RESEARCH Revista Mexicana de Física 66 (2) 187–191 MARCH-APRIL 2020

Analysis of fractional Duffing oscillator

S.C. Eze
Department of Mathematics, University of Nigeria, Nsukka, Nigeria,
e-mail: sundaychristianeze@gmail.com
Tel: +2347032354775

Received 16 October 2019; accepted 27 November 2019

In this contribution, a simple analytical method (which is an elegant combination of well known methods; perturbation and Laplace methods) for solving non-linear and non-homogeneous fractional differential equations is proposed. In particular, the proposed method was used to analyze the fractional Duffing oscillator. The technique employed can be used to analyze other non-linear fractional differential equations and can also be extended to non-linear partial fractional differential equations. The performance of this method is reliable, effective, and gives more general solutions.

Keywords: Fractional calculus; fractional Duffing oscillator; analytical method.

PACS: 02.30.Jr; 02.30.Vv; 41.20.Cv; 45.10.Hj. DOI: https://doi.org/10.31349/RevMexFis.66.187

1. Introduction

The subject of fractional calculus has a long history dating back to the time when the derivative of order $1/2$ was described by Leibniz to L' Hopital in 1695 [1]. In recent years, fractional calculus has found its way in many areas of applied sciences and engineering, and it has been demonstrated that modeling physical systems using fractional calculus is more accurate than using integer calculus. This is because the behaviour of many physical phenomena depends on the previous time history (see Mohammed *et al.* [2]).

There are several works on fractional calculus, such as the work of Eze and Oyesanya [3], who presented a fractional order climate model in the Pacific Ocean. Eze and Oyesanya [4] also used a fractional model to study the impact of climate change with dominant earth's fluctuations. In the work of Mohamed and Amnah [5], a numerical technique for the estimation of stochastic response of the Duffing oscillator with fractional order damping driven by white noise excitation was introduced. An approach for solving fractional differential equations using exp-function and (G'/G)-expansion method was used in Ahmet *et al.* [6]. Djurdjica *et al.* [7] constructed the exact and the approximate solutions of Fuzzy fractional differential equation in the sense of Caputo Hukuhara differentiability with a Fuzzy condition. Rand *et al.* [8] reviewed the concept of fractional derivatives and derived expressions for the transition curves. Yin [9] applied Legendre spectral-collocation method to obtain approximate solutions of non-linear multi-order fractional differential equations. Pranay and Rubayyi [10] used Smudu transform and the variable method to solve differential equations and fractional differential equations related to entropy wavelets.

In this work, we shall propose a simple analytical method (which is an elegant combination of a well known methods; perturbation and Laplace methods) to solve non-linear and non-homogeneous fractional differential equations since Laplace method alone is incapable of handling non-linear and non-homogeneous fractional differential equations because of non-linear terms. In particular, we shall analyze the fractional Duffing oscillator using the proposed method.

The advantage of this proposed method over others such as, Adomian decomposition, Modified Laplace decomposition method, (G'/G)-expansion, Homotopy perturbation, Yang-Laplace, and so on, is its ability to handle the complicated non-homogeneous part of the fractional differential equation.

2. Definition of special functions in fractional calculus

In this section, we shall give some definitions related to Caputo's fractional derivative since in Caputo definition of Fractional calculus, we can find a link between what is possible and what is practical. We shall therefore note that our concentration in this presentation will be on Caputo fractional derivative.

Definition 2.1 *The Left Caputo Fractional Derivative (LCFD) is defined by* [11]

$$ {}^{c_0}_{a_0}D^{\alpha}_{t}u(t) = \frac{1}{\Gamma(q-\alpha)} \int_{a_0}^{t} (t-\tau)^{q-\alpha-1} u^{q}(\tau) d\tau, \qquad (1)$$

where $0 < q-1 \leq \alpha < q$, $q \in Z_{+}$, t_0 is the initial time and $\Gamma(.)$ is the Gamma function.

