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Projective method for spinorial techniques: unifying calculational framework for Dirac amplitudes

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There have been numerous methodologies to perform the calculation of spin-dependent amplitudes for Dirac particles. All of them have their own advantages and properties, but there is no general framework to address the analytic calculation of such amplitudes. In this work, we use the closure property of massive spinors to present a new and general approach to compute transition amplitudes for general spin states. We argue that this perspective can be used to reformulate all other techniques and to relate all of them. Particularly, it is shown that the massless spinor and the helicity spinor techniques can be formulated through this language. Finally, we give an example of this calculation as a procedure by computing the spin-dependent amplitude for the Compton process and comment some of the strengths of our method.

Keywords: Spinor techniques; Dirac amplitudes; compton scattering.

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1. Introduction

The search for new physics and standard-model tests in high energy physics, more than ever, demands the exploration of highly sophisticated processes [1-8]. This brings along experimental challenges as, for example, the implementation of multiparticle detection and measurement. In this context, the theoretical study of the spin-dependent observables and the analysis of high-order processes require the calculation of transition amplitudes of increasing complexity. Because these amplitudes involve a large number of Dirac particles, coupled through interaction terms with several gamma matrices [9-12], the textbook methods become cumbersome to calculate with.

The algebraic structure of physical amplitudes demands the use of bold and clever methods to provide an analytical value. Modern techniques for amplitude calculation have been improved, especially for interactions with many particles [13-16], and are usually optimized for some computational implementations [17-20]. However, almost all of these techniques are devoted to helicity states for fermions [21-29] and only a few can be regarded as an analysis tool for general spin directions [30].

Our goal in this work is to develop a fully covariant method which allows obtainong efficient and compact analytical expressions for the spin-dependent amplitude, so that the framework can be employed within a computational program. Also, we want to suggest that a simple unifying framework to many other approaches can be formulated with our expressions.

The organization of this work is as follows. Section 2 sets the basic notation and contains the central idea of our procedure: the covariance of Dirac spinors and its implications to the closure property, along with the decomposition of an operator in terms of spinors. Section 3 presents a systematic framework for spin amplitude calculation, we call this scheme the projective method. The method is exemplified using the spinor rest frame basis. Section 4 is devoted to formulate the massless-spinor method [18] using the structures of the Sec. 2. In Sec. 5, it is shown that the helicity-spinor method can also be structured within this framework. We give an example of all of these procedures by computing the Compton Scattering amplitude in Sec. 6. Finally, Sec. 7 contains the conclusions. Bjorken and Drell [31] notation and units will be adopted throughout this text.

2. Completeness of the Dirac spinors

The Dirac equation for a free particle of mass m

$$(\not{p} - \epsilon \mathrm{m})\, \psi_\epsilon(x) = 0, \tag{1}$$

where $\epsilon = \pm 1$ is the sign of the energy (Ref. 31, pag. 28), uses two different representations of the Lorentz group. On one hand, the four-vector operator $\hat{p}^\mu = i\left((\partial/\partial t), -\bar{\nabla}\right)$ which transforms as the fundamental representation establishes. On the other hand, the massive-spinor representation acts on the four-component objects (Dirac spinors) which are used to solve Eq. (1). Namely, the positive- and negative-energy plane-wave solutions for Eq. (1) are

$$\psi_+(x) = u(p,s)\, e^{-ip\cdot x}, \quad \psi_-(x) = v(p,s)\, e^{ip\cdot x}, \tag{2}$$

where $x^\mu = (t, \bar{\mathrm{x}})$, $u(p,s)$ is a positive-energy Dirac spinor and $v(p,s)$ is the negative energy one. The energy E and the three-component momentum $\bar{\mathrm{p}}$ form the four-momentum vector $p^\mu = (\mathrm{E}, \bar{\mathrm{p}})$, and $s^\mu = (\mathrm{s}^0, \bar{\mathrm{s}})$, the four-spin vector, fulfill the properties $p^2 = \mathrm{m}^2$ $s \cdot p = 0$ and $s^2 = -1$.

The operator $\not{p}$ contains the Dirac matrices γ^μ which, due to the Lorentz invariance of the Dirac equation (1), transform as both spinor and four-vector object. To fully appreciate what this implies, it is useful to recall the properties of a Dirac spinor.

The spinors $u(p, \pm s)$ and $v(p, \pm s)$ satisfy the set of eigenequations

$$\begin{aligned} \not{p}\, u(p, \pm s) &= \mathsf{m}\, u(p, \pm s), \\ \not{p}\, v(p, \pm s) &= -\mathsf{m}\, v(p, \pm s), \\ \gamma^5 \not{s}\, u(p, \pm s) &= \pm\, u(p, \pm s), \\ \gamma^5 \not{s}\, v(p, \pm s) &= \pm\, v(p, \pm s), \end{aligned} \tag{3}$$

and form a complete basis for any four-component spinor, *i.e*. an arbitrary Dirac spinor w, with four-momentum p', four-spin s' and mass m' can be expanded in the form

$$\begin{aligned} w(p', s') &= \sum_s u(p, s)\, \bar{u}(p, s) w(p', s') \\ &\quad - \sum_s v(p, s)\, \bar{v}(p, s) w(p', s'), \end{aligned} \tag{4}$$

where it is used the notation $\bar{U}(p, s) = U^\dagger(p, s)\gamma^0$ for the adjoint spinor of $U(p, s)$.

The expansion (4) will not mix spinors of opposite energy if it connects spinors with the same four-momenta. This means that the representation of $u(p, s')$ or $v(p, s')$ will only have non-zero terms for coefficients $\bar{u}(p, s)u(p, s')$ or $\bar{v}(p, s)v(p, s')$, respectively.

The orthogonality between spinors $u(p, s)$ and $v(p, s')$ is evident in their rest reference frame, where the Dirac spinors of different energies are mutually orthogonal, no matter which spin directions are chosen. Using a Lorentz transformation $\mathrm{S}(\bar{\beta})$, one goes to the rest frame of the spinor with momentum $\bar{\mathrm{p}} = \bar{\beta}\,\mathrm{E}$, and then the equality

$$\begin{aligned} \bar{v}(p, s)u(p, s') &= \bar{v}(p, s)\mathrm{S}(-\bar{\beta})\mathrm{S}(\bar{\beta})u(p, s') \\ &= \bar{v}(p_0, s_0)u(p_0, s_0') = 0, \end{aligned} \tag{5}$$

is evident and shows that it will hold in any reference frame. Also, formula (4) can be proven by taking advantage of its fully covariant structure, as it is noticed in (Ref. 31, pag. 31), and this suggests that its use will, in turn, generate covariant expressions.

The transformation rule of Dirac matrices (Ref. 31, pag. 20), can be further appreciated with the insight that equation (4) provides. Using it, one can demonstrate the expansion

$$\gamma^\mu = \sum_{rr'} C^\mu_{rr'}(Q, Q')\, \omega_r(Q, S)\bar{\omega}_{r'}(Q', S'), \tag{6}$$

where $r, r' = 1, 2, 3, 4$, the spinor $\omega_r(Q, S) = u(Q, (-1)^{r+1}S)$, for $r = 1, 2$; and $\omega_r(Q, S) = v(Q, (-1)^r S)$, for $r = 3, 4$. Expression (6) shows a twofold composition of representations of the Lorentz group. While the coefficients $C^\mu_{rr'}(Q, Q')$ transform according to the four-vector representation, the operator $\omega_r(Q, S)\bar{\omega}_{r'}(Q', S')$ is a pure spinorial object, transforming with the Dirac-spinor representation of the Lorentz group. Some remarks are in order. Firstly, the selection of the dynamical variables Q and S may seem arbitrary. This is not the case since their free choice reveals that Eqs. (3) constrain the spinors which constitute the basis. Secondly, the anticommutation and antihermitian properties of the gamma matrices are strictly fulfilled by this expansion, although it does not seem obvious from (6). Thirdly, analogous expressions to (6) follow but for products of gamma matrices such as γ^5, $\gamma^5\gamma^\mu$ and $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$.

A general spinorial operator $\mathbf{O}$ will be covariant if it transforms under the Lorentz group as $\mathbf{O}' = \mathrm{S}\,\mathbf{O}\,\mathrm{S}^{-1}$. If $\mathbf{O}$ is written as

$$\mathbf{O} = \mathsf{s} + \mathsf{p}\gamma^5 + v_\mu\gamma^\mu + a_\mu\gamma^5\gamma^\mu + \Omega_{\mu\nu}\sigma^{\mu\nu}, \tag{7}$$

the covariance is verified if the s, p, v^μ, a^μ, and $\Omega^{\mu\nu}$ are objects that transform under the four-vector representation as a scalar, pseudo-scalar, vector, axial-vector, and tensor quantities, respectively. If a covariant operator acts upon any spinor $\omega_r(Q)$, a new spinor $\mathbf{O}\,\omega_r(Q)$ is obtained. In general, this spinor is not necessarily an eigenstate of Eqs. (3), but the result can be expressed as a linear combination of a complete spinorial basis. This idea will be extensively used in the following sections, but the relevance of the completeness in the spinor space becomes now clear.

In the next section, the expressions (4) and (6) will be used to compute a general Dirac amplitude, and that will enable us to understand other techniques systematically.

3. Projective method for the amplitude computation: a General Framework

A Dirac amplitude is a matrix element of the, usually, covariant operator $\mathbf{\Gamma}$. Its Lorentz scalar nature is revealed when one looks at its explicit form. For example, with positive energy states, it looks as

$$M(p', s', p, s) = \bar{u}(p', s')\mathbf{\Gamma}u(p, s). \tag{8}$$

The main purpose of the spinorial techniques is to obtain the analytical value of expression (8) in terms of simple elements, such as inner products between four-vectors, or the volume subtended by four-vectors $\varepsilon^{\alpha\beta\gamma\delta}a_\alpha b_\beta c_\gamma d_\delta$ (where $\varepsilon_{0123} = +1$ is the Levi-Civita symbol of four indices). A common trick is to rewrite Eq. (8) as

$$M(p', s', p, s) = \mathrm{tr}\, u(p, s)\bar{u}(p', s')\,\mathbf{\Gamma}. \tag{9}$$

This procedure has the advantage of explicitly displaying the two covariant objects that constitute the amplitude. Furthermore, using the trace properties of gamma matrices it is possible to obtain an analytical expression for the amplitude. While the $\mathbf{\Gamma}$ operator is usually given in the representation that is shown in Eq. (7), the operator $u(p, s)\bar{u}(p', s')$ is not usually represented this way. Associated with expansion (7), or a similar decomposition (general procedures to expand any operator in terms of covariant bilinears can be found in [32-34]), we can point out a couple of weaknesses of the operator $u(p,s)\bar{u}(p',s')$. On one hand, the procedure requires a considerable amount of work, the expressions obtained are not as compact as one would desire and are not trivially reduced to manifestly covariant functions of p', s', p, and s. On the other hand, the resulting elements do not have a direct physical interpretation and this obscures the overall outcome. To deal with this, a specific basis to overcome these difficulties is presented in the next subsection.

