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Exact solutions of vector bosons in the presence of the Aharonov-Bohm and Coulomb potentials in the gravitational field of topological defects in non-commutative space-time

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In this work, we analyze the relativistic quantum motion of a charged vector particle in the presence of an Aharonov-Bohm (AB) and Coulomb potentials in the non-commutative space-time produced by idealized cosmic strings and global monopoles via the well-know Duffin-Kemmer-Petiau equation. With the help of Wigner functions, we have solved the system in both $P = (-1)^{j+1}$ and $P = (-1)^{j}$ parity. The expressions for the bound state energies and wave functions in both commutative and non-commutative spaces have been obtained.

Keywords: DKP equation; Aharonov-Bohm potential; topological defects; non-commutative-space.

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1. Introduction

The analysis of gravitational interactions with a quantum mechanical system has recently attracted attention in particle physics and has been an active field of research.The general way to understand the interaction between relativistic quantum mechanical particles and gravity is to solve the general relativistic form of their wave equations. These solutions are valuable tools for examining and improving models and numerical methods for solving complicated physical problems.

In the conventional relativistic approach, the interaction of $S = 0$ and $S = 1$ hadrons with different nuclei has been described by the second-order Klein-Gordon (KG) equation for S=0 and Proca equation for S=1 particles. It is well known that is very difficult to tackle these second-order equations mathematically and to derive the physics behind them. Therefore, considerable interest in recent years has been devoted to examining the interactions of S=0 and S=1 hadrons with nuclei by using the first-order relativistic Duffin-Kemmer-Petiau (DKP) equation [1–4].

One important question related to the DKP equation concerns the equivalence between its spin 0 and 1 sectors and the theories based on the second-order KG and Proca equations. Historically, the loss of interest in the DKP stems from the equivalence of the DKP approach to the Klein-Gordon (KG) and Proca descriptions in on-shell situations, in addition to the greater algebraic complexity of the DKP formulation. However, in the 1970s, this supposed equivalence was questioned in several situations involving the breaking of symmetries and hadronic possess, showing that in some cases, the DKP and KG theories can give different results. Moreover, the DKP equation appears to be richer than the KG equation if the interactions are introduced. In this context, alternative DKP-based models were proposed for the study of meson-nucleus interactions, yielding a better adjustment to the experimental data when compared to the KG-based theory. In the same direction, approximation techniques formerly developed in the context of nucleon-nucleus scattering, were generalized, giving a good description for experimental data of meson-nucleus scattering. The deuteron-nucleus scattering was also studied using the DKP equation, motivated by the fact that this theory suggests a spin1 structure from combining two spin-$1/2$. Also, we can cite the works on the meson-nuclear interaction and the relativistic model of $\alpha-$nucleus elastic scattering where they have been treated by the formalism of the DKP theory. Recently, there is a renewed interest in the DKP equation. It has been studied in the context of quantum chromodynamics (QCD), covariant Hamiltonian formalism, in the causal approach, in the context of five-dimensional Galilean invariance, in the scattering of K^+ nucleus, in the presence of the Aharonov-Bohm potential, in the Dirac oscillator interaction, in the study of thermodynamics properties, on the supersymmetric, and finally in the presence of some shape of interactions. These examples, in some cases, break the equivalence between the theories based on the DKP equation and KG and Proca equations (see Refs. [13, 14] and the references therein).

The study of quantum systems in curved space-times goes back to the end of the 1920s and to the beginning of the 1930s, when the generalization of the Schrödinger and Dirac equations to curved spaces was discussed, motivated by the idea of constructing a theory which combines quantum physics and general relativity. Spinor fields and particles interacting with gravitational fields have been the subject of many investigations. Among of them, we can mention those concerning the determination of the renormalized vacuum expectation value of the energy-momentum tensor and the problem of the creation of particles in expanding universes, and those connected with quantum mechanics in different background space-times [5]. The analysis of gravitational interactions with a quantum mechanical system has recently attracted attention in particle physics and has been an active field of research.The general way to understand the interaction between relativistic quantum mechanical particles and gravity is to solve the general relativistic form of their wave equations. These solutions are valuable tools for examining and improving models and numerical methods for solving complicated physical problems. In addition, the influence of the gravitational field on quantum mechanical systems attracted attention in particle physics several years ago. As an example, the analysis of the hydrogen atom in curved space-time has been considered in an arbitrarily curved space-time [6–12].

