The Frenet-Serret description of Born rigidity and its application to the Dirac equation

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The role played by non-inertial frames in physics is one of the most interesting subjects that we can study when dealing with a physical theory. This is especially true for special relativity and the Dirac theory In the case of special relativity, a problem with the concept of rigidity emerged as soon as Max Born gave a reasonable definition of rigid motion: the Herglotz-Noether theorem imposes a strong restriction on the possible rigid motions. In this paper, the equivalence of this theorem with another one that is formulated with the help of Frenet-Serret formalism is proved, showing the connection between the rigid motion and the curvatures of the observer's trajectory in spacetime. Besides, the Dirac equation in the Frenet-Serret frame for an arbitrary observer is obtained and applied to the rotating observers. The solution in the rotating frame is given in terms of that of an inertial one.

Keywords: Born rigidity; accelerated frames; Dirac equation; rotating observers; Frenet-Serret formalism.

1. Introduction

Reference frames that are constructed out of the worldline of a particular observer are of great importance in both general and special relativity (SR). They play the role of the observer's rest frame and can sometimes even simplify the field equations. In general, these frames are taken to be orthonormal, and one of their vectors is chosen to coincide with the observer's 4-velocity. A well-known procedure to obtain a frame that rotates only in the plane formed by the 4-velocity and the 4-acceleration is the so-called Fermi-Walker transport [1]. In some sense, this frame is the closest we can get to an inertial frame that follows the observer. Nonetheless, there exists another frame that also possesses interesting features, namely, the Frenet-Serret frame[i] [3]. This frame not only follows the observer but also gives important information about the observer's worldline. Because of these properties, it has been used to understand many different types of systems [4–11]. For example, in Refs. [6, 9, 10], Bini *et al* use the Frenet-Serret tetrads to analyze circular orbits in black hole spacetimes and to study centripetal acceleration and centrifugal forces in general relativity. Another interesting example is Ref. [11], where Iyer and Vishveshwara study the Frenet-Serret description of gyroscopic precession. Because of the importance of the Frenet-Serret formalism, we will use it in this paper to deal with the rigid motion problem.

Rigidity is an idea that is present in practically all physical theories: we use rigid rods as a standard length, the laboratory frame is always thought of as a rigid frame, etcetera. The situation in SR is not different. However, unlike what happens in Newtonian mechanics, the concept of rigidity in SR is not that simple. The notion of a rigid body is inconsistent with the fact that no signal can propagate with a speed greater than that of light. To avoid this problem, one can work with the concept of rigid motion (RM). The ordinary concept of RM is the idea that two "points" move in such a way that their distance is kept constant by means of a choreographed motion. But even in this case, relativity is problematic. First, we have the problem with the measurement of time. The fact that in an RM the distance is kept constant does not mean that the velocity is the same. Think, for example, of two observers that describe a circular motion with the same angular velocity. If their distance to the center of the motion is not the same, their velocity will be different. In this case, we cannot be sure that they will agree with time measurements. The second problem lies in the fact that if A is at rest with respect to B, and C is at rest with respect to A, in general, we cannot guarantee that C is at rest with respect to B. Nevertheles, a reasonable definition of RM was given by Born [12] and will be considered here.

Although reasonable, Born rigidity imposes a strong restriction on the possible motions. This restriction is described by the Herglotz-Noether theorem (HNT) [13, 14]. Although we can state this theorem in terms of the 4-acceleration and rotation tensors [15], it is possible to have a better intuition of its geometrical properties by using the curvatures of the observer's worldline, as is done in Ref. [16]. In this reference, a similar theorem based on the Serret-Frenet formalism was proved. The main purpose of this paper is to prove the equivalence of these two theorems (Sec. 3.1) and apply the Frenet-Serret formalism to the Dirac equation. In doing so, we write the Frenet-Serret tetrad (FST) in the local coordinate system of a general observer (Sec. 2.2), obtain the Dirac equation in this frame (Sec. 4) and solve it for the case of rotating observers (Sec. 4.1). Finally, we present a summary of the results in Sec. 5.