3.1. The rest basis

In the rest frame of the Dirac spinors, we rewrite Eqs. (3) as

$$\left(\not{\mathfrak{q}} + (-1)^{\epsilon}\right)\Phi_{\epsilon\tau}(\mathfrak{s}) = 0,$$
$$\left(\gamma^5\not{\mathfrak{s}} + (-1)^{\tau}\right)\Phi_{\epsilon\tau}(\mathfrak{s}) = 0, \tag{10}$$

with the spinors at rest $\Phi_{11}(\mathfrak{s}) = \mathrm{S}(\bar{\beta})u(p,s)$, $\Phi_{12}(\mathfrak{s}) = \mathrm{S}(\bar{\beta})u(p,-s)$, $\Phi_{22}(\mathfrak{s}) = \mathrm{S}(\bar{\beta})v(p,-s)$, $\Phi_{21}(\mathfrak{s}) = \mathrm{S}(\bar{\beta})v(p,s)$ and the four-vectors $\mathfrak{q}^{\mu} = (1/\mathsf{m})\Lambda^{\mu\nu}(\bar{\beta})\,p_{\nu} = (1,\bar{0})$, $\mathfrak{s}^{\mu} = \Lambda^{\mu\nu}(\bar{\beta})\,s_{\nu} = (0,\hat{s})$. The Matrix $\Lambda(\bar{\beta})$ is the four-vector representation of the Lorentz transformation to a frame moving with velocity $\bar{\beta}$, *i.e.*, the frame where the spinors are at rest.

It is easy to see that the representation of the Dirac spinors in (Ref. 31, pag. 30) allows for a projective decomposition of the Dirac spinors in terms of the spinors at rest

$$u(p,s) = \sqrt{\frac{2\mathsf{m}}{\mathrm{E}+\mathsf{m}}}u(p,s)\bar{u}(p,s)\ \Phi_{11}(\mathfrak{s}),$$
$$u(p,-s) = \sqrt{\frac{2\mathsf{m}}{\mathrm{E}+\mathsf{m}}}u(p,-s)\bar{u}(p,-s)\ \Phi_{12}(\mathfrak{s}),$$
$$v(p,-s) = -\sqrt{\frac{2\mathsf{m}}{\mathrm{E}+\mathsf{m}}}v(p,-s)\bar{v}(p,-s)\ \Phi_{22}(\mathfrak{s}),$$
$$v(p,s) = -\sqrt{\frac{2\mathsf{m}}{\mathrm{E}+\mathsf{m}}}v(p,s)\bar{v}(p,s)\ \Phi_{21}(\mathfrak{s}). \tag{11}$$

Note that expressions (11) are easily derived using the standard gamma-matrices representation (Ref. 31, pag. 30). They remain valid in an arbitrary representation since two sets of gamma matrices, γ^{μ} and $\tilde{\gamma}^{\mu}$, are equivalent under similarity transformations ([35]).

Using expressions (11), Eq. (9) takes the form

$$M(p',s',p,s) = \sqrt{\frac{2\mathsf{m}'}{\mathrm{E}'+\mathsf{m}'}}\sqrt{\frac{2\mathsf{m}}{\mathrm{E}+\mathsf{m}}}\mathrm{tr}\ \Phi_{11}(\mathfrak{s}) \times \bar{\Phi}_{11}(\mathfrak{s}')\mathrm{P}_{++}(p',s')\,\mathbf{\Gamma}\,\mathrm{P}_{++}(p,s), \tag{12}$$

where both particles can have different masses and the operator $\mathrm{P}_{\alpha\beta}(p,s) = \Pi(\alpha,p)\Sigma(\beta,s) = = \Sigma(\beta,s)\Pi(\alpha,p)$ is constituted with $\Pi(\alpha,p) = (\alpha\not{p} + \mathsf{m})/2m$ and $\Sigma(\beta,s) = (\beta\gamma^5\not{s} + 1)/2$, the energy and spin projectors, respectively, pertaining to positive spin and energy for $\alpha = \beta = +1$. A transition amplitude for a process between an initial spinor of energy sign $\alpha = (-1)^{\epsilon+1}$ and spin eigenvalue in the frame at rest $(-1)^{\epsilon+\tau} = \beta\,(-1)^{\epsilon+1}$, and a final spinor of energy sign $\alpha' = (-1)^{\epsilon'+1}$ and spin eigenvalue in the frame at rest $(-1)^{\epsilon'+\tau'} = \beta'(-1)^{\epsilon'+1}$ is written as

$$M(\epsilon',p',\tau',s',\epsilon,p,\tau,s) = (-1)^{\epsilon'+\epsilon}\sqrt{\frac{2\mathsf{m}'}{\mathrm{E}'+\mathsf{m}'}}\sqrt{\frac{2\mathsf{m}}{\mathrm{E}+\mathsf{m}}} \times \mathrm{tr}\ \Phi_{\epsilon\tau}(\mathfrak{s})\bar{\Phi}_{\epsilon'\tau'}(\mathfrak{s}')\mathrm{P}_{\alpha'\beta'}(p',s')\,\mathbf{\Gamma}\,\mathrm{P}_{\alpha\beta}(p,s). \tag{13}$$

To be useful, the expression in (13) needs the operator expansion of $\Phi_{\epsilon\tau}(\mathfrak{s})\bar{\Phi}_{\epsilon'\tau'}(\mathfrak{s}')$ in terms of equation (7). The task is simple compared to the work required to compute $u(p,s)\bar{u}(p',s')$, and the reason for this lies in the simplifications allowed by the rest frame, particularly the implications for operator expansions. To display this, it is convenient to note that the four-component spinors $\Phi_{\epsilon\tau}$ can be presented as two couples of angular-momentum-like eigenstates, with total angular momentum $1/2$, each for different complementary algebras.

This can be seen from the spin equation in (10)

$$\gamma^5\bar{\gamma}\cdot\hat{\mathfrak{s}}\ \Phi_{\epsilon\tau}(\mathfrak{s}) = (-1)^{\tau}\Phi_{\epsilon\tau}(\mathfrak{s}). \tag{14}$$

Equation (14) implies, via the eigenvalue equation $\gamma^0\Phi_{\epsilon\tau}(\mathfrak{s}) = -(-1)^{\epsilon}\Phi_{\epsilon\tau}(\mathfrak{s})$, that the spinors $\Phi_{\epsilon 1}(\mathfrak{s})$ and $\Phi_{\epsilon 2}(\mathfrak{s})$ are eigenstates of the operator $-\frac{1}{2}\gamma^0\gamma^5\bar{\gamma}\cdot\hat{\mathfrak{s}} = \bar{\mathbf{J}}\cdot\hat{\mathfrak{s}}$ with eigenvalues $-(-1)^{\epsilon}/2$ and $+(-1)^{\epsilon}/2$ respectively. Operators $\bar{\mathbf{J}}$ fulfill the angular momentum commutation rules,

$$\left[-\gamma^0\gamma^5\gamma^i, -\gamma^0\gamma^5\gamma^j\right]\Phi_{\epsilon\tau}(\mathfrak{s}) = -\left[\gamma^i,\gamma^j\right]\Phi_{\epsilon\tau}(\mathfrak{s}) = 2i\varepsilon^{ijk}(-\gamma^0\gamma^5\gamma^k)\Phi_{\epsilon\tau}(\mathfrak{s}), \tag{15}$$

but operators $\gamma^0\bar{\mathbf{J}}$ do not. Using the projectors $\mathbf{P}_{\epsilon} = (1-(-1)^{\epsilon}\gamma^0)/2$, one can decompose the algebra as a direct sum of two representations $\bar{\mathbf{J}}_1 = \mathbf{P}_1\bar{\mathbf{J}}$ and $\bar{\mathbf{J}}_2 = \mathbf{P}_2\bar{\mathbf{J}}$, in such a way that

$$\bar{\mathbf{J}} = \bar{\mathbf{J}}_1 + \bar{\mathbf{J}}_2, \tag{16}$$

and $[\mathbf{J}_1^l, \mathbf{J}_2^m] = 0\ \forall\ l,m$. This shows that the four-component spinors $\Phi_{\epsilon\tau}$ are a basis for a reducible representation of a group $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$.

From Eq. (16) and the definition of $\mathbf{P}_{\epsilon}$, an operator acting in the subspace of spinors of definite energy ϵ can be decomposed in terms of the operator basis with four elements $\mathbf{V}_{\epsilon}^{k} = \left\{\mathbf{P}_{\epsilon}, \bar{\mathbf{J}}_{\epsilon}\right\}$, $k = 0,1,2,3$. Then, it is possible to expand $\Phi_{\epsilon\tau}(\mathfrak{s})\bar{\Phi}_{\epsilon\tau'}(\mathfrak{s}')$ as

$$\Phi_{\epsilon\tau}(\mathfrak{s})\bar{\Phi}_{\epsilon\tau'}(\mathfrak{s}') = \sum_{k} -(-1)^{\epsilon}c^{k}_{\epsilon\tau\tau'}\,\mathbf{V}_{\epsilon}^{k}, \tag{17}$$

where no summation over the ϵ index is implied and the coefficients $c^{k}_{\epsilon\tau\tau'} = c^{k}_{\epsilon\tau\tau'}(\mathfrak{s},\mathfrak{s}')$ will be obtained as the expression $c^{k}_{\epsilon\tau\tau'}(\mathfrak{s},\mathfrak{s}') = -(-1)^{\epsilon}(1/2)\mathrm{tr}\,\mathbf{V}_{\epsilon}^{k}\,\Phi_{\epsilon\tau}(\mathfrak{s})\bar{\Phi}_{\epsilon\tau'}(\mathfrak{s}')$ establishes. For some representation of the gamma matrices, Eqs. (14) are reduced to two-component eigenequations

$(1/2)\hat{\mathbb{s}}\cdot\bar{\sigma}\,\phi_{\epsilon\tau} = (-1)^{\epsilon+\tau}(1/2)\,\phi_{\epsilon\tau}$ and, if we use the rest spin vector parametrization $(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, the coefficients can be easily obtained up to a nonphysical phase. Their explicit expressions can be found in Appendix A.

