mu\nu} - \frac{4\Lambda}{(d-2)(d-1)}\bar{h}_{\mu\nu} = 0, \quad (36)$$

where d is the dimension of the space-time. It is important to observe that in (36), $\Box^2 = \eta^{(+)}_{\mu\nu}\partial^\mu\partial^\nu$ is now generalized to the form $\Box^2 = g^{(0)}{}_{\mu\nu}\mathcal{D}^\mu\mathcal{D}^\nu$.

At this point, considering the $(4+6)$-signature (which can be splitted into a de Sitter and an anti-de Sitter space according to $(4+6) = (1+4)+(3+2)$) one has to set $d = 8$ since there are two constraints, one given by the de Sitter world and another from the anti-de Sitter world. Consequently, the Eq. (36) becomes

$$\Box^2\bar{h}_{\mu\nu} - \frac{2}{21}\Lambda\bar{h}_{\mu\nu} = 0. \quad (37)$$

One recognizes this expression as the equation of a gravitational wave in $d = 8$.

5. Constraints in de Sitter and anti-de Sitter space

When one considers the de Sitter space, one assumes the constraint (14). However, one may notice that actually there are eight possible constraints corresponding to the two metrics $\eta^{(+)}_{\mu\nu}$ and $\eta^{(-)}_{\mu\nu}$ mentioned in section 3. In fact, for the metric $\eta^{(+)}_{\mu\nu}$ one has following possibilities:

$$x^\mu x^\nu\eta^{(+)}_{\mu\nu} + (x^{d+1})^2 = r_0^2, \quad (38)$$

$$x^\mu x^\nu\eta^{(+)}_{\mu\nu} - (x^{d+1})^2 = r_0^2, \quad (39)$$

$$x^\mu x^\nu\eta^{(+)}_{\mu\nu} + (x^{d+1})^2 = -r_0^2 \quad (40)$$

and

$$x^\mu x^\nu\eta^{(+)}_{\mu\nu} - (x^{d+1})^2 = -r_0^2. \quad (41)$$

While for the metric $\eta^{(-)}_{\mu\nu}$ one finds

$$x^\mu x^\nu\eta^{(-)}_{\mu\nu} + (x^{d+1})^2 = r_0^2, \quad (42)$$

$$x^\mu x^\nu\eta^{(-)}_{\mu\nu} - (x^{d+1})^2 = r_0^2, \quad (43)$$

$$x^\mu x^\nu\eta^{(-)}_{\mu\nu} + (x^{d+1})^2 = -r_0^2 \quad (44)$$

and

$$x^\mu x^\nu\eta^{(-)}_{\mu\nu} - (x^{d+1})^2 = -r_0^2. \quad (45)$$

Now, since one has the relation $\eta^{(+)}_{\mu\nu} = -\eta^{(-)}_{\mu\nu}$, one sees that two sets of constraints (38)-(41) and (42)-(45) are equivalents. Hence, we shall focus only on the first set of constraints (38)-(41). Let us now rewrite (39) and (40) as

$$x^\mu x^\nu\eta^{(+)}_{\mu\nu} = [(x^{d+1})^2 + r_0^2] \quad (46)$$

and

$$x^\mu x^\nu\eta^{(+)}_{\mu\nu} = -[(x^{d+1})^2 + r_0^2]. \quad (47)$$

Observe that one may assume that $x^\mu x^\nu\eta^{(+)}_{\mu\nu} \lesseqgtr 0$, with $\eta^{(+)}_{\mu\nu} = (-1, 1, ..., 1)$. So, since right hand side of (46) is strictly positive, by consistency this constraint must be omitted. Similarly since the right hand side of (47) is strictly negative then this constraint must also be omitted. Thus, one only needs to consider the two constraints, (38) and (41) which corresponds to polynomial equations over the reals whose set of solutions can be associated with classically algebraic varieties. When this constraints are used at the level of line element, one discovers that they can be associated with manifolds for the de Sitter and anti-de Sitter space-time. It turns out that the constraints (38) and (41) can be rewritten as