We use the following notations and conventions. A Cartesian coordinate system that is homogeneous and isotropic and in which Newton's laws are valid are represented by $x^{\mu} = (t, x, y, z)$, where $\mu = 0, 1, 2, 3$. A tetrad field will

be denoted by e_a, whose components in the Cartesian basis $\partial_\mu = \partial/\partial x^\mu$ are e_a^μ, where $a = (0), (1), (2), (3)$. When convenient, the coordinate basis ∂_μ will be represented as a coordinate tetrad in the form $\mathring{e}_a = \delta_a^{(\mu)} \partial_\mu$ or $\mathring{e}_{(\mu)} = \partial_\mu$. Latin letters in the middle of the alphabet stand for spatial indices only, *i.e.*, $i, j, k, \ldots = (1), (2), (3)$. The dual basis of ∂_μ is denoted by dx^μ. The metric tensor η takes the form of a diagonal matrix when written in the basis ∂_μ. To be more precise, the elements of η are $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$. The metric components in a general tetrad basis also have the same values, *i.e.*, $\eta_{(0)(0)} = 1$ and $\eta_{(1)(1)} = \eta_{(2)(2)} = \eta_{(3)(3)} = -1$. The metric $\eta_{\mu\nu}$ is used to raise and lower coordinate indices, which are denoted by Greek letters, while η_{ab} is used to raise and lower tangent space indices. In this paper, we use $c = 1$ and $\hbar = 1$.

2. Accelerated Frames

2.1. Fermi-Walker Transport

The main idea behind the Fermi-Walker transport is to eliminate as many rotations as possible while the transported frame follows an observer's worldline. In general, this frame will not be inertial and will possess a rotation in the plane formed by the 4-velocity **u** and the 4-acceleration **a**. To understand this idea, think of the non-relativistic case. Given an arbitrary accelerated observer, which rest frame would approach an inertial frame the most? Of course, it would be a frame that does not rotate with respect to an inertial frame at all, but whose origin is accelerated in the same way as the observer. In relativity, the situation is more involved because the variation of **u** forces the frame to rotate in the mentioned plane. Therefore, it is natural to demand no rotation other than that.

A general rotation can be expressed as (see, *e.g.*, p. 174 of Ref. [1])

$$\frac{de_a^\mu}{d\tau} = \Phi^{\mu\nu} e_{a\nu} \tag{1}$$

with

$$\Phi^{\mu\nu} = a^\mu u^\nu - a^\nu u^\mu + \Omega^{\mu\nu},$$
$$\Omega^{\mu\nu} = u_\alpha \Omega_\beta \varepsilon^{\alpha\beta\mu\nu}, \tag{2}$$

where Ω_β are the components of what we might call the 4-angular velocity vector, τ is the proper time, and $\varepsilon_{\alpha\beta\mu\nu}$ is the Levi-Civita tensor ($\varepsilon_{0123} = +1$). The Fermi-Walker transport corresponds to the case $\Omega_\beta = 0$ and $e_{(0)}^\mu = u^\mu$.

2.2. Frenet-Serret Tetrad

Given the observer's worldline $x^\mu(\tau)$, where τ is its proper time, we can construct a tetrad basis attached to the observer by using the formulas

$$e_{(0)}^\mu = \frac{dx^\mu}{d\tau}, \quad \frac{de_a^\mu}{d\tau} = \Sigma_a^b e_b^\mu, \tag{3}$$

$$\Sigma_a^b = k_1 \delta_a^{(0)} \delta_{(1)}^b + \delta_a^{(1)} \left(k_1 \delta_{(0)}^b + k_2 \delta_{(2)}^b \right)$$
$$+ \delta_a^{(2)} \left(k_3 \delta_{(3)}^b - k_2 \delta_{(1)}^b \right) - k_3 \delta_a^{(3)} \delta_{(2)}^b, \tag{4}$$

where the ks are called curvatures (sometimes k_2 and k_3 are also called the first and second torsions, respectively), and e_a^μ are the components of the FST written in an inertial frame with Cartesian coordinates (t, x, y, z).

To study Born rigidity, we may use the local coordinate system (τ, ξ, χ, ζ), which, is related to $x^\mu = (t, x, y, z)$ by [1, 16, 17]

$$x^\mu(\tau, \xi, \chi, \zeta) = x_n^\mu(\tau) + r^j e_j^\mu(\tau), \tag{5}$$

where $x_n^\mu(\tau)$ is the worldline of an observer n, which is at the origin of the new coordinate system, and $r^j = (\xi, \chi, \zeta)$. The coordinate time τ corresponds to this observer's proper time, while e_j^μ are the Cartesian components of the vector field constructed out of this observer's FST (by means of parallel transport). It should be clear that the inertial observers use the coordinates $x^\mu = (t, x, y, z)$, while the accelerated ones use $x^{\bar\mu} = (\tau, \xi, \chi, \zeta)$ (note the overbar).