Definition 2.2 *The Right Caputo Fractional Derivative (RCFD) is defined by* [11]

$$ {}^{c_0}_{t}D^{\alpha}_{b_0}u(t) = \frac{1}{\Gamma(q-\alpha)} \int_{t}^{b_0} (\tau-t)^{q-\alpha-1} (-1)^{q} u^{q}(\tau) d\tau, \qquad (2)$$

where $0 < q-1 \leq \alpha < q$, $q \in Z_{+}$, t_0 is the initial time and $\Gamma(.)$ is the Gamma function.

Definition 2.3 In a Caputo's sense, the Laplace transform of fractional derivative with order α for a function $u(t)$ is defined by [12]

$$\int_0^\infty e^{-vt} D_{t_0}^\alpha u(t)dt = v^\alpha U(v) - \sum_{m=0}^{n} v^{\alpha-m-1} u^m(0),$$

$$(n-1 < \alpha \leq n). \quad (3)$$

Definition 2.4 *A two parameter function of the Mittag-Leffler (ML) function is defined by*

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{(\alpha j + \beta)}, \quad \alpha > 0, \quad \beta > 0. \quad (4)$$

Definition 2.5 *The Laplace transform of ML is given by*

$$\mathcal{L}\{t^{am+\alpha-1} E_{\alpha,\beta}^{(m)}\} = \frac{m! s^{\alpha-\beta}}{(s^\alpha - a)^{m+1}}. \quad (5)$$

3. Methodology of the new solution

Now we propose a simple analytical method for solving nonlinear and non-homogeneous fractional differential equations.

To show the basic idea, let us consider the following fractional differential equation:

$$D_t^\alpha u(t;\epsilon) + Nu(t;\epsilon) + Lu(t;\epsilon) = f(t;\epsilon), \quad (6)$$

with intial conditions

$$D_t^\beta u(0;\epsilon) = u(0;\epsilon) = 0, \alpha > \beta, \quad (7)$$

where $\alpha = [\alpha_1, \alpha_2, ..., \alpha_n]$, $\beta = [\beta_1, \beta_2, ..., \beta_n]$, indicating fractional orders, $D_\star^\alpha = [D_\star^{\alpha_1} D_\star^{\alpha_2}, ..., D_\star^{\alpha_n}]$ $0 < \alpha_i \leq 2$, $i = 1, 2, ..., n$, N is a nonlinear operators, L is a Linear operator and f is a known function.

Now, we seek to find the solution in order of ϵ, say, power of ϵ, that is

$$u(t;\epsilon) = \sum_{n=0}^{\infty} u_n \epsilon^n. \quad (8)$$

We now substitute Eqs. (8) into (6) and (7) and carry out the expansions in terms of the small parameter to obtain the following system of linear fractional differential equations:

$$D_t^\alpha u_n(t) + Au_n(t) = f(t), \quad (9)$$

with intial conditions

$$D_t^\beta u_n(0) = u_n(0) = 0, \alpha > \beta, \quad (10)$$

where A is a linear operator.

Therefore, the linear fractional differential Eqs. (9) with initial conditions (10) can now be solved in succession using the fractional differential property of Laplace transform in Eq. (3).

Note that in this method, it is necessary to express the fractional differential equation in dimensionless form to bring out the important dimensionless parameter that govern the behaviour of the system.

To demostrate the effectiveness of the proposed method, let us consider the general form of Duffing equation which is given by [13,14]:

$$\frac{d^2 x(\bar{t})}{d\bar{t}^2} + a\frac{dx(\bar{t})}{d\bar{t}} + bx(\bar{t}) + cx^3(\bar{t}) = F(\bar{t}), \quad (11)$$

with initial conditions

$$x(0) = \frac{d}{d\bar{t}} x(0) = 0. \quad (12)$$

Now, we shall use the proposed fractional operator in Rosales *et al*. [15,16] to construct a fractional Duffing oscillator. The idea is to introduce an auxiliary parameter (σ) in the system such that this σ must have a dimension in seconds and consistent with the dimension of the ordinary derivative operator. In the work [16], the parameter σ was introduced for the first time, but its geometric interpretation is not treated. After that, this method was applied in the work [15].