$$x^A x^B\eta^{(+)}_{AB} = r_0^2 \quad (48)$$

and

$$x^A x^B \eta^{(-)}_{AB} = \rho_0^2, \tag{49}$$

where

$$\eta^{(+)}_{AB} = (-1, 1, ..., 1, 1) \tag{50}$$

and

$$\eta^{(-)}_{AB} = (1, -1, -1, ..., -1, 1). \tag{51}$$

Here, one is assuming that (49) allows for a different radius ρ_0. This is useful for emphasizing that r_0^2 refers to the de Sitter world and ρ_0^2 to the anti-de Sitter world. Note that using ρ_0^2 the expression (49) can also be rewritten as

$$x^i x^j \eta^{(-)}_{ij} + (x^{d+1})^2 = \rho_0^2. \tag{52}$$

Now, using (50) and (51) one can write the line elements in the form

$$ds^2_{(+)} = dx^A dx^B \eta^{(+)}_{AB} = dx^\mu dx^\nu \eta^{(+)}_{\mu\nu} + (dx^{d+1})^2 \tag{53}$$

and

$$ds^2_{(-)} = dx^A dx^B \eta^{(-)}_{AB} = dx^\mu dx^\nu \eta^{(-)}_{\mu\nu} + (dx^{d+1})^2. \tag{54}$$

From (38) one obtains

$$x^{d+1} = \pm (r_0^2 - x^\alpha x^\beta \eta^{(+)}_{\alpha\beta})^{\frac{1}{2}}. \tag{55}$$

So, differentiating (55) one obtains

$$dx^{d+1} = \mp \frac{x_\mu}{r_0^2 - x^\alpha x^\beta \eta^{(+)}_{\alpha\beta}} dx^\mu. \tag{56}$$

Similarly, from (52) one gets

$$dx^{d+1} = \mp \frac{x_\mu}{\rho_0^2 - x^\alpha x^\beta \eta^{(-)}_{\alpha\beta}} dx^\mu. \tag{57}$$

Hence, with the help of (56) and (57), one can rewrite (53) and (54) as

$$ds^2_{(+)} = \left(\eta^{(+)}_{\mu\nu} + \frac{x_\mu x_\nu}{r_0^2 - x^\alpha x^\beta \eta^{(+)}_{\alpha\beta}} \right) dx^\mu dx^\nu \tag{58}$$

and

$$ds^2_{(-)} = \left(\eta^{(-)}_{\mu\nu} + \frac{x_\mu x_\nu}{\rho_0^2 - x^\alpha x^\beta \eta^{(-)}_{\alpha\beta}} \right) dx^\mu dx^\nu, \tag{59}$$

respectively.

Thus, one learns that the metrics associated with (58) and (59) are

$$g^{(+)}_{\mu\nu} = \eta^{(+)}_{\mu\nu} + \frac{x_\mu x_\nu}{r_0^2 - x^\alpha x^\beta \eta^{(+)}_{\alpha\beta}} \tag{60}$$

and

$$g^{(-)}_{\mu\nu} = \eta^{(-)}_{\mu\nu} + \frac{x_\mu x_\nu}{\rho_0^2 - x^\alpha x^\beta \eta^{(-)}_{\alpha\beta}}, \tag{61}$$

respectively.

Using (60) and (61) one sees that according to (17) the Riemann tensors $R^{(+)}_{\mu\nu\alpha\beta}$ and $R^{(-)}_{\mu\nu\alpha\beta}$ become

$$R^{(+)}_{\mu\nu\alpha\beta} = \frac{1}{r_0^2} \left(g^{(+)}_{\mu\alpha} g^{(+)}_{\nu\beta} - g^{(+)}_{\mu\beta} g^{(+)}_{\nu\alpha} \right) \tag{62}$$

and

$$R^{(-)}_{\mu\nu\alpha\beta} = -\frac{1}{\rho_0^2} \left(g^{(-)}_{\mu\alpha} g^{(-)}_{\nu\beta} - g^{(-)}_{\mu\beta} g^{(-)}_{\nu\alpha} \right). \tag{63}$$