According to the modern concepts of theoretical physics, topological defects, where they have been formed by the vacuum phase transition in the early Universe, play an important role in the physical properties of systems, and they appear in gravitation as monopoles, strings, and walls. Among them, cosmic strings and monopoles seem to be the best candidates to be observed. The former are linear defects, and the space-time produced by an idealized cosmic string is locally flat, however, globally conical, with a planar angle deficit determined by the string tension. So, they do not produce local gravitational interaction; however, they modify the geometry of the space-time producing planar and solid angle deficit, respectively (See Refs [13, 14] and references therein).

The idea of non-commutative space–time geometry was at first proposed by Heisenberg and revived again by Snyder [15]: this non-commutativity of space has played an important role in understanding various phenomena for physics. In this way, we can cite the study of the thermal properties of both Klein-Gordon and Duffin-Kemmer-Petiau (DKP) oscillators [16, 17]. The study of non-commutative spaces and their implications in physics is an extremely active area of research. It has been argued in various instances that non-commutativity should be considered as a fundamental feature of space-time at the Planck scale. On the other side, the study of quantum systems in a non-commutative (NC) space has been the subject of much interest in last years, assuming that non-commutativity may be, in fact, a result of quantum gravity effects. In these studies, some attention has been given to the models of non-commutative quantum mechanics (NCQM). The interest in this approach lies in the fact that NCQM is a fruitful theoretical laboratory where we can get some insight into the consequences of non-commutativity in field theory by using standard calculation techniques of quantum mechanics. Various non-commutative field theory models have been discussed as well as many extensions of quantum mechanics. Of particular interest are the so-called phase space non-commutativity, which has been investigated in the context of quantum cosmology, black holes physics and the singularity problem. This specific formulation is necessary to implement the Bose-Einstein statistics in the context of NCQM [18–25].

In our case, one way to deal with the NC space is to change the standard product of the fields by the star product (alternatively called Weyl-Moyal product) [53] with

$$(f * g)(x) = \exp\left[i\Theta^{ij}\partial_{x_i}\partial_{x_j}\right] f(x)\, g(x) \,. \quad (1)$$

where $\Theta^{ij} = \Theta\varepsilon^{ij}$ and Θ is the non-commutative parameter [18–25]. The position and momentum satisfying the following commutation relations

$$[\hat{x_i}, \hat{x_j}] = i\Theta_{ij}, \quad [\hat{p_i}, \hat{p_j}] = 0, \quad [\hat{x_i}, \hat{p_j}] = i\delta_{ij} \ . \quad (2)$$

The Aharonov–Bohm effect [26], is a quantum mechanical phenomenon in which an electrically charged particle is affected by an electromagnetic potential $(V, \boldsymbol{A})$, despite being confined to a region in which both the magnetic field $\mathbf{B}$ and electric field $\mathbf{E}$ are zero. In 1959, Aharonov and Bohm, using the Schrödinger equation, considered the scattering of an electron in an external static magnetic field produced by an infinitely long solenoid and found an effect that does not depend on the depth of penetration of the electrons into the region of magnetic force lines. This showed that in quantum mechanics, the electromagnetic field acts on charged particles even when the particles cannot reach the region where the field is localized. The interest in the Aharonov-Bohm effect lies in the overthrow of the classical dictum that the vector potential is only an auxiliary quantity which facilitates the calculation of the observable magnetic and electric fields. The effect will also manifest itself when we remove the slits and study the scattering of the electron waves off the solenoid. In the idealized limit of an infinitely thin solenoid where we would expect that the geometric cross- section should vanish, the Aharonov-Bohm effect ensures that the electrons still scatter. Finally, we note that the **AB** effect has been considered by different authors for different situations [26–29].

Before continuing, an important remark concerning the **AB** effect can be made. Henneberg concluded [27], based on Pauli's criterion [29, 33], that the Aharonov-Bohm effect exists, and the problem if the Pauli criterion can be applied to the Aharonov-Bohm effect has not yet been considered carefully. It can be considered as playing a purely mathematical role. However, a series of experiments presents evidence for the reality of the Aharonov-Bohm effect [27, 29, 32]. The eigenproblems of the kinetic angular momentum (KAM) of the electron in the **AB** effect has been solved by Kretzschmar [33]. The total Hilbert space of the eigenfunctions is split into two subspaces. the symmetry of the motion of the electron around the magnetic flux makes Pauli's criterion inapplicable. Many authors have ignored the inapplicability of the criterion of Pauli [34], for example, see Ref [41].