Now using the proposed idea, we have

$$\frac{d}{d\bar{t}} \to \frac{1}{\sigma^{1-\beta}} \frac{d^\beta}{d\bar{t}^\beta}, \ 0 < \beta \leq 1,$$
$$\frac{d^2}{d\bar{t}^2} \to \frac{1}{\sigma^{2-\alpha}} \frac{d^\alpha}{d\bar{t}^\alpha}, \ 1 < \alpha \leq 2\,. \quad (13)$$

Using (13), the Eq. (11) can be written in terms of fractional time derivatives as:

$$\frac{1}{\sigma^{2-\alpha}} \frac{d^\alpha x}{d\bar{t}^\alpha} + a\frac{1}{\sigma^{1-\beta}} \frac{d^\beta x}{d\bar{t}^\beta} + bx + cx^3 = F(\bar{t}). \quad (14)$$

Now expressing Eq. (14) in dimensionless form, we take initial displacement $x_0(\bar{t})$ as a characteristic distance, and the system's angular frequency $\omega_0 = \sqrt{k}$ as a characteristic time. Thus, we let

$$u = \frac{x}{x_0}, \quad t = \frac{\bar{t}}{\omega_0}, \quad (15)$$

where u and t denote dimensionless quantities.

Therefore, substituting (15) in (14), we have

$$\frac{x_0^\alpha}{\omega_0^\alpha \sigma^{2-\alpha}} \frac{d^\alpha u}{dt^\alpha} + a\frac{x_0^\beta}{\omega_0^\beta \sigma^{1-\beta}} \frac{d^\beta u}{dt^\beta} + bx_0 u + cx_0^3 u^3 = F(\omega_0 t)$$

$$\implies D_t^\alpha u + a^\star D_t^\beta u + b^\star u + c^\star u^3 = \epsilon\lambda\cos\omega(\delta t), \quad (16)$$

with initial conditions

$$u(0) = D_t^\beta u(0) = 0, \quad (17)$$

where

$$a^\star = a\frac{x_0^{\beta-\alpha}}{\omega_0^{\beta-\alpha}\sigma^{\alpha-\beta-1}}, \quad b^\star = bx_0^{1-\alpha}\omega_0^\alpha\sigma^{2-\alpha},$$

$$c^\star = cx_0^{3-\alpha}\omega_0^\alpha\sigma^{2-\alpha}, \quad \epsilon = \frac{\omega_0^\alpha\sigma^{2-\alpha}}{x_0^\alpha} \quad (0 < \epsilon \ll 1)$$

$$\text{and} \quad F(\omega_0 t) = \lambda\cos\omega(\delta t).$$

The function $\omega(\delta t)$ is a slowly varying (sv) function which is defined as follows:
Definition 3.1 (see R. Bojanic and E. Seneta [17]): ω is a sv function if it is a real-valued, positive, and measurable function on $[A,\infty)$, $A>0$, and if

$$\lim_{t\to\infty}\frac{\omega(\delta t)}{\omega(t)}=1, \tag{18}$$

for every $\delta>0$.

Therefore, we shall choose $\omega(\delta t)=1-e^{-\delta t}$, ($0<\delta\ll 1$) and let ω to be a sv function of t with right hand derivatives of all orders at $t=0$, such that $\omega(0)=0$ and $|\omega|\ll 1$,

The symbols D_t^α and D_t^β are describing the orders of Caputo fractional time derivatives, which is defined in Eqs. (1) and (2).

Now let

$$u(t;\epsilon,\delta)=U(\hat{t},\tau;\epsilon,\delta), \tag{19}$$

so that

$$\tau=\delta t, \tag{20}$$

and

$$\hat{t}=t+\frac{1}{\delta}[\mu_2(\tau)\epsilon^2+\mu_3(\tau)\epsilon^3+\ldots],$$
$$\mu_i(0)=0(i=2,3,4,...). \tag{21}$$