The corresponding curvature scalars associated with (62) and (63) are

$$R^{(+)} = \frac{d(d-1)}{r_0^2} \tag{64}$$

and

$$R^{(-)} = -\frac{d(d-1)}{\rho_0^2}. \tag{65}$$

Now, let us consider the Einstein gravitational field equation (see Eq. (34))

$$R^{(+)}_{\mu\nu} - \frac{1}{2} g^{(+)}_{\mu\nu} R^{(+)} + \Lambda^{(+)} g^{(+)}_{\mu\nu} = 0, \tag{66}$$

for $g^{(+)}_{\mu\nu}$. Multiplying this expression by $g^{\mu\nu(+)}$ one sees that (66) leads to

$$R^{(+)} - \frac{1}{2} d R^{(+)} + \Lambda^{(+)} d = 0. \tag{67}$$

Solving (67) for $\Lambda^{(+)}$ leads to

$$\Lambda^{(+)} = \frac{(d-2)(d-1)}{2 r_0^2}, \tag{68}$$

where the Eq. (64) was used.

In analogous way, by considering the Einstein gravitational field equations for $g^{(-)}_{\mu\nu}$

$$R^{(-)}_{\mu\nu} - \frac{1}{2} g^{(-)}_{\mu\nu} R^{(-)} + \Lambda^{(-)} g^{(-)}_{\mu\nu} = 0, \tag{69}$$

one obtains

$$\Lambda^{(-)} = -\frac{(d-2)(d-1)}{2 \rho_0^2}. \tag{70}$$

Note that, since $\Lambda^{(-)}$ refers to the anti-de Sitter space, (70) agrees with the fact that $\Lambda^{(-)} < 0$.

6. The signature of the space-time in linearized gravity

In the previous section, the Einstein gravitational field equations were considered for $g^{(+)}_{\mu\nu}$ and $g^{(-)}_{\mu\nu}$. Such equations lead us to find a relations for $\Lambda^{(+)}$ and $\Lambda^{(-)}$. Now, if one substitutes the Eqs. (68) and (70) into (36) one obtains

$$\left({}^{(+)}\Box^2 - \frac{4\Lambda^{(+)}}{(d-2)(d-1)}\right)\bar{h}^{(+)}_{\mu\nu} = 0, \tag{71}$$

and

$$\left({}^{(-)}\Box^2 + \frac{4\Lambda^{(-)}}{(d-2)(d-1)}\right)\bar{h}^{(-)}_{\mu\nu} = 0. \tag{72}$$

Here, ${}^{(\pm)}\Box^2 = g^{(\pm)(0)}_{\mu\nu}\mathcal{D}^\mu\mathcal{D}^\nu$.

Let us now to consider, in the context of linearized gravity, the vielbein formalism for $g^{(+)}_{\mu\nu}$ and $g^{(-)}_{\mu\nu}$. One introduces the vielbein field e^a_μ and writes

$$g^{(\pm)}_{\mu\nu} = e^a_\mu e^b_\nu \eta^{(\pm)}_{ab}, \tag{73}$$

where

$$e^a_\mu = b^a_\mu + h^a_\mu. \tag{74}$$

If one replaced (74) into (73), the metric $g^{(\pm)}_{\mu\nu}$ becomes

$$g^{(\pm)}_{\mu\nu} = (b^a_\mu + h^a_\mu)(b^b_\nu + h^b_\nu)\eta^{(\pm)}_{ab}. \tag{75}$$

Thus, one obtains

$$\begin{aligned} g^{(\pm)}_{\mu\nu} &= b^a_\mu b^b_\nu \eta^{(\pm)}_{ab} + b^a_\mu h^b_\nu \eta^{(\pm)}_{ab} \\ &\quad + h^a_\mu b^b_\nu \eta^{(\pm)}_{ab} + h^a_\mu h^b_\nu \eta^{(\pm)}_{ab}. \end{aligned} \tag{76}$$