Since $x^{\bar\mu} = x^{\bar\mu}(t, x, y, z)$, we can use Eq. (5) to obtain the frame e_a and the coframe θ^a (the dual basis) in the coordinate system $x^{\bar\mu}$. Differentiating Eq. (5) with respect to ξ, we find that

$$\partial_\xi = \frac{\partial t}{\partial \xi} \partial_t + \frac{\partial x}{\partial \xi} \partial_x + \frac{\partial y}{\partial \xi} \partial_y + \frac{\partial z}{\partial \xi} \partial_z$$
$$= e_{(1)}^t \partial_t + e_{(1)}^x \partial_x + e_{(1)}^y \partial_y + e_{(1)}^z \partial_z = e_{(1)}. \tag{6}$$

Doing the same thing for ∂_χ, ∂_ζ, and ∂_τ, we arrive at

$$e_{(0)} = f(\tau, \xi) \Big[\partial_\tau + k_2(\tau) \left(\chi \partial_\xi - \xi \partial_\chi \right)$$
$$+ k_3(\tau) \left(\zeta \partial_\chi - \chi \partial_\zeta \right) \Big], \quad e_{(1)} = \partial_\xi,$$
$$e_{(2)} = \partial_\chi, \quad e_{(3)} = \partial_\zeta, \quad f(\tau, \xi) = 1/(1 + \xi k_1(\tau)), \tag{7}$$

where we have also used Eqs. (3)-(4) to obtain $e_{(0)}$. On the other hand, the dual basis[ii] (coframe) is

$$\theta^{(0)} = d\tau / f(\tau, \xi), \quad \theta^{(1)} = -\chi k_2(\tau) d\tau + d\xi,$$
$$\theta^{(2)} = [\xi k_2(\tau) - \zeta k_3(\tau)] \, d\tau + d\chi,$$
$$\theta^{(3)} = \chi k_3(\tau) d\tau + d\zeta. \tag{8}$$

Since $ds^2 = \eta_{ab} \theta^a \otimes \theta^b$, we see that the metric in the new coordinates becomes

$$ds^2 = \left[(1 + k_1 \xi)^2 - (k_3^2 + k_2^2)\chi^2 - (k_2 \xi - k_3 \zeta)^2 \right] d\tau^2$$
$$+ 2 \Big[k_2 \left(\chi d\xi - \xi d\chi \right) + k_3 \left(\zeta d\chi - \chi d\zeta \right) \Big] d\tau$$
$$- d\xi^2 - d\chi^2 - d\zeta^2, \tag{9}$$

where

$$(1 + k_1\xi)^2 - (k_3^2 + k_2^2)\chi^2 - (k_2\xi - k_3\zeta)^2 > 0. \quad (10)$$

The affine connection coefficients in the tetrad field e_a, denoted by $\omega^a{}_{bc}$, do not vanish. As is well known, these coefficients can be obtained through the relation

$$\omega^a{}_{bc} = e^a{}_{\bar\lambda} e_b{}^{\bar\mu} \left(\partial_{\bar\mu} e_c^{\bar\lambda} + \Gamma^{\bar\lambda}_{\bar\mu\bar\nu} e_c^{\bar\nu} \right), \quad (11)$$

where $e_a^{\bar\lambda}$ and $e^a{}_{\bar\lambda}$ are the components of e_a and θ^a in the coordinate bases $\partial_{\bar\mu} = (\partial_\tau, \partial_\xi, \partial_\chi, \partial_\zeta)$ and $dx^{\bar\mu} = (d\tau, d\xi, d\chi, d\zeta)$, respectively. The object $\Gamma^{\bar\lambda}_{\bar\mu\bar\nu}$ denotes the Christoffel symbols of the metric (9). After a straightforward, but tedious, calculation, we find that

$$\omega_{(0)(0)(1)} = -\omega_{(1)(0)(0)} = f(\tau, \xi)k_1(\tau),$$

$$\omega_{(1)(0)(2)} = -\omega_{(2)(0)(1)} = f(\tau, \xi)k_2(\tau),$$

$$\omega_{(2)(0)(3)} = -\omega_{(3)(0)(2)} = f(\tau, \xi)k_3(\tau). \quad (12)$$

Note from Eqs. (12) and (7) that, along the worldline of the observer n ($\xi = \chi = \zeta = 0$), $\omega^a{}_{bc}$ is given solely by the curvatures of the curve.

One of the advantages of using the FST lies on the following results [3]:

1. The curvatures determine the observer's worldline up to a Poincaré transformation.

2. When k_3 vanishes, the curve lies inside a hyperplane (a three-dimensional volume in Minkowski spacetime). In this case, the 3-velocity is inside a plane.