Expression (17) can be arranged in a covariant form

$$\Phi_{\epsilon\tau}(\mathbb{s})\bar{\Phi}_{\epsilon\tau'}(\mathbb{s}') = -(-1)^{\epsilon}\check{c}\cdot q + \not{\check{c}} - (-1)^{\epsilon} i\,\gamma^5\sigma^{\mu\nu}q_\mu \c{c}_\nu + \gamma^5 \not{\c{c}}, \tag{18}$$

with the four-vectors $\check{c}^\mu = \check{c}^\mu(\epsilon,\tau,\tau') = (1/2)(c^0_{\epsilon\tau\tau'}, \bar{0})$, $\c{c}^\mu = \c{c}^\mu(\epsilon,\tau,\tau') = (1/2)(0, c^1_{\epsilon\tau\tau'}, c^2_{\epsilon\tau\tau'}, c^3_{\epsilon\tau\tau'})$ and $q^\mu = (\mathbb{q}^\mu + \mathbb{q}'^\mu)/2$.

The expansion of the operator $\Phi_{\epsilon\tau}(\mathbb{s})\bar{\Phi}_{\epsilon'\tau'}(\mathbb{s}')$, for $\epsilon \neq \epsilon'$, can be obtained with the operator γ^5. After multiplying the first equation in (10) by γ^5, the energy effectively changes its sign but, the eigenvalue for the spin equation keeps its sign

$$\not{\mathbb{q}}\,\gamma^5\Phi_{\epsilon\tau}(\mathbb{s}) = (-1)^{\epsilon}\gamma^5\Phi_{\epsilon\tau}(\mathbb{s}),$$
$$\bar{\mathbf{J}}\cdot\hat{\mathbb{s}}\,\gamma^5\Phi_{\epsilon\tau}(\mathbb{s}) = (-1)^{\epsilon+\tau}\frac{1}{2}\,\gamma^5\Phi_{\epsilon\tau}(\mathbb{s}). \tag{19}$$

Notice that, up to an unphysical phase, the operators (18) transform under γ^5 as

$$\begin{aligned} &\gamma^5\Phi_{11}(\mathbb{s})\bar{\Phi}_{1\tau'}(\mathbb{s}') \to \Phi_{22}(\mathbb{s})\bar{\Phi}_{1\tau'}(\mathbb{s}'),\\ &\gamma^5\Phi_{12}(\mathbb{s})\bar{\Phi}_{1\tau'}(\mathbb{s}') \to \Phi_{21}(\mathbb{s})\bar{\Phi}_{1\tau'}(\mathbb{s}'),\\ &\Phi_{1\tau}(\mathbb{s})\bar{\Phi}_{11}(\mathbb{s}')\gamma^5 \to -\Phi_{1\tau}(\mathbb{s})\bar{\Phi}_{22}(\mathbb{s}'),\\ &\Phi_{1\tau}(\mathbb{s})\bar{\Phi}_{12}(\mathbb{s}')\gamma^5 \to -\Phi_{1\tau}(\mathbb{s})\bar{\Phi}_{21}(\mathbb{s}'). \end{aligned} \tag{20}$$

It is important to note that Eqs. (13), (18) and (B.1) (see appendix B) have no arbitrary phases; this is relevant when one is interested in the relative phases of a multiparticle process.

As an elementary application of expressions (13) and (18), the transition amplitude for a vector operator $\mathbf{\Gamma} = v_\mu\gamma^\mu$ and for spinors with the same energy sign is shown in Appendix B. However, by using the formulas (13), (A.1), (18), and (20), it is possible to perform calculations for any kind of transition amplitude with a systematic methodology. Nevertheless, this is a special case of the more general projective method.

3.2. General projective method

When one deals with an analytic amplitude calculation, like the one in equation (13), the implementation of a specific basis is useful to optimize the overall procedure. Particularly, the rest basis is a powerful tool to obtain the spin-dependent amplitude. A specific interaction term $\mathbf{\Gamma}$ could be successfully treated with a suitable election of the spinorial basis. Such a basis will be used in the generalized formula (13)

$$M(\alpha'p', \beta's', \alpha p, \beta s) = \sum_{rr'} \kappa_r\kappa_{r'}\; c^*_{r'}(Q', S'; \alpha'p', \beta's') \times c_r(Q, S; \alpha p, \beta s)\;\mathrm{tr}\,\omega_r(Q,S)\bar{\omega}_{r'}(Q',S')\mathbf{\Gamma}, \tag{21}$$

where the spinors $\omega_r(Q,S)$ were defined above and the notation of the spin-energy projectors are used, with

$$\begin{aligned} c_r(Q,S;+p,\beta s) &= \bar{\omega}_r(Q,S)u(p,\beta s),\\ c_r(Q,S;-p,\beta s) &= \bar{\omega}_r(Q,S)v(p,\beta s). \end{aligned} \tag{22}$$

Formula (21) reflects the main idea of this work. Although it is written with a general basis, it shows formidable characteristics of the completeness of a spinorial basis, *i.e.*, the reduction of the amount of work needed, and the explicit covariance of the expressions obtained. Instead of working with the squared modulus of the amplitude $|M|^2$, formula (21) allows one to deal with the transition amplitude M itself, without invoking non-physical quantities, and using a decomposition of the original transition amplitude in terms of elementary amplitudes $\bar{\omega}_{r'}(Q',S')\Gamma\omega_r(Q,S)$. In the next two sections, we will use two popular bases to express formula (21) and, exploiting their main properties and symmetries, we will obtain closed and compact expressions.

4. The massless-spinor method

Any massive spinorial basis possesses the interesting symmetry provided by the operator γ^5. Multiplying Eqs. (3) by γ^5 shows that the spinors $\gamma^5\omega_r(Q,S)$ are also its solutions. Actually, the spinor $\gamma^5\omega_r(Q,S)$ corresponds to an energy sign $-\kappa_r$ and spin sign $-\delta_r$ as follows

$$\not{p}\,\gamma^5\omega_r(Q,S) = -\kappa_r \mathrm{m}\,\gamma^5\omega_r(Q,S),$$
$$\gamma^5\not{s}\,\gamma^5\omega_r(Q,S) = -\eta_r\,\gamma^5\omega_r(Q,S), \tag{23}$$

where $\kappa_r = +1$, $\eta_r = +1$ for $r = 1,2$ and $\kappa_r = -1$, $\eta_r = -1$ for $r = 3,4$. This is a consequence of the PCT transformation over the spinor part of the wave functions (2). Nothing prevents the use of the set of eight linearly dependent massive spinors $\{\omega_r(Q,S), \gamma^5\omega_r(Q,S)\}$ as an overcomplete basis.

There are multiple ways to reduce this set to a complete basis. For example, the linear combinations $\omega_r(Q,S) \pm \gamma^5\omega_{r+2}(Q,S)$, with $r = 1,2$, form a complete basis. However, some useful linear combinations are

$$W^{(r)}_\lambda = W^{(r)}_\lambda(Q,S) = \frac{1}{2}\left(1+\lambda\gamma^5\right)\omega_r(Q,S), \tag{24}$$

with $\lambda = \pm 1$. The eight states (24) are not anymore eigenstates of Eqs. (3), but they fulfill

$$\gamma^5 W^{(r)}_\lambda = \lambda W^{(r)}_\lambda, \tag{25}$$

and, with an appropriate phase selection, have the nontrivial orthogonality properties

$$\bar{W}^{(r')}_{\lambda'} W^{(r)}_\lambda = \frac{\kappa_r}{2}\delta_{-\lambda'\lambda}\left(\delta_{r'r} + \lambda(\delta_{r'\,r+2} + \delta_{r'+2\,r})\right). \tag{26}$$

Equations (24), (25) and (26) suggest that a decomposition of the massive spinor space into two complementary spaces can be made. This approach emerges when one reinterprets equation (24) as the result of projecting the state $\omega_r(Q,S)$ with the chiral projector operators $\mathbf{p}_\lambda = (1+\lambda\gamma^5)/2$. Effectively, an operator $\mathbf{O}$ can be decomposed into the direct sum of two operators, which are acting, separately, on the $\mathbf{p}_\lambda$ and $\mathbf{p}_{-\lambda}$ subspaces. This can be seen explicitly in each splitting of the covariant operators

$$\begin{aligned}
&\mathsf{s} \to \mathsf{s}\,\mathbf{p}_\lambda + \mathsf{s}\,\mathbf{p}_{-\lambda},\\
&\mathsf{p}\,\gamma^5 \to \mathsf{p}\,\mathbf{p}_\lambda\gamma^5\mathbf{p}_\lambda + \mathsf{p}\,\mathbf{p}_{-\lambda}\gamma^5\mathbf{p}_{-\lambda},\\
&v_\mu\gamma^\mu \to v_\mu\,\mathbf{p}_{-\lambda}\gamma^\mu\mathbf{p}_\lambda + v_\mu\,\mathbf{p}_\lambda\gamma^\mu\mathbf{p}_{-\lambda},\\
&a_\mu\gamma^5\gamma^\mu \to a_\mu\,\mathbf{p}_{-\lambda}\gamma^5\gamma^\mu\mathbf{p}_\lambda + a_\mu\,\mathbf{p}_\lambda\gamma^5\gamma^\mu\mathbf{p}_{-\lambda},\\
&\Omega_{\mu\nu}\sigma^{\mu\nu} \to \Omega_{\mu\nu}\,\mathbf{p}_\lambda\sigma^{\mu\nu}\mathbf{p}_\lambda + \Omega_{\mu\nu}\,\mathbf{p}_{-\lambda}\sigma^{\mu\nu}\mathbf{p}_{-\lambda},
\end{aligned} \tag{27}$$

and it means that two independent bases can be constructed in each of these subspaces.

A general eigenstate of Eq. (25) can be dentoed as π^ζ_λ. The respective basis will be generated when the equation for the quantum number ζ is found and the degeneration in state π^ζ_λ is avoided. Since $[\alpha^i,\gamma^5]=0$, one can impose the equation

$$\bar{\mathrm{k}}\cdot\bar{\alpha}\;\pi^\zeta_\lambda = \zeta\,\mathrm{k}_0\;\pi^\zeta_\lambda, \tag{28}$$

where $k^2 = (\zeta\,\mathrm{k}_0)^2 - \mathrm{k}_1^2 - \mathrm{k}_2^2 - \mathrm{k}_3^2 = 0$, $\mathrm{k}_0 > 0$ and $\zeta = \pm1$. Equation (28) is known as the Weyl equation and it can be interpreted as a (dynamical) constriction over the spinors $\pi^\zeta_\lambda = \pi^\zeta_\lambda(k)$. Due to the condition (dispersion relation) $k^2=0$, it is common to use the term *massless spinor* to designate π^ζ_λ.