Since one is assuming that $h^a_\mu \ll 1$ then $h^a_\mu h^b_\nu \eta^{(\pm)}_{ab} \sim 0$ and therefore (76) is reduced to

$$g^{(\pm)}_{\mu\nu} = b^a_\mu b^b_\nu \eta^{(\pm)}_{ab} + b^a_\mu h^b_\nu \eta^{(\pm)}_{ab} + h^a_\mu b^b_\nu \eta^{(\pm)}_{ab}. \tag{77}$$

If one establishes the identifications $g^{(\pm)(0)}_{\mu\nu} = b^a_\mu b^b_\nu \eta^{(\pm)}_{ab}$ and $h^{(\pm)}_{\mu\nu} = h^a_\mu b^b_\nu \eta^{(\pm)}_{ab} + b^a_\mu h^b_\nu \eta^{(\pm)}_{ab}$ one obtains

$$g^{(\pm)}_{\mu\nu} = g^{(\pm)(0)}_{\mu\nu} + h^{(\pm)}_{\mu\nu}, \tag{78}$$

which is the expression (22) but with the signatures $+$ or $-$ in $g^{(\pm)(0)}_{\mu\nu}$ and $h^{(\pm)}_{\mu\nu}$ identified.

Now, we shall compare the Eqs. (71) and (72) with (12) and (13), respectively. Since $\Lambda^{(+)} > 0$ and $\Lambda^{(-)} < 0$ one can introduces the two mass terms

$$m^{(+)2} = \frac{4\Lambda^{(+)}}{(d-2)(d-1)} \tag{79}$$

and

$$m^{(-)2} = -\frac{4\Lambda^{(-)}}{(d-2)(d-1)}. \tag{80}$$

For $d = 4$, corresponding to the observable universe, and for ordinary matter one has

$$m^{(+)2} = \frac{2}{3}\Lambda^{(+)}. \tag{81}$$

This mass expression must be associated with a systems traveling lower than the light velocity $(c > v)$. In the case of particles traveling faster than light velocity $(v > c)$, corresponding to tachyons, one obtains

$$m^{(-)2} = -\frac{2}{3}\Lambda^{(-)}. \tag{82}$$

Note that since $\Lambda^{(+)} > 0$ and $\Lambda^{(-)} < 0$, both rest masses $m^{(+)2}$ and $m^{(-)2}$ are positives.

7. Linearized gravity in (4+6)-dimensions

The key idea in this section is to split the $(4+6)$-dimension as $(4+6) = (1+4) + (3+2)$. This means that the $(4+6)$-dimensional space is splitted in two parts the de Sitter world of $(1+4)$-dimensions and anti de Sitter world of $(3+2)$-dimensions. In this direction, let us write the line elements in (3) and (7) in the form

$$\begin{aligned} d\mathcal{S}^2_{(+)} &= -c^2 dt^2_{0(+)} = -c^2 dt^2_{(+)} \\ &\quad + dx^2_{(+)} + dy^2_{(+)} + dz^2_{(+)} + dw^2_{(+)} \end{aligned} \tag{83}$$

and

$$\begin{aligned} d\mathcal{S}^2_{(-)} &= -c^2 dt^2_{0(-)} = +c^2 dt^2_{(-)} \\ &\quad - dx^2_{(-)} - dy^2_{(-)} - dz^2_{(-)} + d\tau^2_{(-)}, \end{aligned} \tag{84}$$

respectively. One can drop the parenthesis notation in the coordinates $x^\mu_{(+)}$ and $x^\mu_{(-)}$ if one makes the convention that the index μ in $x^\mu_{(+)}$ runs from 1 to 4, while the index μ in $x^\mu_{(-)}$ is changed to an index a running from 5 to 8. Thus, in tensorial notation one may write (83) and (84) as

$$d\mathcal{S}^2_{(+)} = \eta^{(+)}_{\mu\nu} dx^\mu dx^\nu + dw^2_{(+)} \tag{85}$$

and

$$d\mathcal{S}^2_{(-)} = \eta^{(-)}_{ab} dx^a dx^b + d\tau^2_{(-)}. \tag{86}$$

Here, $\eta^{(+)}_{\mu\nu}$ and $\eta^{(-)}_{ab}$ denotes flat Minkowski metrics with the signatures $(-1,+1,+1,+1)$ and $(+1,-1,-1,-1)$, respectively. It seems evident that one can reach a unified framework by adding (85) and (86) in the form

$$\begin{aligned} d\mathcal{S}^2 &= d\mathcal{S}^2_{(+)} + d\mathcal{S}^2_{(-)} = \eta^{(+)}_{\mu\nu} dx^\mu dx^\nu \\ &\quad + dw^2_{(+)} + \eta^{(-)}_{ab} dx^a dx^b + d\tau^2_{(-)}. \end{aligned} \tag{87}$$