3. When k_2 vanishes, the curve lies in a plane (3-velocity inside a line).

4. For $k_1 = 0$, the world line is a straight-line (constant 3-velocity).

It is worth noting that the FST will be Fermi-Walker transported only if the observer's worldline has $k_2 = k_3 = 0$. This suggests a relation between these curvatures and the last term in Eq. (2), that is, k_2 and k_3 are somehow related to spatial rotations with respect to the Fermi-Walker transported frame. This relation will be presented in Sec. 3.1

3. Rigid Motion

Despite the fact that Einstein and Born used the expression "rigid body" in their work [12, 19], it is well known now that the concept of a rigid body in SR is problematic[iii]. In fact, even in classical mechanics, this concept is also problematic because there is no medium where a pulse can propagate with a speed greater than the finite value of the speed of sound. But, in classical mechanics, we can at least have a totally satisfactory definition of an RM by assuming that a system of particles describes an RM when the distance between any pair of particles remains constant. This is not to say that the system is a rigid body. In fact, this is just an idealized motion[iv], where the forces acting on the particles are such that the strain always vanishes. In this context, a rigid body would be just an approximation of a body whose motion is very close to an RM regardless of the force acting on it. The problem in SR is that measurements of distance are not invariant, which means that they depend on the reference frame used.

To solve this problem, we can assume that the distance between two infinitesimally separated parts of the body should remain constant in the rest frame of these parts. This is known as the "Born rigidity" [12], which can be formally defined as follows:

Definition 3.1 *Let u^μ be the 4-velocity field associated with the motion of the system, $g_{\mu\nu}$ the Minkowski metric in an arbitrary coordinate system, and $h_{\mu\nu} = u_\mu u_\nu - g_{\mu\nu}$ the restriction of $-g_{\mu\nu}$ to $u^\perp$. The motion is said to be rigid if*

$$L_u h_{\mu\nu} = -(\nabla_\nu u_\mu + \nabla_\mu u_\nu)$$
$$+ u^\alpha u_\nu \nabla_\alpha u_\mu + u^\alpha u_\mu \nabla_\alpha u_\nu = 0, \quad (13)$$

where $L_u h_{\mu\nu}$ is the Lie derivative of $h_{\mu\nu}$, and $\nabla_\nu u_\mu$ stand for the components of the covariant derivative.

This is the same as saying that the strain rate tensor vanishes [15, 21] and that Killing motions are rigid [22].

While the RM in Newtonian spacetime has six degrees of freedom, three translations, and three rotations, which means that we can give any trajectory we want to a particular point of the body, Born rigidity possesses only three degrees and, therefore, does not allow for an arbitrary motion [13, 14]. The possible motions are given by the HNT, which will be stated in the next section.

3.1. Possible motions

The HNT restricts the RMs to the following class [15]:

Theorem 3.1 *The only possible RMs in the sense of Born rigidity are those with $\Omega_{ij} = 0$ or $(d/d\tau)a^i = 0$ and $(d/d\tau)\Omega_{ij} = 0$.*

Using the Frenet-Serret formalism, it was also proved the following theorem [16].

Theorem 3.2 *The only possible RMs in the sense of Born rigidity are those with arbitrary k_1 and $k_2 = k_3 = 0$ or $(d/d\tau)k_1 = (d/d\tau)k_2 = (d/d\tau)k_3 = 0$.*

Of course, these theorems must be equivalent. To prove their equivalency, let us assume that the observer n uses a FST e_a, which is given by Eqs. (3) and (4). In this basis, Eq. (1) can be written as

$$\frac{de_i^\mu}{d\tau} = -k_1 \eta_{(1)i} e_{(0)}^\mu + \Omega_i^j e_j^\mu, \quad (14)$$

where we have used $a^\mu = k_1 e_{(1)}^\mu$ and

$$\Omega^j{}_i = e_{(0)\alpha} e_\mu^j e_{i\nu} \varepsilon^{\alpha\beta\mu\nu} \Omega_\beta. \quad (15)$$

By taking $i = (1), (2)$ in Eq. (14) and using Eqs. (3), and (4), we arrive at

$$k_2 = \Omega^{(2)}_{(1)}, \qquad \Omega^{(3)}_{(1)} = 0, \qquad k_3 = \Omega^{(3)}_{(2)}. \qquad (16)$$

It is clear that Ω_{ij} in the FST has only two nonvanishing components. Moreover, we have $\mathbf{a} = a^\mu \partial_\mu = a^b e_b = k_1 e_{(1)}$, *i.e.*, $a^b = (0, k_1, 0, 0)$. From this we see that if $dk_1/d\tau$ vanishes, then so does $da^i/d\tau$. Hence, the theorem 3.2 is equivalent to 3.1.