As a consequence, a basis will be formed by spinors π^{+1}_λ, π^{-1}_λ over the $\mathbf{p}_\lambda$ subspace, while the subspace $\mathbf{p}_{-\lambda}$ will have the basis $\pi^{+1}_{-\lambda}$, $\pi^{-1}_{-\lambda}$. However, the orthogonality rules are now

$$\bar{\pi}^{\zeta'}_{\lambda'}\pi^\zeta_\lambda = \frac{1}{2}\delta_{\zeta'\zeta}\delta_{-\lambda'\lambda}, \tag{29}$$

where we have used the normalization of the states (24); this implies that the expansion of any operator in each subspace will be

$$\mathbf{O}_\lambda = \sum_{\zeta'\zeta} C^{\mathbf{O}}_{\lambda\zeta'\zeta}\;\pi^{\zeta'}_\lambda\,\bar{\pi}^\zeta_{-\lambda}. \tag{30}$$

The different λ signs in the spinors in expression (30) are required by expression (29). This is an evidence that both, an operator $\mathbf{O}$ (acting over the four-dimensional spinor space) and a spinor $\omega_r(Q,S)$, need the four Weyl spinors π^ζ_λ to be expanded. The main reason behind the structure of Eq. (30) follows from the form in which the orthogonality between spinors is imposed. The covariace of Eqs. (29) introduces the operator γ^0, which does not commute with γ^5.

Then, a covariant expression for the expansion of any spinor w in terms of Weyl spinors is

$$w = 2\sum_\zeta\;\pi^\zeta_\lambda\,\bar{\pi}^\zeta_{-\lambda}w + 2\sum_\zeta\;\pi^\zeta_{-\lambda}\,\bar{\pi}^\zeta_\lambda w, \tag{31}$$

where w can be a massive or massless spinor.

Equation (31) is useful to select a complete basis from the eight states (24) but, as can be seen, there is no unique way to form a complete basis using states (24). Equation (28) helps in the task. The combination of the dynamical quantities $k_\mathfrak{t} = p + \mathfrak{t}\,\mathsf{m}\,s$, with $\mathfrak{t}=\pm1$, are two, nonorthogonal, light-like four-vectors. The application of the operators $\not{k}_\mathfrak{t}$ on the states (24) yields

$$\not{k}_\mathfrak{t}W^{(r)}_\lambda = \left(\not{p} + \mathfrak{t}\,\mathsf{m}\not{s}\right)W^{(r)}_\lambda = \mathsf{m}\,\left(\kappa_r - \lambda\mathfrak{t}\eta_r\right)\,W^{(r)}_{-\lambda}. \tag{32}$$

A remarkable property of the operators $\not{k}_\mathfrak{t}$ is shown by the right-hand side of the Eq. (32). Imposing the condition $\kappa_r - \lambda\mathfrak{t}\eta_r = 0$, it is possible to establish a Weyl-like equation over the states (24).

For example, if we explicitly choose the basis $W^{(R)}_\lambda$ with $R=1,2$, two of them, $W^{(1)}_{+1}$ and $W^{(2)}_{-1}$, fulfill the Weyl equation with $\not{k}_{+1}$, while the others, $W^{(1)}_{-1}$ and $W^{(2)}_{+1}$, satisfy a similar equation with $\not{k}_{-1}$. Other selections can be found, for example, in [**?**]. We can change the label $(R)\to R$ and then, the orthogonality rules among them reduce to

$$\bar{W}^{R'}_{\lambda'}W^R_\lambda = \frac{1}{2}\delta_{R'R}\delta_{-\lambda'\lambda}. \tag{33}$$

The expansion of an arbitrary, massive or massless, spinor w can be written as

$$w = 2\sum_R\;W^R_\lambda\,\bar{W}^R_{-\lambda}w + 2\sum_R\;W^R_{-\lambda}\,\bar{W}^R_\lambda w. \tag{34}$$

As a first elementary application of formula (34), we present the massive spinors in terms of massless spinors

$$\begin{aligned}
u(p,s) &= W^1_{+1} + W^1_{-1},\\
u(p,-s) &= W^2_{+1} + W^2_{-1},\\
v(p,-s) &= W^1_{+1} - W^1_{-1},\\
v(p,s) &= W^2_{+1} - W^2_{-1},
\end{aligned} \tag{35}$$

where the same dynamical quantities $Q=p$, $S=s$ have been used in formula (34); this means that $W^R_\lambda = W^R_\lambda(q,s)$ in expression (35). Another application of formula (34) is the expression for a transition amplitude with a vector interaction $\mathbf{\Gamma} = v_\mu\gamma^\mu$,

$$\begin{aligned}
M(r',p',s',r,p,s) = v_\mu\big(&\bar{W}^{\rho'}_{+1}\gamma^\mu W^\rho_{+1}\\
&+ \kappa_{r'}\kappa_r\,\bar{W}^{\rho'}_{-1}\gamma^\mu W^\rho_{-1}\big),
\end{aligned} \tag{36}$$

with $\rho = r + \kappa_r - 1$, and $W^{\rho'}_{\lambda'} = W^{\rho'}_{\lambda'}(p',s')$. A similar procedure as the one used in Sec. 3.1 can be applied to obtain closed formulas for the terms $\bar{W}^{\rho'}_\lambda\gamma^\mu W^\rho_\lambda$, though, it is convenient to use a specific form of spinors W^ρ_λ and calculate it directly, but we do not give explicit expressions for $\bar{W}^{\rho'}_\lambda\gamma^\mu W^\rho_\lambda$ here.

Having established the connection with the massless spinor basis, now we will study the helicity formalism relation with the formalism of this work.

5. Helicity-spinor method

5.1. Helicity amplitude

An helicity spinor [36] $w_\kappa^h(p,s_h)$, of energy sign $\kappa=\pm 1$, is defined through equations

$$\not p\, w_\kappa^h(p,s_h) = \kappa \mathrm{m}\, w_\kappa^h(p,s_h),$$
$$\frac{\bar{\mathrm{p}}\cdot\bar{\Sigma}}{|\bar{\mathrm{p}}|}\, w_\kappa^h(p,s_h) = h\, w_\kappa^h(p,s_h), \tag{37}$$

with $\Sigma^k=\epsilon^{klm}\sigma^{lm}$, the helicity sign $h\pm 1$, and helicity-spin four-vector

$$s_h = \left(\frac{|\bar{\mathrm{p}}|}{\mathrm{m}}, \frac{\mathrm{E}}{\mathrm{m}}\hat{\mathrm{p}}\right). \tag{38}$$

It is essential to be careful with the noncovariant appearance of the second Eq. (37). This equation can be translated to the apparently covariant expression

$$\gamma^5 \not s_h\, w_\kappa^h(p,s_h) = \kappa\, h\, w_\kappa^h(p,s_h), \tag{39}$$

where both Eqs. (37) are involved in its deduction. For Lorentz-boost transformations that do not reverse the direction of $\bar{\mathrm{p}}$, Eq. (39) is a covariant expression. We restrict our analysis to this case.

It is useful to define the helicity associated rest basis of a helicity spinor $w_\kappa^h(p,s_h)$

$$\Phi_\kappa^h(\hat{\mathrm{p}}) = \mathrm{S}(\bar{\beta})\, w_\kappa^h(p,s_h) = \left(\cosh\frac{\chi}{2} - h\,\gamma^5\sinh\frac{\chi}{2}\right) w_\kappa^h(p,s_h). \tag{40}$$

The correspondence between the bases $\Phi_\kappa^h(\hat{\mathrm{p}})$ and $\Phi_{\epsilon\tau}(\hat{\mathrm{p}})$ is $\Phi_{11}\to\Phi_{+1}^{+1}$, $\Phi_{12}\to\Phi_{+1}^{-1}$, $\Phi_{21}\to\Phi_{-1}^{-1}$, $\Phi_{22}\to\Phi_{-1}^{+1}$. Definition (40) allows us to write the helicity amplitude for an operator $\mathbf{\Gamma}$, for final $w_{\kappa'}^{h'}(p',s'_h)$ and initial $w_\kappa^h(p,s_h)$ helicity states, as

$$M(\kappa' p', s'_{h'}, \kappa\, p, s) = \cosh\frac{\chi'}{2}\cosh\frac{\chi}{2} M^{h'h}_{\kappa'\kappa} + h'\sinh\frac{\chi'}{2} \times\cosh\frac{\chi}{2} M^{h'h}_{-\kappa'\kappa} + h\cosh\frac{\chi'}{2} \times\sinh\frac{\chi}{2} M^{h'h}_{\kappa'-\kappa} + h'\,h\sinh\frac{\chi'}{2} \times\sinh\frac{\chi}{2} M^{h'h}_{-\kappa'-\kappa}, \tag{41}$$

where the notation $M^{h'h}_{\kappa'\kappa}=\bar{\Phi}^{h'}_{\kappa'}\mathbf{\Gamma}\Phi^h_\kappa$ and the PCT transformation have been used (see Eq. (19)).

The elements $M^{h'h}_{\kappa'\kappa}$ can be obtained with the formalism developed in Sec. (3.), particularly using formulas (18) and (20). For example, when formula (41) is used for interaction through a vector operator $\mathbf{\Gamma}=v_\mu\gamma^\mu$, we obtain

$$M(\kappa p', s'_{h'}, \kappa\, p, s_h) = 4\left(\cosh\frac{\chi'}{2}\cosh\frac{\chi}{2}\,\check{c}(\epsilon_\kappa,\tau_{\kappa h},\tau_{\kappa h'})\cdot v + h'\sinh\frac{\chi'}{2}\cosh\frac{\chi}{2}\,\c{c}(\epsilon_\kappa,\tau_{\kappa h},\tau_{\kappa h'})\cdot v + h\cosh\frac{\chi'}{2}\sinh\frac{\chi}{2}\,\c{c}(\epsilon_\kappa,\tau_{\kappa h},\tau_{\kappa h'})\cdot v + h'\,h\sinh\frac{\chi'}{2}\sinh\frac{\chi}{2}\,\check{c}(\epsilon_{-\kappa},\tau_{-\kappa h},\tau_{-\kappa h'})\cdot v\right), \tag{42}$$

$$M(-\kappa p', s'_{h'}, \kappa\, p, s_h) = 4\left(\cosh\frac{\chi'}{2}\cosh\frac{\chi}{2}\,\c{c}(\epsilon_\kappa,\tau_{\kappa h},\tau_{\kappa h'})\cdot v + h'\sinh\frac{\chi'}{2}\cosh\frac{\chi}{2}\,\check{c}(\epsilon_\kappa,\tau_{\kappa h},\tau_{\kappa h'})\cdot v + h\cosh\frac{\chi'}{2}\sinh\frac{\chi}{2}\,\check{c}(\epsilon_{-\kappa},\tau_{-\kappa h},\tau_{-\kappa h'})\cdot v + h'\,h\sinh\frac{\chi'}{2}\sinh\frac{\chi}{2}\,\c{c}(\epsilon_\kappa,\tau_{\kappa h},\tau_{\kappa h'})\cdot v\right), \tag{43}$$

with $\epsilon_\kappa=(3-\kappa)/2$ and $\tau_{\kappa h}=(3-\kappa h)/2$.