Let us assume that in a world of $(4+6)$-signature one has the two constraints

$$\eta^{(+)}_{\mu\nu}x^\mu x^\nu + w^2 = \frac{3}{\Lambda_{(+)}} \tag{88}$$

and

$$\eta^{(-)}_{ab}dx^a dx^b + \tau^2 = \frac{3}{\Lambda_{(-)}}, \tag{89}$$

where $\Lambda_{(+)} > 0$ and $\Lambda_{(-)} < 0$ again play the role of two cosmological constants. Following similar procedure as in Sec. 5, considering the constraints (88) and (89) one can generalize the the line element (87) in the form

$$d\mathcal{S}^2 = g^{(+)(0)}_{\mu\nu}dx^\mu dx^\nu + g^{(-)(0)}_{ab}dx^a dx^b. \tag{90}$$

where

$$g^{(+)(0)}_{\mu\nu} = \eta^{(+)}_{\mu\nu} + \frac{x_\mu x_\nu}{\frac{3}{\Lambda_{(+)}} - x^i x^j \eta^{(+)}_{ij}} \tag{91}$$

and

$$g^{(-)(0)}_{ab} = \eta^{(-)}_{ab} + \frac{x_a x_b}{\frac{3}{\Lambda_{(-)}} - x^i x^j \eta^{(-)}_{ij}}. \tag{92}$$

Using (91) and (92) one can define a background matrix γ_{AB}, with indexes A and B running from 1 to 8 in the form

$$\gamma^{(0)}_{AB} = \begin{pmatrix} g^{(+)(0)}_{\mu\nu} & 0 \\ 0 & g^{(-)(0)}_{ab} \end{pmatrix}. \tag{93}$$

Thus, one can write the linearized metric associated with (93) as

$$\gamma_{AB} = \gamma^{(0)}_{AB} + h_{AB}. \tag{94}$$

Hence, following a analogous procedure as the presented in section 4, one obtains the equation for h_{AB} in $d = 4+4 = 8$-dimensions,

$$\Box^2 \bar{h}_{AB} - \frac{2}{21}\Lambda \bar{h}_{AB} = 0. \tag{95}$$

Here, one has

$$\Box^2 = \gamma^{(0)}_{AB}\mathcal{D}^A\mathcal{D}^B. \tag{96}$$

One can split $\Box^2$ in the form

$$\Box^2 = \Box^{(+)^2} + \Box^{(-)^2}, \tag{97}$$

where $\Box^{(+)^2} = g^{(+)(0)}_{\mu\nu}\mathcal{D}^\mu\mathcal{D}^\nu$ and $\Box^{(-)^2} = g^{(0)(-)}_{ab}\mathcal{D}^a\mathcal{D}^b$. One shall use now the separation of variables method. For this purpose let us assume that the perturbation $\bar{h}_{AB} = \bar{h}_{AB}(x^\mu, x^a)$ can be splitted in the form

$$\bar{h}_{AB} = \bar{h}^{(+)}{}_{AC}(x^\mu)\bar{h}^{(-)C}{}_{B}(x^a). \tag{98}$$

Thus, one discovers that (95) becomes

$$\bar{h}^{(-)C}{}_{B}\Box^{(+)^2}\bar{h}^{(+)}{}_{AC} + \bar{h}^{(+)}{}_{AC}\Box^{(-)^2}\bar{h}^{(-)C}{}_{B} - \frac{2}{21}\Lambda\bar{h}^{(+)}{}_{AC}\bar{h}^{(-)C}{}_{B} = 0. \tag{99}$$