To see that k_2 and k_3 are nothing but spatial rotations, we can use the identities (valid for $\det(e^a_\mu) = 1$)

$$e^\beta_{(3)} = e_{(0)\alpha} e_{(1)\mu} e_{(2)\nu} \varepsilon^{\alpha\beta\mu\nu}, \qquad (17)$$

$$e^\beta_{(1)} = e_{(0)\alpha} e_{(2)\mu} e_{(3)\nu} \varepsilon^{\alpha\beta\mu\nu} \qquad (18)$$

in Eq. (15) to obtain the relations $\Omega^{(2)}_{(1)} = \Omega_{(3)}$ and $\Omega^{(3)}_{(2)} = \Omega_{(1)}$. Comparing these relations with Eq. (16), we conclude that $k_2 = \Omega_{(3)}$ and $k_3 = \Omega_{(1)}$. This means that the curvature k_2 is associated with a rotation about the direction defined by $e_{(3)}$ while k_3 is related to a rotation about $e_{(1)}$. For more details about the geometrical meaning of these curvatures, see Refs. [6, 9–11] and references therein.

The advantage of using the theorem 3.2 rather than 3.1 lies on the geometrical intuition that it gives. For instance, if you wish to know whether an observer can be part of a Born RM, you can use the results shown at the end of Sec. 2.2.: If the observer's worldline lies in a plane (in terms of space, its trajectory would be a straight line), then it can be seen as part of an RM (k_2 and k_3 vanish in this case). On the other hand, if its worldline is not inside a plane and its first curvature (k_1) is not constant, then it cannot be part of an RM. In addition, one can use this approach to easily construct RMs, as done in Ref. [16].

As examples of rigid observers we have Rindler observers, (a rigid rod), whose torsions vanish, and the rotating ones ("rigid disk" rotating with a constant angular velocity), whose curvatures are all constant. The worldline of a Rindler observer lies in a plane, while the one of a rotating observer lies in a hyperplane.

4. Dirac equation

Quantum mechanics in noninertial frames has been extensively studied by many authors [23–28]. The most interesting cases studied so far are the Rindler and the rigidly-rotating ones. With respect to the latter case, one can find systems that can behave as rigid disks, such as rapidly-rotating neutron stars [27]. In this section, we use the Frenet-Serret formalism to write the Dirac equation in a arbitrary accelerated frame; then, we apply the result to a rotating motion that generalizes to some extent those that are generally used in the literature.

In a noninertial frame, the Dirac equation can be written as

$$i\gamma^\mu (\partial_\mu + \Gamma_\mu) \Psi - m\Psi = 0, \qquad (19)$$

with

$$\Gamma_\mu = \frac{1}{8} \omega_{a\mu b} \left[\gamma^a, \gamma^b \right], \qquad (20)$$

where $\gamma^\mu = e^\mu_a \gamma^a$, e^μ_a are the components of e_a in the global inertial frame $\mathring{e}_a$, and the γ^as are the gamma matrices. When convenient, we will take the matrices γ^a as the standard Dirac matrices:

$$\gamma^{(0)} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix},$$

$$\sigma_{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (21)$$

From Eqs. (12) and (20), we discover that

$$\Gamma_c = \frac{1}{2} f \left(k_1 \gamma^{(0)} \gamma^{(1)} \right.$$

$$\left. + k_2 \gamma^{(1)} \gamma^{(2)} + k_3 \gamma^{(2)} \gamma^{(3)} \right) \delta^{(0)}_c. \qquad (22)$$

Substitution of Eqs. (22) and (7) into Eq. (19) gives

$$\left\{ \gamma^{(0)} f \left[\partial_\tau + k_2 \left(\chi \partial_\xi - \xi \partial_\chi \right) + k_3 \left(\zeta \partial_\chi - \chi \partial_\zeta \right) \right] \right.$$

$$+ \gamma^{(1)} \partial_\xi + \gamma^{(2)} \partial_\chi + \gamma^{(3)} \partial_\zeta + \frac{1}{2} f \left(k_1 \gamma^{(1)} \right.$$

$$\left. + k_2 \gamma^{(0)} \gamma^{(1)} \gamma^{(2)} + k_3 \gamma^{(0)} \gamma^{(2)} \gamma^{(3)} \right) + im \Bigg\} \Psi = 0, \quad (23)$$

where we have multiplied it by $-i$, and $f = 1/(1 + \xi k_1(\tau))$. Note that this equation holds for any representation of the gamma matrices.