From Eqs. (20) and (21) we have, respectively,

$$\frac{d\tau}{dt}=\delta, \tag{22}$$

and

$$\frac{\partial\hat{t}}{\partial\tau}=\frac{1}{\delta}[\mu_2\prime(\tau)\epsilon^2\mu_3\prime(\tau)\epsilon^3+\ldots], \tag{23}$$

$$\frac{\partial\hat{t}}{\partial t}=1, \tag{24}$$

$$\therefore\frac{du}{dt}=\frac{\partial U}{\partial\hat{t}}\frac{\partial\hat{t}}{\partial t}+\frac{\partial U}{\partial\hat{t}}\frac{\partial\hat{t}}{\partial\tau}\frac{d\tau}{dt}+\frac{\partial U}{\partial\tau}\frac{d\tau}{dt}\implies \tag{25}$$

$$\frac{du}{dt}=\frac{\partial U}{\partial\hat{t}}+\frac{\partial U}{\partial\hat{t}}[\mu_2\prime(\tau)\epsilon^2+\mu_3\prime(\tau)\epsilon^3+...]+\delta\frac{\partial U}{\partial\tau}. \tag{26}$$

Now using Eq. (26), we have

$$D_t^\alpha u=\left\{\frac{\partial U}{\partial\hat{t}}+\frac{\partial U}{\partial\hat{t}}\phi(\epsilon)+\delta\frac{\partial U}{\partial\tau}\right\}^\alpha \tag{27}$$

and

$$D_t^\beta u=\left\{\frac{\partial U}{\partial\hat{t}}+\frac{\partial U}{\partial\hat{t}}\phi(\epsilon)+\delta\frac{\partial U}{\partial\tau}\right\}^\beta, \tag{28}$$

where $\phi(\epsilon)=\mu_2'(\tau)\epsilon^2+\mu_3'(\tau)\epsilon^3+...$

Substituting Eqs. (27) and (28) in Eq. (16), we have

$$D_{\hat{t}}^\alpha U+\alpha\phi(\epsilon)D_{\hat{t}}^\alpha U+\alpha\delta U_\tau D_{\hat{t}}^{\alpha-1}U+...+a^\star[D_{\hat{t}}^\beta U$$
$$+\beta\phi(\epsilon)D_{\hat{t}}^\beta U+\beta\delta U_\tau D_{\hat{t}}^{\beta-1}U+...]+b^\star U+c^\star U^3$$
$$=\epsilon\lambda\cos\omega(\tau). \tag{29}$$

We now assume the solution in the following asymptotic series:

$$U(\hat{t},\tau;\epsilon,\delta)=\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}U^{ij}(\hat{t},\tau)\epsilon^i\delta^j=\epsilon(U^{10}+\delta U^{11}+...)$$
$$+\ \epsilon^2(U^{20}+\delta U^{21}+...)+\epsilon^3(U^{30}+\delta U^{31}+...)$$
$$+... \tag{30}$$

Equating equations of order (ϵ^i,δ^j) in Eq. (29) using the series (30), we obtain the following linear fractional differential equations:

$$O(\epsilon):D_{\hat{t}}^\alpha U^{10}+a^\star D_{\hat{t}}^\beta U^{10}+b^\star U^{10}=\lambda\cos\omega(\tau), \tag{31}$$

$$O(\epsilon,\delta):D_{\hat{t}}^\alpha U^{11}+a^\star D_{\hat{t}}^\beta U^{11}+b^\star U^{11}=0, \tag{32}$$

$$O(\epsilon^2):D_{\hat{t}}^\alpha U^{20}+a^\star D_{\hat{t}}^\beta U^{20}+b^\star U^{20}=0, \tag{33}$$

$$O(\epsilon^2,\delta):D_{\hat{t}}^\alpha U^{21}+a^\star D_{\hat{t}}^\beta U^{21}+b^\star U^{21}$$
$$=-\alpha U_\tau^{10}D_{\hat{t}}^{\alpha-1}U^{10}-a^\star\beta U_\tau^{10}D_{\hat{t}}^{\beta-1}U^{10}, \tag{34}$$

$$O(\epsilon^3):D_{\hat{t}}^\alpha U^{30}+a^\star D_{\hat{t}}^\beta U^{30}+b^\star U^{30}$$
$$=-\alpha\mu_2'D_{\hat{t}}^\alpha U^{10}-a^\star\beta D_{\hat{t}}^\beta U^{10}-c^{\star 3}(U^{10})^3 \tag{35}$$

$$O(\epsilon^3,\delta):D_{\hat{t}}^\alpha U^{31}+a^\star D_{\hat{t}}^\beta U^{31}+b^\star U^{31}=-\alpha\mu_2'D_{\hat{t}}^\alpha U^{11}$$
$$-a^\star\beta D_{\hat{t}}^\beta U^{11}-\alpha U_\tau^{10}D_{\hat{t}}^{\alpha-1}U^{20}$$
$$-a^\star\beta U_\tau^{10}D_{\hat{t}}^{\beta-1}U^{20}-3c^{\star 3}(U^{10})^2U^{11}, \tag{36}$$

etc.