Multiplying the last equation by $\bar{h}^{(+)AE}\bar{h}^{(-)B}{}_{D}$ yields

$$\bar{h}^{(+)AE}\Box^{(+)^2}\bar{h}^{(+)}{}_{AD} - \frac{2}{21}\Lambda\delta^E_D = -\bar{h}^{(-)B}{}_{D} \times \Box^{(-)^2}\bar{h}^{(-)E}_{B}. \tag{100}$$

Thus, one observes that while the left hand side of (100) depends only of x^μ and the right hand side depends only of x^a one may introduce a constant $\hat{\Lambda}$ such that

$$\bar{h}^{(+)AE}\Box^{(+)^2}\bar{h}^{(+)}{}_{AD} - \frac{2}{21}\Lambda\delta^E_D = \hat{\Lambda}\delta^E_D \tag{101}$$

and

$$-\bar{h}^{(-)B}{}_{D}\Box^{(-)^2}\bar{h}^{(-)E}{}_{B} = \hat{\Lambda}\delta^E_D. \tag{102}$$

One may rewrite (101) and (102) in the for

$$\Box^{(+)^2}\bar{h}^{(+)}{}_{AB} - \left(\frac{2}{21}\Lambda + \hat{\Lambda}\right)\bar{h}^{(+)}{}_{AB} = 0 \tag{103}$$

and

$$\Box^{(-)^2}\bar{h}^{(-)}{}_{AB} + \hat{\Lambda}\bar{h}^{(-)}{}_{AB} = 0. \tag{104}$$

According to the formalism presented in Sec. 5, one can identify the tachyonic mass in the anti-de Sitter-world by $m^{(-)2} = -\hat{\Lambda}$, while the mass in the de Sitter-world by $m^{(+)2} = (2/21)\Lambda + \hat{\Lambda}$. Moreover, one can also introduce an effective $M^2 = (2/21)\Lambda$ in the $(4+4)$-world. Note that the effective mass M^2 can be written as $M^2 = m^{(+)2} + m^{(-)2}$. Thus, (103) and (104) can be rewritten as

$$(\Box^{(+)^2} - m^{(+)2})\bar{h}^{(+)}{}_{\mu\nu} = 0 \tag{105}$$

and

$$(\Box^{(-)^2} - m^{(-)2})\bar{h}^{(-)}{}_{ab} = 0. \tag{106}$$

Here, we fixed the gauges $\bar{h}^{(+)}{}_{ab} = 0$ and $\bar{h}^{(-)}{}_{\mu\nu} = 0$. Thus, we have shown that from linearized gravity in $(4+4)$-dimensions one can derive linearized gravities in $(1+3)$-dimensions and in $(3+1)$-dimensions. Moreover, the modes $\bar{h}^{(+)}{}_{\mu\nu}$ can be associated with a massive graviton in the de Sitter space, while the modes $\bar{h}^{(-)}{}_{ab}$ can be associated with the tachyonic graviton in the anti-de Sitter space.

8. Final remarks

In this work we have developed a higher dimensional formalism for linearized gravity in the de Sitter or anti-de Sitter space-background which are characterized by the cosmological constants $\Lambda^{(+)} > 0$ and $\Lambda^{(-)} < 0$, respectively. Our starting point are the higher dimensional Einstein gravitational field equations and the perturbed metric $g^{(\pm)}_{\mu\nu} = g^{(\pm)(0)}_{\mu\nu} + h^{(\pm)}_{\mu\nu}$, where $g^{(\pm)(0)}_{\mu\nu}$ is a background metric associated with the cosmological constants $\Lambda^{(+)}$, $\Lambda^{(-)}$ and the Minkowski flat metric $\eta^{(\pm)}_{\mu\nu}$. After straightforward computations and after imposing a gauge conditions for $h^{(\pm)}_{\mu\nu}$ we obtain the two Eqs. (71) and (72). We proved that these two equations admit an interpretation of massive graviton with mass given by (79) and (80). According to the formalism discussed in Sec. 2, the massive graviton with mass $m^{(+)2}$ can be associated with ordinary graviton which "lives" in the de Sitter space, while the massive graviton with mass $m^{(-)2}$ is a tachyonic graviton which "lives" in the anti-de Sitter space. We should mention that these results agree up to sign from those described by Novello and Neves [17]. The origin of this difference in the signs is that although they consider a version of linearized gravity their approach refers only to four dimensions and rely in a field strength $F_{\mu\nu\alpha\beta}$ which is not used in our case. Here, we get a four dimensional graviton mass $m_g^2 = (2/3)\Lambda$ for de Sitter space and using the Planck 2015 data [18] we can set $m_g \sim 3.0 \times 10^{-69}$ kg, while the current upper bound obtained by the detection of gravitational waves is $m_g \leq 1.3 \times 10^{-58}$ kg [15].