In Sec. 3.1, we saw that the curvatures k_2 and k_3 are related to space rotations of the frame around ∂_ζ and ∂_ξ, respectively. If convenient, one may use $a = \sqrt{-a^\mu a_\mu} = k_1$, $\Omega_{(1)} = k_3$, and $\Omega_{(3)} = k_2$ to recast Eq. (23) as

$$\left\{ \gamma^{(0)} f \left(\partial_\tau + \frac{1}{2} \vec{a} \cdot \vec{\alpha} - i\vec{\Omega} \cdot \vec{J} \right) \right.$$

$$\left. + \gamma^{(1)} \partial_\xi + \gamma^{(2)} \partial_\chi + \gamma^{(3)} \partial_\zeta + im \right\} \Psi = 0, \qquad (24)$$

where $\vec{J} = \vec{L} + \vec{S}$, $\vec{a} = a\hat{\xi}$, $\vec{\alpha} = \alpha^{(1)}\hat{\xi} + \alpha^{(2)}\hat{\chi} + \alpha^{(3)}\hat{\zeta}$ ($\alpha^i = \gamma^{(0)}\gamma^i$), $\vec{\Omega} = \Omega_{(1)}\hat{\xi} + \Omega_{(3)}\hat{\zeta}$, and the triad $(\hat{\xi}, \hat{\chi}, \hat{\zeta})$ are the versions of ∂_ξ, ∂_χ, ∂_ζ in the ordinary vector formalism. The orbital and spin angular momenta are defined as $\vec{L} = -i\vec{X} \times \partial/\partial\vec{X}$, where $\vec{X} = \xi\hat{\xi} + \chi\hat{\chi} + \zeta\hat{\zeta}$, and

$$\vec{S} = (i/2) \left(\gamma^{(2)}\gamma^{(3)}\hat{\xi} + \gamma^{(3)}\gamma^{(1)}\hat{\chi} + \gamma^{(1)}\gamma^{(2)}\hat{\zeta} \right)$$

$$= (1/2) \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix};$$

the gamma matrices are given by Eq. (21). The Dirac equation written in this form is clearly compatible with Eqs. (11)-(14) in Ref. [17].

4.1. Dirac equation in a rotating frame

When writing the Dirac equation in a rotating frame, one uses a frame that is adapted to an observer at the origin of the inertial frame $(x, y, z = 0)$. As a result, the observer's proper time coincides with that of the inertial observers. In a real experiment, however, that is not always possible. For instance, if we perform an experiment on the surface of the Earth and take into account its rotation, we would not be able to use a clock at the center of rotation. Here we write the Dirac equation in this more realistic situation.

Let us take as the observer n the one with coordinates

$$x_n^0 = t_n, \quad x_n^1 = r_0 \cos\theta, \quad x_n^2 = r_0 \sin\theta,$$
$$x_n^3 = 0, \quad (\theta = \Omega t_n + \theta_0). \tag{25}$$

Its FST is given by

$$e_{(0)} = \lambda \mathring{e}_{(0)} - \Omega r_0 \lambda \sin\theta \mathring{e}_{(1)} + \Omega r_0 \lambda \cos\theta \mathring{e}_{(2)},$$
$$e_{(1)} = -\cos\theta \mathring{e}_{(1)} - \sin\theta \mathring{e}_{(2)},$$
$$e_{(2)} = -\Omega r_0 \lambda \mathring{e}_{(0)} + \lambda \sin\theta \mathring{e}_{(1)} - \lambda \cos\theta \mathring{e}_{(2)},$$
$$e_{(3)} = \mathring{e}_{(3)}, \tag{26}$$
$$k_1 = \Omega^2 r_0 \lambda^2, \ k_2 = \Omega \lambda^2, \ k_3 = 0, \tag{27}$$

where $\lambda = 1/\sqrt{1 - \Omega^2 r_0^2}$. Substitution of Eqs. (25)-(27) in Eq. (5) results in

$$t = \lambda(\tau - \Omega r_0 \chi), \tag{28}$$
$$x = (r_0 - \xi)\cos\theta + \chi\lambda\sin\theta, \tag{29}$$
$$y = (r_0 - \xi)\sin\theta - \chi\lambda\cos\theta, \tag{30}$$
$$z = \zeta, \quad \theta = \Omega\lambda\tau + \theta_0, \tag{31}$$

where we have used $t_n = \lambda\tau$ in Eq. (28) and in the angle θ. From Eq. (10), we see that the new coordinate system is restricted by

$$(1 + \Omega^2\lambda^2 r_0\xi)^2 > \Omega^2\lambda^4(\chi^2 + \xi^2). \tag{32}$$