Formula (40) can be used to express a general spinor $\omega_r(Q,S)$ in terms of the helicity associated rest basis

$$\omega_r(Q,S) = \kappa_r\Bigg(\left(\cosh\frac{\chi}{2}K^h_{\kappa_r r} + h\sinh\frac{\chi}{2}K^h_{-\kappa_r r}\right) \times\left(\cosh\frac{\chi}{2}\Phi^h_{\kappa_r} + h\sinh\frac{\chi}{2}\Phi^h_{-\kappa_r}\right) + \left(\cosh\frac{\chi}{2}K^{-h}_{\kappa_r r} - h\sinh\frac{\chi}{2}K^{-h}_{-\kappa_r r}\right) \times\left(\cosh\frac{\chi}{2}\Phi^{-h}_{\kappa_r} - h\sinh\frac{\chi}{2}\Phi^{-h}_{-\kappa_r}\right)\Bigg), \tag{44}$$

where the helicity spinorial basis w^h_κ has the same four-momentum $q^\mu = Q^\mu = (\mathrm{Q}_0,\bar{\mathrm{Q}})$, and $K^h_{\kappa_r r} = \bar{\Phi}^h_{\kappa_r}(\hat{\mathrm{Q}})\omega_r(Q,S)$. In practice, it is straightforward to directly evaluate the symbols $K^h_{\kappa_r r}$ using an explicit representation of the respective spinors. Finally, the formula for a general spin amplitude can be found in Appendix C.

In the next section an elementary use of the formulas developed in Secs. (3-5) is presented. There, multiple expressions for the amplitude of the Compton process are derived.

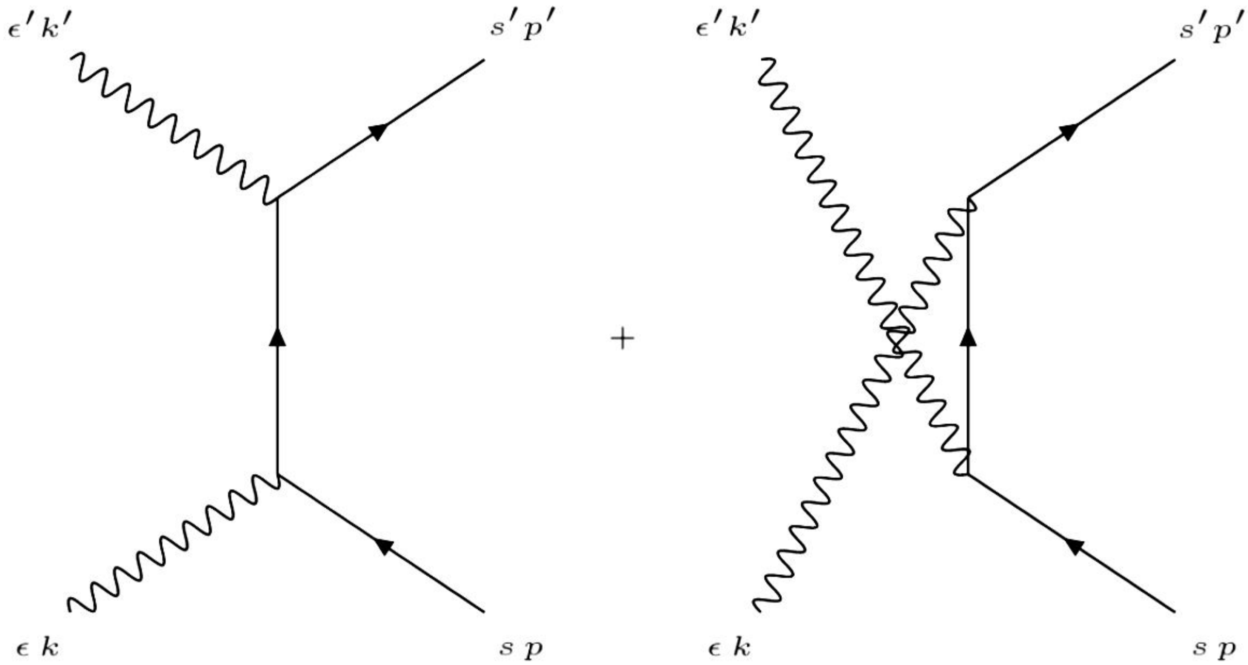

FIGURE 1. Feynman diagram for Compton scattering.

6. An example: the Compton effect

The transition amplitude for the Compton process (see Fig. 1) is

$$M(p',s',p,s) = -ie^2\ \bar{u}(p',s')\Big(\not\epsilon'^*\frac{1}{\not p+\not k-m}\not\epsilon + \not\epsilon\frac{1}{\not p-\not k'-m}\not\epsilon'^*\Big)u(p,s), \tag{45}$$

where $k^\mu = \mathrm{k}(1,\hat{\mathrm{k}})$, $k'^\mu = \mathrm{k}'(1,\hat{\mathrm{k}})$ are the initial and final photon four-momentum, and ϵ^μ, ϵ'^μ are the four-vectors that represent their respective polarizations, which are in a complex-number representation and allow the treatment of elliptical polarization.

Using the Mandelstam variables

$$\begin{aligned} s &= (p+k)^2,\\ u &= (p-k')^2,\\ t &= (k-k')^2,\\ s+u+t &= 2m^2, \end{aligned} \tag{46}$$

and after some manipulations, one can rewrite equation (45) as

$$M(p',s',p,s) = -ie^2\ \bar{u}(p',s')\Big(\frac{2\epsilon\cdot p\,\not\epsilon'^* - \not\epsilon'^*\not\epsilon\not k}{s-m^2} - \frac{2\epsilon\cdot p'\,\not\epsilon'^* + \not k\not\epsilon\not\epsilon'^*}{m^2-u}\Big)u(p,s). \tag{47}$$

The use of the equation $\not p u(p,s) = mu(p,s)$, the four-momentum conservation $p+k = p'+k'$ and the gamma matrices identity $\gamma^\alpha\gamma^\beta\gamma^\delta = g^{\alpha\beta}\gamma^\delta - g^{\alpha\delta}\gamma^\beta + g^{\beta\delta}\gamma^\alpha + i\varepsilon^{\alpha\beta\delta\mu}\gamma^5\gamma_\mu$, reduces the number of gamma matrices. If the final terms are arranged, expression (47) looks as

$$M(p',s',p,s) = -ie^2\bar{u}(p',s')(\not{\mathbb{V}} + \gamma^5\not{\mathbb{A}})u(p,s), \tag{48}$$

where

$$\mathbb{V}^\mu = \frac{(s-u)(-\epsilon\cdot\epsilon'^* k^\mu + \epsilon'^*\cdot k\ \epsilon^\mu) + 2t\epsilon\cdot p'\epsilon'^{*\mu}}{(m^2-u)(s-m^2)} + \frac{2\epsilon\cdot k'\epsilon'^{*\mu}}{s-m^2}, \tag{49}$$

$$\mathbb{A}^\mu = it\frac{\epsilon_\alpha\epsilon'^*_\beta k_\delta\varepsilon^{\alpha\beta\delta\mu}}{(m^2-u)(s-m^2)}. \tag{50}$$

Using the spinor techniques, and the compact expression (48), efficient treatment of the amplitude can be made as it is shown below.

6.1. The helicity-spinor method

With formulas (18) and (42), the helicity amplitude for this process is

$$
\begin{aligned}
M(p', s'_{h'}, p, s_h) = -4ie^2 \Bigg(& \cosh\frac{\chi'}{2}\cosh\frac{\chi}{2}\left(\check{c}(1, \tau_{+1h}, \tau_{+1h'})\cdot \mathbb{V} - \c{c}(1, \tau_{+1h}, \tau_{+1h'})\cdot \mathbb{A}\right) \\
& + h' \sinh\frac{\chi'}{2}\cosh\frac{\chi}{2}\left(\c{c}(1, \tau_{+1h}, \tau_{+1h'})\cdot \mathbb{V} - \check{c}(1, \tau_{+1h}, \tau_{+1h'})\cdot \mathbb{A}\right) \\
& + h \cosh\frac{\chi'}{2}\sinh\frac{\chi}{2}\left(\c{c}(1, \tau_{+1h}, \tau_{+1h'})\cdot \mathbb{V} - \check{c}(1, \tau_{+1h}, \tau_{+1h'})\cdot \mathbb{A}\right) \\
& + h' h \sinh\frac{\chi'}{2}\sinh\frac{\chi}{2}\left(\check{c}(2, \tau_{-1h}, \tau_{-1h'})\cdot \mathbb{V} - \c{c}(2, \tau_{-1h}, \tau_{-1h'})\cdot \mathbb{A}\right)\Bigg).
\end{aligned} \tag{51}
$$