Now using the scales Eqs. (19)-(21) and the series Eq. (30) on initial conditions Eq. (17), we obtain the following initial conditions:

$$U^{ij}(0,0)=0\forall ij, \tag{37}$$

$$O(\epsilon):D_{\hat{t}}^\beta U^{1j}(0,0)=0\forall j, \tag{38}$$

$$O(\epsilon^2):D_{\hat{t}}^\beta U^{20}(0,0)=0, \tag{39}$$

$$O(\epsilon^2,\delta):D_{\hat{t}}^\beta U^{21}(0,0)=-\beta U_\tau^{10}(0,0)D_{\hat{t}}^{\beta-1}U^{10}(0,0), \tag{40}$$

$$O(\epsilon^3):D_{\hat{t}}^\beta U^{30}(0,0)=-\beta\mu'D_{\hat{t}}^\beta U^{10}(0,0), \tag{41}$$

$$O(\epsilon^3,\delta):D_{\hat{t}}^\beta U^{31}(0,0)=-\beta\mu'D_{\hat{t}}^\beta U^{11}(0,0)$$
$$-\beta U_\tau^{10}(0,0)D_{\hat{t}}^{\beta-1}U^{20}(0,0), \tag{42}$$

etc.

Now, using the fractional differential property of Laplace transform in Eq. (3), we can easly obtain the values of U^{ij} as:

$$U^{10} = \frac{\lambda \cos\omega(\tau)}{b^\star}\left\{1 + \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} b^{\star j} t^{j(\alpha-\beta)-\beta(j+1)-1} E^j_{\alpha-\beta,-\beta(j+1)}(-a^\star \hat{t}^{\alpha-\beta})\right\}$$

$$+ \frac{a^\star \lambda \cos\omega(\tau)}{b^\star} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} b^{\star j} t^{\alpha(j-1)-\beta-2} E^j_{\alpha-\beta,-(\alpha+1)+\beta(1-j)}(-a^\star \hat{t}^{\alpha-\beta}),$$

$$U^{11} = 0, \quad U^{20} = 0, \quad U^{21} = -\mathcal{L}^{-1}\left\{\frac{\mathcal{L}[\alpha U^{10}_\tau D^{\alpha-1}_{\hat{t}} U^{10} + a^\star \beta U^{10}_\tau D^{\beta-1}_{\hat{t}} U^{10}]}{s^\alpha + a^\star s^\beta + b^\star}\right\},$$

$$U^{30} = -\mathcal{L}^{-1}\left\{\frac{\mathcal{L}[\alpha \mu'_2 D^\alpha_{\hat{t}} U^{10} + a^\star \beta D^\beta_{\hat{t}} U^{10} + c^\star (U^{10})^3]}{s^\alpha + a^\star s^\beta + b^\star}\right\}, \quad U^{31} = 0. \tag{43}$$

Now, if we substitute the values of U^{ij} which are given in Eq. (43) into Eq. (30) and plot the displacement-time graphs, then we obtain Figs. 1 and 2.

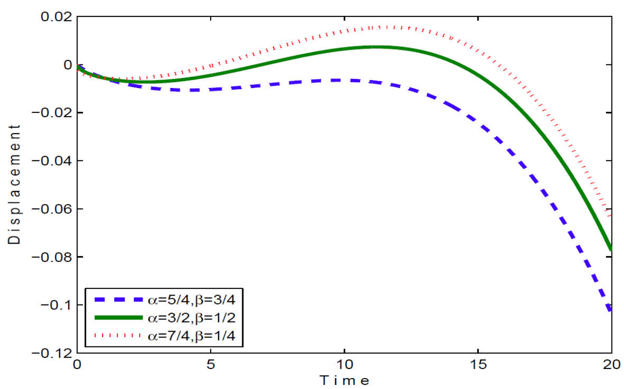

FIGURE 1. Solution in fractional orders, ($\alpha = 5/4$, $\beta = 3/4$), ($\alpha = 3/2$, $\beta = (1/2)$ and ($\alpha = 7/4$, $\beta = 1/4$).