Note that, since the curvatures of the worldline of the observer n are constant, the set of observers characterized by ξ, χ, and ζ constant, the rotating ones, corresponds to an RM. Note also that, unlike what was done in Ref. [23], these coordinates are adapted to an observer that is not necessarily at $x = y = z = 0$: the new coordinate time τ corresponds to the proper time of the observer whose worldline is given by Eq. (25). That is the reason why we have the Lorentz factor λ in the expressions above. The case where this observer is at the origin of the inertial frame is obtained by taking the limit[v] $r_0 \to 0$ in Eqs. (26)-(31).

Using Eq. (27), we see that Eqs. (7) and (9) become

$$e_{(0)} = f\partial_\tau + \Omega\lambda^2 f(\chi\partial_\xi - \xi\partial_\chi), \ e_{(1)} = \partial_\xi,$$
$$e_{(2)} = \partial_\chi, \ e_{(3)} = \partial_z, \ f = 1/(1 + \Omega^2\lambda^2 r_0\xi), \tag{33}$$

$$ds^2 = [(1 + \lambda^2\Omega^2 r_0\xi)^2 - \lambda^4\Omega^2(\xi^2 + \chi^2)] d\tau^2$$
$$+ 2\lambda^2\Omega(\chi d\xi - \xi d\chi)d\tau - d\xi^2 - d\chi^2 - dz^2. \tag{34}$$

In turn, using Eq. (27) in Eq. (22), we find that

$$\Gamma_c = \frac{1}{2}\Omega\lambda^2 f \left(\Omega r_0 \gamma^{(0)}\gamma^{(1)} + \gamma^{(1)}\gamma^{(2)}\right)\delta_c^{(0)}. \tag{35}$$

So, we have

$$i\gamma^b\Gamma_b = \frac{i}{2}\Omega\lambda^2 f \left[\Omega r_0\gamma^{(1)} + \gamma^{(0)}\gamma^{(1)}\gamma^{(2)}\right]. \tag{36}$$

Using Eqs. (33) and (36) in Eq. (19), we finally obtain

$$\left\{\gamma^{(0)}\left[f\partial_\tau + \Omega\lambda^2 f(\chi\partial_\xi - \xi\partial_\chi)\right] + \gamma^{(1)}\partial_\xi\right.$$
$$+ \gamma^{(2)}\partial_\chi + \gamma^{(3)}\partial_z + \frac{1}{2}\Omega\lambda^2 f\left(\Omega r_0\gamma^{(1)}\right.$$
$$\left.\left. + \gamma^{(0)}\gamma^{(1)}\gamma^{(2)}\right) + im\right\}\Psi = 0. \tag{37}$$

4.1.1. Solution

To obtain the solution of Eq. (37) in terms of the solution in the inertial frame, we take $\theta_0 = \pi$ so that the frames e_a and $\mathring{e}_a$ coincide as $\Omega \to 0$. Furthermore, we also use the definitions

$$\lambda = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \Omega r_0, \quad \phi = \Omega\lambda\tau/2. \tag{38}$$

The components of e_a in the inertial frame $\mathring{e}_a = \delta_a^{(\mu)}\partial_\mu$ correspond to the Lorentz matrix given by the relation $e_a = e_a^\mu \partial_\mu = \Lambda_a{}^b \mathring{e}_b$. We can read off the values of $\Lambda^a{}_b$ from Eqs. (26):

$$\Lambda^a{}_b = \lambda\delta_{(0)}^a \left(\delta_b^{(0)} + \Omega r_0\sin\theta\delta_b^{(1)} - \Omega r_0\cos\theta\delta_b^{(2)}\right)$$
$$- \delta_{(1)}^a \left(\cos\theta\delta_b^{(1)} + \sin\theta\delta_b^{(2)}\right)$$
$$+ \lambda\delta_{(2)}^a \left(\Omega r_0\delta_b^{(0)} + \sin\theta\delta_b^{(1)} - \cos\theta\delta_b^{(2)}\right)$$
$$+ \delta_{(3)}^a\delta_b^{(3)}. \tag{39}$$