The outline of this paper is as follows: Sec. 2 is devoted to the DKP equation in curved space-time. In Secs. 3 and 4, we focus on the solutions of the vector bosons in the presence of the Aharonov-Bohm and Coulomb potentials in the gravitational field of topological defects in both commutative and non-commutative space-times. Finally, Sec. 5 will be a conclusion.

2. The DKP equation in curved space

2.1. The formalism

The first-order relativistic DKP equation for a free charged vector bosons of mass M in flat space-time is given by [1–4]

$$\left(i\beta^{\mu}\partial_{\mu} - \frac{Mc}{\hbar}\right)\psi = 0, \quad (3)$$

where $\beta^{\mu}(\mu = 0, 1, 2, 3)$ are the DKP matrices which satisfy the following commutation rules

$$\beta^{\kappa}\beta^{\nu}\beta^{\lambda} + \beta^{\lambda}\beta^{\nu}\beta^{\kappa} = g^{\kappa\nu}\beta^{\lambda} + g^{\nu\lambda}\beta^{\kappa}, \quad (4)$$

and $g^{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$ is the Minkowski metric tensor. For the flat space, the beta matrices are chosen as in Ref. [36, 37] (see Appendix A).

In curved space-time, (3) can be written by [38–43]

$$\left\{i\tilde{\beta}^{\mu}\left(\partial_{\mu} + \frac{1}{2}\omega_{\mu ab}S^{ab} - \frac{ie}{\hbar c}A_{\mu}\right) - \frac{Mc}{\hbar}\right\}\Psi = 0, \quad (5)$$

with A_{μ} denotes the vector potential associated with the electromagnetic field, $S^{ab} = \left[\beta^{a}, \beta^{b}\right]$ and $\tilde{\beta}^{\mu}$ are the Kemmer matrices: these matrices are related to their Minkowski counterparts via

$$\tilde{\beta}^{\mu} = e^{\mu}_{(a)}\beta^{a}. \quad (6)$$

The spin connection is calculated by using the following relation

$$\omega_{\mu ab} = e_{\prime(a)l}e^{j}_{(b)}\Gamma^{l}_{j\mu} - e^{j}_{(b)}\partial_{\mu}e_{\prime(a)j}, \quad (7)$$

where

$$\Gamma^{\mu}_{\nu\lambda} = \frac{g^{\mu\rho}}{2}\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\right), \quad (8)$$

are the Christoffel symbols [43]. with the aids of Eqs. (6), (7) and (8), spin connection coefficients are:

- For the metric corresponding to the cosmic strings [44–48]

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - a'^2 r^2 \sin^2\theta d\varphi^2, \quad (9)$$

where$-\infty < \text{t} < +\infty$, $0 \leqslant r$, $0 \leqslant \theta \leqslant \pi$, and $0 \leqslant \varphi \leqslant 2\pi$, the spin connection coefficients are [49]

$$\omega_{\theta ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\omega_{\varphi ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a'\text{sin}\theta \\ 0 & 0 & 0 & a'\text{cos}\theta \\ -a'\text{sin}\theta & -a'\text{cos}\theta & 0 & 0 \end{pmatrix}. \quad (10)$$

The parameter a' is the deficit angle associated with conical geometry obeying $a' = 1 - 4\eta$, and η is the linear mass density of the string. It is defined in the range $(0, 1]$, and corresponds to a deficit angle $\Omega = 2\pi\,(1 - a')$. Here, the tetrad $e^{\mu}_{(a)}$ are chosen to be

$$e^{\mu}_{(a)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{a' r\text{sin}\theta} \end{pmatrix}. \quad (11)$$

- Now, in the case of the global monopoles, with the following metric [50–52]

$$ds^2 = dt^2 - dr^2 - b'^2 r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad (12)$$

where $b'^2 = 1 - 8\pi G\eta^2$ and the parameter η being the energy scale of symmetry breaking, the spin connection coefficients are [49]

$$\omega_{\theta ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b' & 0 \\ 0 & -b' & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\omega_{\varphi ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b'\text{sin}\theta \\ 0 & 0 & 0 & \text{cos}\theta \\ -b'\text{sin}\theta & -\text{cos}\theta & 0 & 0 \end{pmatrix}, \quad (13)$$

with the tetrad $e^{\mu}_{(a)}$ are chosen to be

$$e^{\mu}_{(a)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{b' r} & 0 \\ 0 & 0 & 0 & \frac{1}{b' r\text{sin}\theta} \end{pmatrix}. \quad (14)$$

Now, we are ready to discuss the problem of the applicability of the Pauli criterion in the presence of **AB** potential for both cosmic strings and global monopoles.