Equation (51) explicitly shows three parts: kinematic elements through the hyperbolic functions, the spin direction elements $\c{c}^\mu$, $\check{c}^\mu$, and the dynamical elements $\mathbb{V}^\mu$, $\mathbb{A}^\mu$. This separation could be useful, for example, to analyze some kinematic configuration of interest, as the center of mass high energy limit, in which this process preserves helicity. If one takes the approximations $\cosh\chi/2 \approx \cosh(\chi'/2)$ and $\sinh(\chi/2) \approx \sinh(\chi'/2)$ in Eq. (51), the desired result $M(p', s'_{-h}, p, s_h) = 0$ is easily obtained. There exist systems where this limit is interesting, as the inverse Compton Light Source (details of this machine can be found in the literature, for example in [37]). Then, in this limit, and assuming helicity conservation, equation (51) reads

$$
\begin{aligned}
M(p', s'_h, p, s_h) = -4ie^2 \Bigg(& \cosh\chi\Big(\check{c}(1, \tau_{+1h}, \tau_{+1h})\cdot \mathbb{V} - \c{c}(1, \tau_{+1h}, \tau_{+1h})\cdot \mathbb{A}\Big) \\
& + h \sinh\chi\Big(\c{c}(1, \tau_{+1h}, \tau_{+1h})\cdot \mathbb{V} - \check{c}(1, \tau_{+1h}, \tau_{+1h})\cdot \mathbb{A}\Big)\Bigg).
\end{aligned} \tag{52}
$$

Using expressions (48), (49), (50) and formula (C.1), the amplitude for general spin directions is

$$
\begin{aligned}
M(r', p', s', r, p, s) = \Bigg(& a^{h'}_{+1r'}(\chi')a^{h}_{+1r}(\chi)\left(\cosh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{h'h}_{+1+1} + h' \sinh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{h'h}_{-1+1}\right. \\
& \left. + h \cosh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{h'h}_{+1-1} + h'h \sinh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{h'h}_{-1-1}\right) + a^{-h'}_{+1r'}(\chi')a^{h}_{+1r}(\chi)\left(\cosh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{-h'h}_{+1+1}\right. \\
& \left. - h' \sinh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{-h'h}_{-1+1} + h \cosh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{-h'h}_{+1-1} - h'h \sinh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{-h'h}_{-1-1}\right) \\
& + a^{h'}_{+1r'}(\chi')a^{-h}_{+1r}(\chi)\left(\cosh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{h'-h}_{+1+1} + h' \sinh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{h'-h}_{-1+1} - h \cosh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{h'-h}_{+1-1}\right. \\
& \left. - h'h \sinh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{h'-h}_{-1-1}\right) + a^{-h'}_{+1r'}(\chi')a^{-h}_{+1r}(\chi)\left(\cosh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{-h'-h}_{+1+1} - h' \sinh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{-h'-h}_{-1+1}\right. \\
& \left. - h \cosh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{-h'-h}_{+1-1} + h'h \sinh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{-h'-h}_{-1-1}\right)\Bigg),
\end{aligned} \tag{53}
$$

with $M^{h'h}_{+1+1} = -4ie^2\left(\check{c}(1, \tau_{+1h}, \tau_{+1h'})\cdot \mathbb{V} - \c{c}(1, \tau_{+1h}, \tau_{+1h'})\cdot \mathbb{A}\right)$, $M^{h'h}_{-1-1} = -4ie^2\left(\check{c}(2, \tau_{-1h}, \tau_{-1h'})\cdot \mathbb{V} - \c{c}(2, \tau_{-1h}, \tau_{-1h'})\cdot \mathbb{A}\right)$ and $M^{h'h}_{-1+1} = M^{h'h}_{+1-1} = -4ie^2\left(\c{c}(1, \tau_{+1h}, \tau_{+1h'})\cdot \mathbb{V} - \check{c}(1, \tau_{+1h}, \tau_{+1h'})\cdot \mathbb{A}\right)$.

6.2. The massless-spinor method

Using equations (34) and (36), the transition amplitude is

$$
M(r', p', s', r, p, s) = -ie^2\Bigg((\mathbb{V}_\mu - \mathbb{A}_\mu)\bar{W}^{r'}_{+1}\gamma^\mu W^{r}_{+1} + (\mathbb{V}_\mu + \mathbb{A}_\mu)\bar{W}^{r'}_{-1}\gamma^\mu W^{r}_{-1}\Bigg). \tag{54}
$$

As stated above, it is easy to obtain expressions for $\bar{W}_{\lambda}^{r'}\gamma^{\mu}W_{\lambda}^{r}$ with an explicit representation of the spinors. For example, if one uses the standard representation (Ref. 31, pag. 30), the first element looks like

$$\bar{W}_{\lambda}^{r'}\gamma^{0}W_{\lambda}^{r} = \frac{1}{2}\sqrt{\frac{\mathrm{E}+\mathrm{m}}{2\mathrm{m}}}\sqrt{\frac{\mathrm{E}'+\mathrm{m}}{2\mathrm{m}}}\left(a^{r}a^{r'*}\left(1+\frac{\lambda\mathrm{p}_z}{\mathrm{E}+\mathrm{m}}+\frac{\lambda\mathrm{p}'_z}{\mathrm{E}'+\mathrm{m}}+\frac{\mathrm{p}'_z\mathrm{p}_z+(\mathrm{p}'_x-i\mathrm{p}'_y)(\mathrm{p}_x+i\mathrm{p}_y)}{(\mathrm{E}+\mathrm{m})(\mathrm{E}'+\mathrm{m})}\right)\right.$$
$$+b^{r}b^{r'*}\left(1-\frac{\lambda\mathrm{p}_z}{\mathrm{E}+\mathrm{m}}-\frac{\lambda\mathrm{p}'_z}{\mathrm{E}'+\mathrm{m}}+\frac{\mathrm{p}'_z\mathrm{p}_z+(\mathrm{p}'_x+i\mathrm{p}'_y)(\mathrm{p}_x-i\mathrm{p}_y)}{(\mathrm{E}+\mathrm{m})(\mathrm{E}'+\mathrm{m})}\right)+\lambda b^{r}a^{r'*}\left(\frac{\mathrm{p}_x-i\mathrm{p}_y}{\mathrm{E}+\mathrm{m}}+\frac{\mathrm{p}'_x-i\mathrm{p}'_y}{\mathrm{E}'+\mathrm{m}}\right. \quad (55)$$
$$\left.\left.+\lambda\frac{\mathrm{p}'_z(\mathrm{p}_x-i\mathrm{p}_y)-\mathrm{p}_z(\mathrm{p}'_x-i\mathrm{p}'_y)}{(\mathrm{E}+\mathrm{m})(\mathrm{E}'+\mathrm{m})}\right)+\lambda a^{r}b^{r'*}\left(\frac{\mathrm{p}_x+i\mathrm{p}_y}{\mathrm{E}+\mathrm{m}}+\frac{\mathrm{p}'_x+i\mathrm{p}'_y}{\mathrm{E}'+\mathrm{m}}+\lambda\frac{\mathrm{p}_z(\mathrm{p}'_x+i\mathrm{p}'_y)-\mathrm{p}'_z(\mathrm{p}_x+i\mathrm{p}_y)}{(\mathrm{E}+\mathrm{m})(\mathrm{E}'+\mathrm{m})}\right)\right),$$

where $a^1 = e^{-i(\varrho/2)}A$, $a^2 = -e^{-i(\varrho/2)}B^*$, $b^1 = e^{i(\varrho/2)}B$, $b^2 = e^{i(\varrho/2)}A^*$ and (Ref. [31], pag. 547)

$$A = \sqrt{\frac{1-\gamma\beta_3\mathrm{s}^0+\frac{(\gamma-1)\beta_1\beta_3}{\beta^2}\mathrm{s}_1+\frac{(\gamma-1)\beta_2\beta_3}{\beta^2}\mathrm{s}_2+\left(1+\frac{(\gamma-1)(\beta_3)^2}{\beta^2}\right)\mathrm{s}_3}{2}}$$
$$B = \sqrt{\frac{1+\gamma\beta_3\mathrm{s}^0-\frac{(\gamma-1)\beta_1\beta_3}{\beta^2}\mathrm{s}_1-\frac{(\gamma-1)\beta_2\beta_3}{\beta^2}\mathrm{s}_2-\left(1+\frac{(\gamma-1)(\beta_3)^2}{\beta^2}\right)\mathrm{s}_3}{2}}, \quad (56)$$
$$\tan\varrho = \frac{-\gamma\beta_2\mathrm{s}^0+\frac{(\gamma-1)\beta_1\beta_2}{\beta^2}\mathrm{s}_1+\frac{(\gamma-1)\beta_2\beta_3}{\beta^2}\mathrm{s}_3+\left(1+\frac{(\gamma-1)\beta_2^2}{\beta^2}\right)\mathrm{s}_2}{-\gamma\beta_1\mathrm{s}^0+\frac{(\gamma-1)\beta_1\beta_2}{\beta^2}\mathrm{s}_2+\frac{(\gamma-1)\beta_1\beta_3}{\beta^2}\mathrm{s}_3+\left(1+\frac{(\gamma-1)\beta_1^2}{\beta^2}\right)\mathrm{s}_1}, \quad (57)$$

where $\bar{\beta} = (\beta_1,\beta_2,\beta_3) = (\bar{\mathrm{p}}/\mathrm{E})$, $\gamma = (1/\sqrt{1-\bar{\beta}^2})$, $\mathrm{s}^0 = (\bar{\beta}\cdot\hat{\mathrm{s}}/\sqrt{1-(\bar{\beta}\cdot\hat{\mathrm{s}})^2})$ and $\bar{\mathrm{s}} = (\mathrm{s}_1,\mathrm{s}_2,\mathrm{s}_3) = (\hat{\mathrm{s}}/\sqrt{1-(\bar{\beta}\cdot\hat{\mathrm{s}})^2})$.

6.3. The projective method: The rest basis

Because this formula is too long, it can be found in Appendix D.