This Lorentz transformation can be split into two interesting ones:

$$\Lambda_1{}^a{}_b = \delta_{(0)}^a\delta_b^{(0)} + \delta_{(1)}^a \left(\cos 2\phi\delta_b^{(1)} + \sin 2\phi\delta_b^{(2)}\right)$$
$$+ \delta_{(2)}^a \left(-\sin 2\phi\delta_b^{(1)} + \cos 2\phi\delta_b^{(2)}\right) + \delta_{(3)}^a\delta_b^{(3)}, \tag{40}$$

$$\Lambda_2{}^a{}_b = \delta_{(0)}^a \left(\lambda\delta_b^{(0)} + \beta\lambda\delta_b^{(2)}\right) + \delta_{(1)}^a\delta_b^{(1)}$$
$$+ \delta_{(2)}^a \left(\beta\lambda\delta_b^{(0)} + \lambda\delta_b^{(2)}\right) + \delta_{(3)}^a\delta_b^{(3)}, \tag{41}$$

where $\Lambda^a_{\ b} = \Lambda^a_{2\ c}\Lambda^c_{1\ b}$. Together, they induce the transformation $\Psi = S\mathring{\Psi}$, where $\mathring{\Psi}$ is the solution of the Dirac equation in the inertial frame.

Following the standard procedure for computing S (see, *e.g.*, p. 70 of Ref. [29]), we find that

$$S = \lambda_+ \left(\cos\phi - \sin\phi\gamma^{(1)}\gamma^{(2)}\right)$$
$$- \lambda_- \left(\sin\phi\gamma^{(0)}\gamma^{(1)} - \cos\phi\gamma^{(0)}\gamma^{(2)}\right), \quad (42)$$

where $\lambda_\pm = (\lambda \pm 1)^{1/2}/\sqrt{2}$. The inverse transformation is

$$S^{-1} = \lambda_+ \left(\cos\phi + \sin\phi\gamma^{(1)}\gamma^{(2)}\right)$$
$$+ \lambda_- \left(\sin\phi\gamma^{(0)}\gamma^{(1)} - \cos\phi\gamma^{(0)}\gamma^{(2)}\right). \quad (43)$$

The solution of Eq. (37) is $\Psi = S\mathring{\Psi}$ with $\mathring{\Psi}$ satisfying the Dirac equation in a global inertial frame of reference, *i.e.*, $i\mathring{\gamma}^\mu\partial_\mu\mathring{\Psi} - m\mathring{\Psi} = 0$, where $\mathring{\gamma}^\mu = \mathring{e}^\mu_a\gamma^a$ and $\partial_\mu = (\partial_t, \partial_x, \partial_y, \partial_z)$. Note that the $\mathring{\gamma}^\mu$s are the ordinary gamma matrices and $\mathring{e}^\mu_a = \delta^{(\mu)}_a$.

5. Summary

In this paper, we have seen that the HNT can be formulated in terms of the Frenet-Serret curvatures, which allowed us to use the FST to deal with the Born rigidity. This approach turned out to be very fruitful because of the geometrical meaning of the curvatures of the observer's worldline.

It was shown in Sec. 3.1. that the curvatures (torsions) k_2 and k_3 correspond to the rotations $\Omega_{(3)}$ and $\Omega_{(1)}$, respectively. These relations helped us to see the connection between the rotation of the FST with respect to a Fermi-Walker transported frame and the geometrical properties of the observer's motion.

We obtained the Dirac equation in the rest frame of a particle that describes an arbitrary motion using the Frenet-Serret formalism. We have seen that the final expression can be easily converted to physical parameters such as angular and spin momenta. The resultant equation was the same as that of Ref. [17]. As an application, we wrote this equation for the case of rotating observers and found its solution in terms of the solution in an inertial frame.

i. For a generalization of the Frenet formulae, see Sec. 2.7 of [2].

ii. The dual basis is such that $\langle\theta^a, e_b\rangle = \delta^a_b$. For more details, see Ref. [18].

iii. See, *e.g.*, p. 132 of Ref. [20].

iv. To understand the difference between RM and rigid body, imagine that two dancers looking at the audience run to their right at the same speed for some seconds. Their distance was kept constant and, if they knew what to do before the movement starts, they would not have to interact with each other to ensure this. Therefore, no signal is propagating from one to the other.

v. Since in this limit the worldline of the observer n becomes the straight line $(t, 0, 0, 0)$, the curvatures k_2 and k_3 their geometrical meaning. Nevertheless, we can still work with them and, as we will see, they are related to rotations of the frame, which keeps them meaningful even in this situation.

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