2.2. The Pauli criterion

As we know, the KAM satisfy the following relations

$$\boldsymbol{J} \times \mathbf{J} = i\hbar\boldsymbol{J}, \quad (15)$$

where $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$, and with the following standard commutations rules [53, 54]

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k, \quad (16)$$

$$[J_z, J_{\pm}] = \pm\hbar J_{\pm},\ \left[J^2, J_{\pm}\right] = 0, \quad (17)$$

where

$$J_z = -i\hbar\frac{\partial}{\partial\varphi}. \quad (18)$$

Here $J_{\pm} = J_x \pm iJ_y$ are the ladder operators. The KAM is based on these commutation relations where Pauli required that the appropriate eigenfunctions be those which are square-integrable and are closed under the operation of the ladder operators. This condition is called the Pauli criterion, and consequently, the commutations relations in Eqs. (15), (16) and (17) are pertinent for the Pauli criterion to be applicable.

In the presence of the **AB** potential with the following components

$$A_r = A_\theta = 0,\ A_\varphi = \frac{\phi}{2\pi r \sin\theta}, \tag{19}$$

the KAM becomes

$$\mathbf{J} \times \mathbf{J} = i\hbar\mathbf{J} + \mathbf{2i}\alpha\hbar^{\mathbf{2}}\cos\theta\delta\big(\cos^2\theta - 1\big)\mathbf{e_r}. \tag{20}$$

From this equation we have

$$[J_z, J_\pm] = \pm\hbar J_\pm \tag{21}$$

$$\big[J^2, J_\pm\big] = \mp 2\alpha\hbar^2\big[J_\pm\delta\left(\cos^2\theta - 1\right) + \delta\left(\cos^2\theta - 1\right)J_\pm\big]. \tag{22}$$

$$J_z = -i\hbar\left(\frac{\partial}{\partial\varphi} - i\alpha\right). \tag{23}$$

In this stage, an important remark about the applicability of Pauli criterion in the presence of **AB** can be made: in the non-relativistic case, Cheng [28] shows that, in the presence of **AB**, the KAM does not satisfy the fundamental commutation relations (15), (16) and (17), and instead of this equation we have Eqs. (20), (21) and (22). These rules are different from Eqs. (15), (16) and (17). As described in [28], in the region of the existence of the magnetic field which is inaccessible to the electron, the commutation relations of the KAM should take it into account. This type of commutations relations as said to be global [28], and consequently, the Pauli criterion is inapplicable.

The eigensolutions of J_z and J^2 under the following boundary condition at $\theta = 0, \pi$,

$$\psi\left(\theta, \varphi\right)_{|\theta = o, \pi} = 0. \tag{24}$$

are presented by Kretzschmar [33]. This condition means that the particle is restricted to the doubly-connected region of $\theta \neq 0, \pi$: the topological explanation of the **AB** effect assumes that the presence of a solenoid makes the configuration space non-simply connected. The Aharonov-Bohm effect specifies that there can be a shift in the interference fringes whenever there is a superposition of waves with different winding numbers about the hole, and it gives that phase shift in the interference pattern as

$$\oint \boldsymbol{A} d\boldsymbol{l}. \tag{25}$$

The presence of the hole-in-space at the solenoid enables the existence of a vector potential $\boldsymbol{A}(r)$ with nonzero circulation, while keeping a zero magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ everywhere in space. However, it does not require that this circulation be nonzero, - and, in fact, it can be arbitrary, and among the possible arbitrary values of the circulation is the real number $\oint \boldsymbol{A} d\boldsymbol{l} = 0$. So, the presence of a doubly-connected region of space is perfectly consistent with a null value of the magnetic flux at the solenoid: so, by using the following substitutions [53]