Furthermore, in the previous section, we discuss the case of the $(4+6)$ signature where we identify $m^{(+)2}$ and $m^{(-)2}$ as a contribution to an effective mass M^2 in the unified framework of $(4+4)$-dimensions. It would be interesting for a future work to have a better understanding of the meaning of the mass M^2. Also, it may be interesting to extent this work to a higher dimensional cosmological model with a massive graviton.

On the other hand, it is worth mentioning that our proposed formalism in $(4+4)$-dimensions may be related to the so called double field theory [19]. This is a theory formulated with $x^A = (x^\mu, x^a)$ coordinates corresponding to the double space $R^4 \times T^4$, with $A = 1, 2, ..., 8$ and $D = 8 = 4 + 4$. In this case the constant metric is given by

$$ds^2 = \eta_{AB} dx^A dx^B. \tag{107}$$

Moreover, the relevant group in this case is $O(4,4)$ which is associated with the manifold M^8. It turns out that M^8 can be compactified in such a way that becomes the product $R^4 \times T^4$ of flat space and a torus. In turn the group $O(8,8)$ is broken into a group containing $O(4,4) \times O(4,4;Z)$. A detail formulation of this possible relation will be present elsewhere.

Finally, it is inevitable to mention that perhaps the formalism developed in this work may be eventually useful for improvements of the direct detection of gravitational waves. This is because recent observations [20] established that the cosmological value has to be small and positive and that the observable universe resembles to a de Sitter universe rather than an anti de Sitter universe. Also, it will be interesting to explore a link between this work and the electromagnetic counterpart of the gravitational waves [21].

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Appendix

A. Negative mass squared term - tachyon association

This appendix is dedicated to clarify why the expression (80) refers to a tachyon system. In some sense the below presentation is the reverse argument as the one presented in Sec. 2.

Consider the Klein-Gordon equation

$$(\Box^2 + m_0^2)\varphi = 0. \tag{A.1}$$

If one considers a plain wave solution for (A.1), the solution can be written as

$$\varphi = A e^{p^\mu x_\mu}, \tag{A.2}$$

where A is a constant. Therefore, using (A.2) one can verify that (A.1) is reduced to

$$(p^2 + m_0^2)\varphi = 0, \tag{A.3}$$

which implies that

$$p^2 + m_0^2 = 0. \tag{A.4}$$

Since $p^\mu = m_0 u^\mu$ one discovers that (A.4) leads to a relation of the form

$$dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}}. \tag{A.5}$$

Which implies that $v < c$ and therefore the system moves with velocities less than the light velocity. Similarly if instead of (A.1) one considers the expression

$$(\Box^2 - m_0^2)\varphi = 0. \tag{A.6}$$

A plain wave solution would imply the classical expression

$$p^2 - m_0^2 = 0, \tag{A.7}$$

which again considering the relation $p^\mu = m_0 u^\mu$ one finds that instead of the relation (A.5) one has

$$dt = \frac{d\tau}{\sqrt{v^2/c^2 - 1}}. \tag{A.8}$$

This implies that $v > c$ and therefore describes a tachyon system. If instead of the field φ one considers the $h_{\mu\nu}$ and assume a plain wave solution of the form $h_{\mu\nu} = A_{\mu\nu} e^{p^\mu x_\mu}$, one may be able to obtain the corresponding expression (A.5) and (A.8) for linearized gravity.

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