6.4. The projective method: Using the Lorentz-invariant property of the amplitude

The amplitude (48) is a Lorentz scalar and it allows us to compute its analytical value in the reference frame where the electron is initially at rest. The result is general because it is possible to rewrite all the variables in another frame using a Lorentz transformation. With the aid of equation (21) and the covariance of amplitude (48), we get

$$M(p',s',p,s) = -ie^2\left(a\ \mathrm{tr}\ \phi\phi'^{\dagger}+i\ \mathrm{tr}\ \phi\phi'^{\dagger}\bar{\sigma}\cdot\bar{b}\right). \quad (58)$$

The (two-component) spinors ϕ', ϕ have their spin quantization directions defined by the final $\hat{\mathrm{s}}'$ and initial $\hat{\mathrm{s}}$ spin three-dimensional vectors in their respective rest frames. The notation for a and $\bar{b}$ is

$$a = \frac{\mathrm{m}}{(\mathrm{m}^2-u)(s-\mathrm{m}^2)\sqrt{4\mathrm{m}^2-t}}$$
$$\times\left(-(s-u)^2\hat{\epsilon}_f^*\cdot\hat{\epsilon}_i+2\,t\,\hat{\epsilon}_f^*\cdot\bar{\mathrm{k}}_i\hat{\epsilon}_i\cdot\bar{\mathrm{k}}_f\right), \quad (59)$$

$$\bar{b} = \frac{1}{\sqrt{4\mathrm{m}^2-t}}\left((\bar{\mathrm{k}}_f-\bar{\mathrm{k}}_i)\times\left(-\hat{\epsilon}_f^*\cdot\hat{\epsilon}_i\,\hat{\mathrm{k}}_i+\hat{\epsilon}_f^*\cdot\hat{\mathrm{k}}_i\,\hat{\epsilon}_i\right.\right.$$
$$\left.-\hat{\epsilon}_f^*\cdot\hat{\epsilon}_i\,\hat{\mathrm{k}}_f+\hat{\epsilon}_f^*\,\hat{\epsilon}_i\cdot\hat{\mathrm{k}}_f\right)+(\hat{\mathrm{k}}_i\cdot\hat{\epsilon}_f^*\times\hat{\epsilon}_i-\hat{\mathrm{k}}_f\cdot\hat{\epsilon}_f^*\times\hat{\epsilon}_i)$$
$$\left.\times(\bar{\mathrm{k}}_f-\bar{\mathrm{k}}_i)\right), \quad (60)$$

where we have used the radiation gauge $\epsilon = (0,\hat{\epsilon})$. Since the spin dependencies $\phi\phi'^{\dagger}$ are clearly separated from any other kind of term, expression (58) can be useful when trying to analyze the spin properties of the amplitude.

7. Conclusions

The notion of covariant-spinorial operator helps to understand the importance of massive spinors' closure property. An important consequence of this is shown by Eq. (6). Although we do not take advantage of this equation, Eq. (6) clarifies the arguments that are employed in the text. It shows the origin of the covariance and generality of our results.

Our procedure, the projective method, allows computing transition amplitudes using a decomposition of a (massive or massless) spinor as a linear combination of others (massive or massless) spinors which, in principle, are not related to the problem. However, to avoid unnecessary phases, it is helpful to use a spinorial basis related to the problem.

We have shown that most common spinor techniques, as the helicity- and massless-spinor methods, can be formulated using the idea of a complete spinorial basis. This framework allows us to exploit the symmetries and properties of the spinors, without the need for a specific representation for them or for the gamma matrices, allowing a wider understanding of the formalism. The framework thus proposed is free of unphysical singularities, therefore it could be fruitful to implement it within a computational code.

For instance, the spinorial basis at rest allows for obtaining closed formulas, which are fully covariant, for transition amplitudes. Thus, without requiring a particular four-component spinor representation and with no need to fix arbitrary phases. The use of this particular basis is a great example of how spinor symmetries help on the obtaining of useful formulas; at the same time, it permits to get analytical values of any amplitude using, simultaneously, two or more different bases. This is relevant, particularly if one has an interest in evaluating the efficiency, or shortcomings, of different schemes, like those related to singularity issues in the amplitude.

As an example of our procedures, formulas for the Compton amplitude, for helicity states and general spin states, were obtained. It can be noted that Eq. (51) has not been computed for a particular frame as our procedure does not depend on such an election. Also, the structure of Eq. (51) shows separately the kinematic, spin direction and dynamical factors, which are useful in the analysis of any process. The covariance of the framework is exploited with the formula (58), a compact expression for general spin configurations. It could be useful studying polarization effects, such as the electron spin asymmetry using polarized photons, or the photon helicity asymmetry using unpolarized electrons.

Appendix

A.

The coefficients $c^k_{\epsilon\tau\tau'}$ look explicitly as

$$\begin{aligned}
c^0_{111} = c^0_{222} = c^{0*}_{122} = c^{0*}_{211} &= \frac{1}{2}\left(\cos\frac{\vartheta}{2}e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta'}{2}e^{i\frac{\varphi'}{2}} + \sin\frac{\vartheta}{2}e^{i\frac{\varphi}{2}}\sin\frac{\vartheta'}{2}e^{-i\frac{\varphi'}{2}}\right),\\
c^1_{111} = c^1_{222} = -c^{1*}_{122} = -c^{1*}_{211} &= \frac{1}{2}\left(\cos\frac{\vartheta}{2}e^{-i\frac{\varphi}{2}}\sin\frac{\vartheta'}{2}e^{-i\frac{\varphi'}{2}} + \sin\frac{\vartheta}{2}e^{i\frac{\varphi}{2}}\cos\frac{\vartheta'}{2}e^{i\frac{\varphi'}{2}}\right),\\
c^2_{111} = c^2_{222} = -c^{2*}_{122} = -c^{2*}_{211} &= \frac{i}{2}\left(\cos\frac{\vartheta}{2}e^{-i\frac{\varphi}{2}}\sin\frac{\vartheta'}{2}e^{-i\frac{\varphi'}{2}} - \sin\frac{\vartheta}{2}e^{i\frac{\varphi}{2}}\cos\frac{\vartheta'}{2}e^{i\frac{\varphi'}{2}}\right),\\
c^3_{111} = c^3_{222} = -c^{3*}_{122} = -c^{3*}_{211} &= \frac{1}{2}\left(\cos\frac{\vartheta}{2}e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta'}{2}e^{i\frac{\varphi'}{2}} - \sin\frac{\vartheta}{2}e^{i\frac{\varphi}{2}}\sin\frac{\vartheta'}{2}e^{-i\frac{\varphi'}{2}}\right),\\
c^0_{121} = c^0_{212} = -c^{0*}_{112} = -c^{0*}_{221} &= \frac{1}{2}\left(\cos\frac{\vartheta}{2}e^{i\frac{\varphi}{2}}\sin\frac{\vartheta'}{2}e^{-i\frac{\varphi'}{2}} - \sin\frac{\vartheta}{2}e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta'}{2}e^{i\frac{\varphi'}{2}}\right),\\
c^1_{121} = c^1_{212} = c^{1*}_{112} = c^{1*}_{221} &= \frac{1}{2}\left(\cos\frac{\vartheta}{2}e^{i\frac{\varphi}{2}}\cos\frac{\vartheta'}{2}e^{i\frac{\varphi'}{2}} - \sin\frac{\vartheta}{2}e^{-i\frac{\varphi}{2}}\sin\frac{\vartheta'}{2}e^{-i\frac{\varphi'}{2}}\right),\\
c^2_{121} = c^2_{212} = c^{2*}_{112} = c^{2*}_{221} &= -\frac{i}{2}\left(\cos\frac{\vartheta}{2}e^{i\frac{\varphi}{2}}\cos\frac{\vartheta'}{2}e^{i\frac{\varphi'}{2}} + \sin\frac{\vartheta}{2}e^{-i\frac{\varphi}{2}}\sin\frac{\vartheta'}{2}e^{-i\frac{\varphi'}{2}}\right),\\
c^3_{121} = c^3_{212} = c^{3*}_{112} = c^{3*}_{221} &= -\frac{1}{2}\left(\cos\frac{\vartheta}{2}e^{i\frac{\varphi}{2}}\sin\frac{\vartheta'}{2}e^{-i\frac{\varphi'}{2}} + \sin\frac{\vartheta}{2}e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta'}{2}e^{i\frac{\varphi'}{2}}\right).
\end{aligned} \tag{A.1}$$

B.

The explicit view of the formula for the transition amplitude between particle states and the interaction term $\not{v}$ is

$$\begin{aligned}
M(\epsilon,p',\tau',s',\epsilon,p,\tau,s) = \frac{1}{4\mathrm{m}\mathrm{m}'}\sqrt{\frac{2\mathrm{m}'}{\mathrm{E}'+\mathrm{m}'}}\sqrt{\frac{2\mathrm{m}}{\mathrm{E}+\mathrm{m}}}\Big(&-(-1)^{\epsilon}\check{c}\cdot q(-(-1)^{\tau'}i[s'p'vp] - (-1)^{\tau'+\epsilon+\tau}\mathrm{m}\langle p's'vs\rangle\\
&-(-1)^{\tau}i[p'vsp] - (-1)^{\epsilon}\mathrm{m}\,p'\cdot v + -(-1)^{\epsilon}\mathrm{m}'p\ \cdot\ v + (-1)^{\tau'+\tau+\epsilon}\mathrm{m}'\langle s'vps\rangle)\\
&-(-1)^{\tau'+\tau}\langle\langle\check{c}p's'vps\rangle\rangle + (-1)^{\tau'+\epsilon}i\mathrm{m}[\check{c}p's'v] + \langle\check{c}p'vp\rangle - (-1)^{\epsilon+\tau}i\mathrm{m}[\check{c}p'vs]\\
&-(-1)^{\tau'+\epsilon}i\mathrm{m}'[\check{c}s'vp] + (-1)^{\tau'+\tau}\mathrm{m}'\mathrm{m}\langle\check{c}s'vs\rangle - (-1)^{\tau+\epsilon}i\mathrm{m}'[\check{c}vps] + \mathrm{m}'\mathrm{m}\,\check{c}\cdot v\\
&-(-1)^{\tau'+\tau}i[[\c{c}p's'vps]] + i[\c{c}p'vp] + (-1)^{\tau'+\epsilon}\mathrm{m}\langle\c{c}p's'v\rangle - (-1)^{\epsilon+\tau}\mathrm{m}\langle\c{c}p'vs\rangle
\end{aligned}$$

$$
\begin{aligned}
&- (-1)^{\tau'+\epsilon}\mathsf{m}'\langle \text{\d{ç}} s'vp\rangle + (-1)^{\tau'+\tau} i\mathsf{m}'\mathsf{m}[\text{\d{ç}} s'vs] - (-1)^{\tau+\epsilon}\mathsf{m}'\langle \text{\d{ç}} vps\rangle + (-1)^{\epsilon}((-1)^{\tau'}\langle\langle q\text{\d{ç}} p's'vp\rangle\rangle \\
&- (-1)^{\epsilon+\tau'+\tau} i\mathsf{m}[[q\text{\d{ç}} p's'vs]] + (-1)^{\tau}\langle\langle q\text{\d{ç}} p'vps\rangle\rangle + (-1)^{\tau'+\epsilon+\tau} i\mathsf{m}'[[\text{\d{ç}} s'vps]] \\
&- (-1)^{\epsilon} i\mathsf{m}[q\text{\d{ç}} p'v] - (-1)^{\tau'}\mathsf{m}'\mathsf{m}\langle q\text{\d{ç}} s'v\rangle - (-1)^{\epsilon} i\mathsf{m}'[q\text{\d{ç}} vp] + (-1)^{\tau}\mathsf{m}'\mathsf{m}\langle q\text{\d{ç}} vs\rangle)\Big),
\end{aligned} \tag{B.1}
$$

with the notation $[abcd] = \varepsilon_{\mu_1\mu_2\mu_3\mu_4}a^{\mu_1}b^{\mu_2}c^{\mu_3}d^{\mu_4}$, $[[abcdef]] = -a{\cdot}b[cdef]+a{\cdot}c[bdef]-b{\cdot}c[adef]+d{\cdot}e[abcf]-d{\cdot}f[abce]+e{\cdot}f[abcd]$, $\langle abcd\rangle = a{\cdot}b\; c{\cdot}d - a{\cdot}c\; b{\cdot}d + a{\cdot}d\; b{\cdot}c$, $\langle\langle abcdef\rangle\rangle = a{\cdot}b\langle cdef\rangle - a{\cdot}c\langle bdef\rangle + a{\cdot}d\langle bcef\rangle - a{\cdot}e\langle bcdf\rangle + a{\cdot}f\langle bcde\rangle$.