$$J_z\psi_{j\lambda}\left(\theta, \varphi\right) = \lambda\hbar\psi_{j\lambda}\left(\theta, \varphi\right), \tag{26}$$

$$J^2\psi_{j\lambda}\left(\theta, \varphi\right) = j\left(j+1\right)\hbar^2\psi_{j\lambda}\left(\theta, \varphi\right), \tag{27}$$

with

$$\lambda = m - \alpha, \quad m = (0, \pm 1, \pm 2, \ldots), \tag{28}$$

$$j = |\lambda| + n', \ \ n' = 0, 1, 2, \ldots. \tag{29}$$

the normalized eigenfunctions are of the form

$$\psi_{\lambda j}\left(\theta, \varphi\right) = c_{\lambda, j} P_j^{-|\lambda|}\left(\cos\theta\right) e^{im\varphi}, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{30}$$

where

$$c_{\lambda, j} = e^{(i\pi/2)\lambda + (i\pi/2)|\lambda|}\left(\frac{2j{+}1}{4\pi}\frac{\Gamma\left(j{+}\left|\lambda\right|{+}1\right)}{\Gamma\left(j{-}\left|\lambda\right|{+}1\right)}\right)^{1/2}. \tag{31}$$

According to the sign of λ, we have

$$\left\{\begin{array}{c} \psi_{\lambda_1 j_1}\left(\theta, \varphi\right) = c_{\lambda_1, j_1} P_{j_1}^{-\lambda_1}\left(\cos\theta\right) e^{im\varphi}, \\ \lambda_1 = m - \alpha > 0, \ j_1 = \lambda_1 + n' \\ \psi_{\lambda_2 j_2}\left(\theta, \varphi\right) = c_{\lambda_2, j_2} P_{j_2}^{\lambda_2}\left(\cos\theta\right) e^{im\varphi}, \\ \lambda_2 = m - \alpha < 0, \ j_2 = -\lambda_2 + n'. \end{array}\right. \tag{32}$$

The inapplicability of the Pauli criterion modified completely the total Hilbert space S: the total Hilbert space S is split into two subspaces, S_+, and S_-. S_+ is spanned by all the eigenfunctions $\psi_{\lambda_1 j_1}\left(\theta, \varphi\right)$, and S_- is spanned by all the eigenfunctions $\psi_{\lambda_2 l_2}\left(\theta, \varphi\right)$. These two subspaces are not connected by the ladder operators.

In what follows, we calculate the KAM for both cosmic strings and global monopoles in non-commutative space: the case of commutative space-time is well treated in [49].

- ***case of cosmic strings***

In non-commutative cosmic strings, replacing the vector **AB** by the relation

$$A_\varphi = \frac{\Phi}{2\pi a' r \sin\theta}, \tag{33}$$

we have the following relation

$$\boldsymbol{J} \times \mathbf{J} = i\hbar\mathbf{J} + \frac{2i\hbar^2\alpha}{\boldsymbol{a'}}\cos\theta\delta\big(\cos^2\theta - 1\big)e_r. \tag{34}$$

- case of global monopoles

The case of non-commutative global monopoles, with

$$A_\varphi = \frac{\Phi}{2\pi b' r \sin\theta}, \tag{35}$$

Eq. (19) becomes

$$\mathbf{J}\times\mathbf{J} = i\hbar\mathbf{J} + \frac{\mathbf{2i\hbar^2}\alpha}{\mathbf{b'^2}}\cos\theta\delta\left(\cos^2\theta - 1\right)e_r \tag{36}$$

Here, we note the following:(i) the non-commutative-space does not, affect the KAM relations of commutations in both cosmic strings and global monopoles cases. In the limit where $a' \to 1$ ($b' \to 1$), we recover the same results as in the case of flat space, (ii) as the topologies of the configuration spaces in both cosmic strings and global monopoles models are different however the respective **AB** vectors turn out to be the same. This situation can be explained as follows: generally, when we investigate the effect of curvature of space on the **AB** effect, we consider the situation in which there is not only a tube of magnetic force lines but also an external static cylindrically symmetric gravitational field with symmetry axis of that coincides with the axis of the magnetic tube. Nevertheless in the large distances from the symmetry axis, the space becomes locally flat, and the region of spatial curvature (the gravitational tube) may either coincide with the magnetic tube or include it or, finally, be included in it [54, 55]. Besides, we can see that Eqs. (19), (20) and (21) are obviously different from Eqs. (19), (16) and (17). According to Cheng [28], the KAM of the particle, in both cases, does not satisfy the fundamental commutation relations of the angular momentum even when the particle is restricted to the doubly-connected space where it does not touch the magnetic field on the z-axis. The region where the magnetic field exists and is inaccessible to the particle should also be taken into account in the physically meaningful commutation relations.