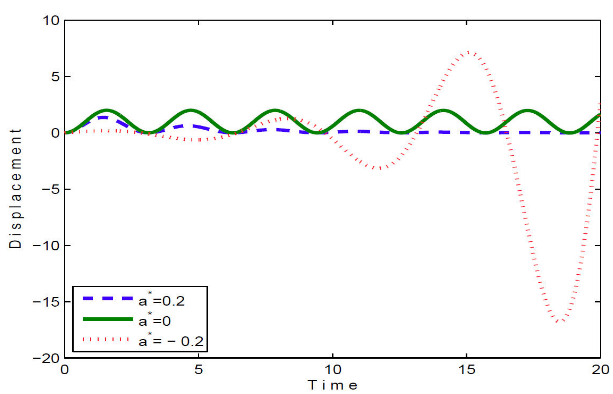

FIGURE 2. Solution in integer orders, $\alpha = 2$, $\beta = 1$, $a^* = 0.2$, $a^* = 0$ and $a^* = -0.2$.

4. Conclusion

In this paper, a new analytical method to solve non-linear and non-homogeneous fractional differential equations was proposed. This new analytical method is an elegant combination of well known methods; perturbation and Laplace methods. This method was used to analyze the fractional Duffing oscillator. The performance of this method is reliable, effective and can be used to solve other non-linear and non-homogeneous fractional differential equations; it can also be extended to solve non-linear and non-homogeneous partial fractional differential equations.

1. B. Ross, *Hist. Math*. **4** (1977) 75.
2. M. Al-Refai, M. Ali Hajji, and M. I. Syam, *Abstr. Appl. Anal.* **2014** (2014) 891837.
3. S. C. Eze and M.O. Oyesanya, *J. Fract. Cal. Appl.* **10** (2019) 10.
4. S. C. Eze and M.O. Oyesanya, *Math. Clim. Weather Forecast*. **5** (2019) 1.
5. A.E. Mohamed and S.A. Amnah, *J. Fract. Calc. Appl. 4* (2013) 357.

6. B. Ahmet, G. Özkan, H. Ali and B. Anjan, *Romaninan J. Phys.* **60** (2014) 360.
7. T. Djurdjica, T. Arpal and T. Aleresandur, *TWMS J. App. Eng. Math*. **4** (2014) 98.
8. R. H. Rand, S. M. Sah, and M. K. Suchorsky, in *Proceedings of the ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, Montreal, 2010* (ASME, New York, 2010), p. 873.
9. Y. Yin, *Appl. Math.* **4** (2013) 113.
10. G. Pranay and T.A. Rubayyi, *J. Nonlinear Sci. Appl.* **9** (2016) 1944.
11. M. O. Oyesanya, *Trans. Of NAMP*, **1** (2015) 133; **2** (2016) 325.
12. H. Jafari, C.M. Khalique and M. Nazari, *Appl. Math. Lett.* **24** (2011) 1799.
13. A. Okasha El-Nady and M. M. A. Lashin, *J. Mech. Eng. Autom.* **6** (2016) 110.
14. R. Venkatesh, C. Subhadeep and R. Asok, *Signal Process.* **88** (2008) 339.
15. J. J. Rosales, J. F. Gómez, M. Guia, and V. I. Tkach, *in Proceedings of the 11th International Conference on Laser and Fiber-Optical Networks Modeling, Kharkov, 2011*, edited by I. A. Sukhoivanov and O. V. Shulika (IEEE, Kharkov, 2011), p. 1.
16. J. F. Gómez Aguilar, J. J. Rosales-García, J.J. Bernal-Alvarado, T. Córdova-Fraga and R. Guzmán-Cabrera, *Rev. Mex. Fis*. **58** (2012) 348.
17. R. Bojanic and E. Seneta, *J. Math. Anal. Appl.* **34** (1971) 302.