C.

The formula for a general spin amplitude using our approach to the helicity formalism is

$$
\begin{aligned}
M(r',p',s',r,p,s) = \kappa_{r'}\kappa_r\Bigg(& a^{h'}_{\kappa_{r'}r'}(\chi')a^{h}_{\kappa_r r}(\chi)\Bigg(\cosh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{h'h}_{\kappa_{r'}\kappa_r} + h'\sinh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{h'h}_{-\kappa_{r'}\kappa_r} \\
&+ h\cosh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{h'h}_{\kappa_{r'}-\kappa_r} + h'h\sinh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{h'h}_{-\kappa_{r'}-\kappa_r}\Bigg) + a^{-h'}_{\kappa_{r'}r'}(\chi')a^{h}_{\kappa_r r}(\chi) \\
&\times\Bigg(\cosh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{-h'h}_{\kappa_{r'}\kappa_r} - h'\sinh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{-h'h}_{-\kappa_{r'}\kappa_r} + h\cosh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{-h'h}_{\kappa_{r'}-\kappa_r} \\
&- h'h\sinh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{-h'h}_{-\kappa_{r'}-\kappa_r}\Bigg) + a^{h'}_{\kappa_{r'}r'}(\chi')a^{-h}_{\kappa_r r}(\chi)\Bigg(\cosh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{h'-h}_{\kappa_{r'}\kappa_r} \\
&+ h'\sinh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{h'-h}_{-\kappa_{r'}\kappa_r} - h\cosh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{h'-h}_{\kappa_{r'}-\kappa_r} - h'h\sinh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{h'-h}_{-\kappa_{r'}-\kappa_r}\Bigg) \\
&+ a^{-h'}_{\kappa_{r'}r'}(\chi')a^{-h}_{\kappa_r r}(\chi)\Bigg(\cosh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{-h'-h}_{\kappa_{r'}\kappa_r} - h'\sinh\frac{\chi'}{2}\cosh\frac{\chi}{2}M^{-h'-h}_{-\kappa_{r'}\kappa_r} \\
&- h\cosh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{-h'-h}_{\kappa_{r'}-\kappa_r} + h'h\sinh\frac{\chi'}{2}\sinh\frac{\chi}{2}M^{-h'-h}_{-\kappa_{r'}-\kappa_r}\Bigg)\Bigg),
\end{aligned} \tag{C.1}
$$

where the notation $a^{h}_{\kappa_r r}(\chi) = \cosh(\chi/2)\,K^{h}_{\kappa_r r} + h\sinh(\chi/2)\,K^{h}_{-\kappa_r r}$ has been used.

D.

With the aid of expressions (20) and (B.1), the amplitude is written as

$$
\begin{aligned}
M(1,p',\tau',s',1,p,\tau,s) = \frac{-ie^2}{4\mathsf{m}\mathsf{m}'}\sqrt{\frac{2\mathsf{m}'}{\mathrm{E}'+\mathsf{m}'}}\sqrt{\frac{2\mathsf{m}}{\mathrm{E}+\mathsf{m}}}\Bigg(& \check{c}\cdot q(-(-1)^{\tau'} i[s'p'\mathbb{V}p] + (-1)^{\tau'+\tau}\mathsf{m}\langle p's'\mathbb{V}s\rangle - (-1)^{\tau} i[p'\mathbb{V}sp] \\
&+ \mathsf{m}\,p'\cdot\mathbb{V} + \mathsf{m}'p{\cdot}\mathbb{V} - (-1)^{\tau'+\tau}\mathsf{m}'\langle s'\mathbb{V}ps\rangle) - (-1)^{\tau'+\tau}\langle\langle \check{c}p's'\mathbb{V}ps\rangle\rangle - (-1)^{\tau'} i\mathsf{m}[\check{c}p's'\mathbb{V}] \\
&+ \langle \check{c}p'\mathbb{V}p\rangle + (-1)^{\tau} i\mathsf{m}[\check{c}p'\mathbb{V}s] + (-1)^{\tau'} i\mathsf{m}'[\check{c}s'\mathbb{V}p] + (-1)^{\tau'+\tau}\mathsf{m}'\mathsf{m}\langle \check{c}s'\mathbb{V}s\rangle + (-1)^{\tau} i\mathsf{m}'[\check{c}\mathbb{V}ps] \\
&+ \mathsf{m}'\mathsf{m}\,\check{c}{\cdot}\mathbb{V} - (-1)^{\tau'+\tau} i[[\text{\d{ç}} p's'\mathbb{V}ps]] + i[\text{\d{ç}} p'\mathbb{V}p] - (-1)^{\tau'}\mathsf{m}\langle \text{\d{ç}} p's'\mathbb{V}\rangle + (-1)^{\tau}\mathsf{m}\langle \text{\d{ç}} p'\mathbb{V}s\rangle \\
&+ (-1)^{\tau'}\mathsf{m}'\langle \text{\d{ç}} s'\mathbb{V}p\rangle + (-1)^{\tau'+\tau} i\mathsf{m}'\mathsf{m}[\text{\d{ç}} s'\mathbb{V}s] + (-1)^{\tau}\mathsf{m}'\langle \text{\d{ç}}\mathbb{V}ps\rangle - ((-1)^{\tau'}\langle\langle q\text{\d{ç}} p's'\mathbb{V}p\rangle\rangle \\
&+ (-1)^{\tau'+\tau} i\mathsf{m}[[q\text{\d{ç}} p's'\mathbb{V}s]] + (-1)^{\tau}\langle\langle q\text{\d{ç}} p'\mathbb{V}ps\rangle\rangle - (-1)^{\tau'+\tau} i\mathsf{m}'[[\text{\d{ç}} s'\mathbb{V}ps]] + i\mathsf{m}[q\text{\d{ç}} p'\mathbb{V}] \\
&- (-1)^{\tau'}\mathsf{m}'\mathsf{m}\langle q\text{\d{ç}} s'\mathbb{V}\rangle + i\mathsf{m}'[q\text{\d{ç}}\mathbb{V}p] + (-1)^{\tau}\mathsf{m}'\mathsf{m}\langle q\text{\d{ç}}\mathbb{V}s\rangle) + (-1)^{\tau}\mathsf{m}\langle \check{c}p'\mathbb{A}s\rangle + \check{c}\cdot q(-(-1)^{\tau'}\langle s'p'\mathbb{A}p\rangle \\
&+ (-1)^{\tau'+\tau} i\mathsf{m}[p's'\mathbb{A}s] + (-1)^{\tau}\langle p'\mathbb{A}sp\rangle + (-1)^{\tau'}\mathsf{m}'\mathsf{m}\,s'\cdot\mathbb{A} + (-1)^{\tau}\mathsf{m}'\mathsf{m}s\cdot\mathbb{A} - (-1)^{\tau'+\tau} i\mathsf{m}'[s'\mathbb{A}ps]) \\
&+ (-1)^{\tau'+\tau} i[[\check{c}p's'\mathbb{A}ps]] + (-1)^{\tau'}\mathsf{m}\langle \check{c}p's'\mathbb{A}\rangle + i[\check{c}p'\mathbb{A}p] + (-1)^{\tau'}\mathsf{m}'\langle \check{c}s'\mathbb{A}p\rangle + (-1)^{\tau'+\tau} i\mathsf{m}'\mathsf{m}[\check{c}s'\mathbb{A}s]
\end{aligned}
$$

$$- (-1)^{\tau} \mathsf{m}' \langle \check{c} \mathbb{A} p s \rangle - \mathsf{m}' \mathsf{m}\, \c{c} \cdot \mathbb{A} + (-1)^{\tau' + \tau} \langle \langle \c{c} p' s' \mathbb{A} p s \rangle \rangle + \langle \c{c} p' \mathbb{A} p \rangle + (-1)^{\tau'} i \mathsf{m} [\c{c} p' s' \mathbb{A}]$$
$$+ (-1)^{\tau} i \mathsf{m} [\c{c} p' \mathbb{A} s] + (-1)^{\tau'} i \mathsf{m}' [\c{c} s' \mathbb{A} p] + (-1)^{\tau' + \tau} \mathsf{m}' \mathsf{m} \langle \c{c} s' \mathbb{A} s \rangle - (-1)^{\tau} i \mathsf{m}' [\c{c} \mathbb{A} p s]$$
$$- ((-1)^{\tau'} i [[q \c{c} p' s' \mathbb{A} p]] + (-1)^{\tau' + \tau} \mathsf{m} \langle \langle q \c{c} p' s' \mathbb{A} s \rangle \rangle - (-1)^{\tau} i [[q \c{c} p' \mathbb{A} p s]] + (-1)^{\tau' + \tau} \mathsf{m}' \langle \langle q \c{c} s' \mathbb{A} p s \rangle \rangle$$
$$- \mathsf{m} \langle q \c{c} p' \mathbb{A} \rangle + (-1)^{\tau'} i \mathsf{m}' \mathsf{m} [q \c{c} s' \mathbb{A}] + \mathsf{m}' \langle q \c{c} \mathbb{A} p \rangle + (-1)^{\tau} i \mathsf{m}' \mathsf{m} [q \c{c} \mathbb{A} s]) \Big). \tag{D.1}$$

Using equations (56) and (57), the four-vectors $\check{c}^{\mu}$ and $\c{c}^{\mu}$ are built with the correspondence

$$A \to \cos \frac{\vartheta}{2},\ B \to \sin \frac{\vartheta}{2}, \varrho \to \varphi. \tag{D.2}$$

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