3. The vector bosons with the Aharonov-Bohm and Coulomb potentials in the presence of topological defects in the commutative space

3.1. The solutions in the cosmic strings

The components of the **AB** vector potential in the background of cosmic strings are written as

$$A_0 = \tfrac{kq}{r}, \quad A_r = A_\theta = 0, \quad A_\varphi = \tfrac{\Phi}{2\pi a' r\sin\theta}. \tag{37}$$

From Eq. (5), the DKP equation with the **AB** and Coulomb potentials

$$\left\{\frac{\beta^0}{\hbar c}\left(E - \frac{kq}{r}\right) + i\beta^1\partial_r + \frac{i\beta^2}{r}\left(\partial_\theta - \beta^2\beta^1\right)\right\}\psi + \left\{\frac{i\beta^3}{a' r\sin\theta}\left[(\partial_\varphi - i\alpha) - a'\sin\theta\beta^3\beta^1 - a'\cos\theta\beta^3\beta^2\right] - \frac{Mc}{\hbar}\right\}\psi = 0, \tag{38}$$

where the wave function ψ [36, 37, 56–58] has the following form

$$\psi(r,\theta,\varphi) = e^{-\frac{iEt}{\hbar}}\left(\varPhi_0 D_0, \varPhi_1 D_{-1}, \varPhi_2 D_0, \varPhi_3 D_{+1}, E_1 D_{-1}, E_2 D_0, E_3 D_{+1}, H_1 D_{-1}, H_2 D_0, H_3 D_{+1}\right)^T. \tag{39}$$

Here D denotes the Wigner functions [36, 37, 59], by using some properties of these functions such as $D_\sigma = D^j_{-m,\sigma}$, $\sigma = 0, +1, -1$, and with the help of recurrent formulas

$$\partial_\theta D_{-1} = \frac{1}{2}\left(aD_{-2} - \nu D_0\right),$$

$$\frac{\lambda - \cos\theta}{\sin\theta}D_{-1} = \frac{1}{2}\left(aD_{-2} + \nu D_0\right) \tag{40}$$

$$\partial_\theta D_0 = \frac{1}{2}\left(\nu D_{-1} - \nu D_{+1}\right),$$

$$\frac{\lambda}{\sin\theta}D_0 = \frac{1}{2}\left(\nu D_{-1} + \nu D_{+1}\right) \tag{41}$$

$$\partial_\theta D_{+1} = \frac{1}{2}\left(\nu D_0 - aD_{+2}\right),$$

$$\frac{\lambda + \cos\theta}{\sin\theta}D_{+1} = \frac{1}{2}\left(\nu D_0 + aD_{+2}\right) \tag{42}$$

where $\nu = \sqrt{j(j+1)/2}$ and $a = \sqrt{(j-1)(j+2)}$, the eigenvalue of J_z and J^2 are:

$$J_z D^j_{-\lambda,s} = \lambda D^j_{-\lambda,s} \tag{43}$$

$$J^2 D^j_{-\lambda,s} = j(j+1)D^j_{-\lambda,s},$$

$$j = |\lambda| + n', \ (n' = 0, 1, 2, \ldots) \tag{44}$$

with

$$J_z = -\frac{i\hbar}{a'}\left(\frac{d}{d\varphi} - i\alpha\right), \tag{45}$$

$$\lambda = \frac{m - \alpha}{a'}. \tag{46}$$

Inserting these relations into (38), and after an algebraic calculation, we obtain the following system of equations

$$-\left(\partial_r+\frac{2}{r}\right)E_2-\frac{\nu}{r}\left(E_1+E_3\right)=\frac{Mc}{\hbar}\Phi_0, \tag{47}$$

$$\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)E_1+i\left(\partial_r+\frac{1}{r}\right)H_1+\frac{i\nu}{r}H_2=\frac{Mc}{\hbar}\Phi_1, \tag{48}$$

$$\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)E_2-\frac{i\nu}{r}\left(H_1-H_3\right)=\frac{Mc}{\hbar}\Phi_2, \tag{49}$$

$$\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)E_3-i\left(\partial_r+\frac{1}{r}\right)H_3-\frac{i\nu}{r}H_2=\frac{Mc}{\hbar}\Phi_3, \tag{50}$$

$$-\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)\Phi_1+\frac{\nu}{r}\Phi_0=\frac{Mc}{\hbar}E_1, \tag{51}$$

$$-\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)\Phi_2-\frac{d}{dr}\Phi_0=\frac{Mc}{\hbar}E_2, \tag{52}$$

$$-\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)\Phi_3+\frac{\nu}{r}\Phi_0=\frac{Mc}{\hbar}E_3, \tag{53}$$

$$-i\left(\partial_r+\frac{1}{r}\right)\Phi_1-\frac{i\nu}{r}\Phi_2=\frac{Mc}{\hbar}H_1, \tag{54}$$

$$\frac{i\nu}{r}\left(\Phi_1-\Phi_3\right)=\frac{Mc}{\hbar}H_2, \tag{55}$$

$$i\left(\partial_r+\frac{1}{r}\right)\Phi_3+\frac{i\nu}{r}\Phi_2=\frac{Mc}{\hbar}H_3, \tag{56}$$

To solve this system of equations, we define the operator $\hat{\Pi}$ [36, 37] where:

$$\hat{\Pi}=\begin{vmatrix}1&0&0&0\\0&\Pi_3&0&0\\0&0&\Pi_3&0\\0&0&0&-\Pi_3\end{vmatrix}P,$$

$$\Pi_3=\begin{vmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{vmatrix}. \tag{57}$$

The eigenvalue equation $\widehat{\Pi}\psi=P\psi$ results in two different in parity states: $P=(-1)^{j+1}$ and $P=(-1)^{j}$.

3.1.1. *Solutions with the parity* $P=(-1)^{j+1}$

In this case, we have

$$\Phi_0=0,\Phi_3=-\Phi_1,\Phi_2=0 \tag{58}$$

$$E_3=-E_1,E_2=0,H_3=H_1 \tag{59}$$

Putting them into the system of equations, we obtain

$$\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)E_1+i\left(\partial_r+\frac{1}{r}\right)H_1+\frac{i\nu}{r}H_2=\frac{Mc}{\hbar}\Phi_1, \tag{60}$$

$$-\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)\Phi_1=\frac{Mc}{\hbar}E_1, \tag{61}$$

$$-i\left(\partial_r+\frac{1}{r}\right)\Phi_1=\frac{Mc}{\hbar}H_1, \tag{62}$$

$$\frac{2i\nu}{r}\Phi_1=\frac{Mc}{\hbar}H_2. \tag{63}$$

After an algebraic calculation, we have

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{1}{(c\hbar)^2}\left(E-\frac{kq}{r}\right)^2-\left(\frac{Mc}{\hbar}\right)^2-\frac{j\left(j+1\right)}{r^2}\right\}\phi_1=0. \tag{64}$$

Now, using the following substitutions

$$\rho=\xi r,\ \xi^2=\frac{4\left(M^2c^4-E^2\right)}{\hbar^2c^2}, \tag{65}$$

$$\gamma=\frac{kq}{\hbar c},\quad \varsigma=\frac{2\gamma E}{\hbar c\xi}, \tag{66}$$

$$\Phi_1=\frac{R\left(r\right)}{r}, \tag{67}$$

Eq. (64) becomes

$$\frac{d^2R\left(r\right)}{d\rho^2}+\left(-\frac{j\left(j+1\right)-\gamma^2}{\rho^2}-\frac{\zeta}{\rho}-\frac{1}{4}\right)R\left(r\right)=0. \tag{68}$$

Putting that

$$R\left(r\right)=N\rho^{s+1}\exp\left(-\frac{\rho}{2}\right)H\left(\rho\right), \tag{69}$$

(68) is transformed into

$$\rho^2\frac{d^2H\left(\rho\right)}{d\rho^2}+\left(2\left(s+1\right)\rho-\rho^2\right)\frac{dH\left(\rho\right)}{d\rho}+\Big[\left(s\left(s+1\right)-\left(j\left(j+1\right)-\gamma^2\right)\right)-\left(\zeta+s+1\right)\rho\Big]H\left(\rho\right)=0$$

with

$$s+\varsigma+1=-n,\ \left(n=0,1,2,\ldots\right), \tag{70}$$

or

$$s=-\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\gamma^2}. \tag{71}$$

Finally, the solutions are

$$\Phi_1=\left(\frac{\sqrt{\left(\left(Mc^2\right)^2-E^2\right)}n!}{\left(c\hbar\right)2\left(n+s+1\right)\Gamma\left(n+2s+2\right)}\right)^{\frac{1}{2}}\times\rho^{s+1}\exp\left(-\frac{\rho}{2}\right)L_n^{2s+1}, \tag{72}$$

$$E_{nl} = \frac{Mc^2}{\sqrt{1+\frac{\gamma^2}{\left(n+\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\gamma^2}\right)^2}}}$$
$$= \frac{Mc^2}{\sqrt{1+\frac{\gamma^2}{\left(n+\frac{1}{2}+\sqrt{\left(\left|\frac{m-\alpha}{a'}\right|+n'+\frac{1}{2}\right)^2-\gamma^2}\right)^2}}}, \qquad (73)$$

Equation (73) can be put into another form as

$$E_{nl} = \frac{Mc^2}{\sqrt{1+\gamma^2\kappa^{-2}}}, \qquad (74)$$

where

$$\kappa = n+\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\gamma^2}$$
$$= n+\frac{1}{2}+\sqrt{\left(\left|\frac{m-\alpha}{a'}\right|+n'+\frac{1}{2}\right)^2-\gamma^2}.$$

In the non-relativistic approximation, the behavior of the spectrum of energy, for very small values of the constant γ, can be expanding in a power series in γ as follows [59]

$$E_{nl} \simeq Mc^2\left[1-\frac{\gamma^2}{2N^2}-\frac{\gamma^4}{2N^4}\left(\frac{N}{j+\frac{1}{2}}-\frac{3}{4}\right)+\ldots\right], \qquad (75)$$

where

$$N = \left[n+\underbrace{n'+\left|\frac{m-\alpha}{a'}\right|}_{j}+1\right]$$

is the principal quantum number, and $[N]$ means the biggest integer inferior to N: the different terms in (75) can be interpreted as follows: the first term corresponds to the rest energy of the particle. The second term is the same as the energy of a particle of mass M in a Coulomb field in the non-relativistic approximation. This term depends on geometrical parameters of space a'. The third term determines the relativistic correction to the energy. We see that the correction to the energy depends on the quantum number n, $j = n' + |(m-\alpha)/a'|$, and with the geometric parameter of space-time a'. Finally, in both limits $\alpha \to 0$ (annihilation of the Aharonov-Bohm potential) and $a' \to 1$ (flat space), we obtain

$$E_{nl}^{(NR)} = \frac{Mc^2}{\sqrt{1+\gamma^2\kappa^{-2}}}, \qquad (76)$$

with NR denotes the non-relativistic,

$$\kappa = n+\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\gamma^2}$$

and $j = |m| + n'$. Eq. (76) coincide with the habitual spectrum of the energy of Coulomb potential [59].

Concerning the total wave function, we consider both subspaces S_+ and S_-

- For the subspace S_+, where $\lambda_1 > 0$ and $j_1 = \lambda_1 + n'$, the total spinor is

$$\psi_{n\lambda_1 j_1} = e^{-\frac{iEt}{\hbar}}\begin{pmatrix} 0 \\ D_{-1} \\ 0 \\ -D_{+1} \\ \frac{-i}{Mc^2}\left(E-\frac{kq}{r}\right)D_{-1} \\ 0 \\ \frac{i}{Mc^2}\left(E-\frac{kq}{r}\right)D_{+1} \\ \frac{-i}{Mc^2}\left(\partial_r+\frac{1}{r}\right)D_{-1} \\ \frac{i}{Mc^2}\left(\frac{2i\nu}{r}\right)H_2 D_0 \\ \frac{-i}{Mc^2}\left(\partial_r+\frac{1}{r}\right)D_{+1} \end{pmatrix}\varPhi_1 \qquad (77)$$

with

$$\varPhi_1 = N_{nor}P_{j_1}^{-\lambda_1}(\cos\theta)\,\rho^s e^{-\frac{\rho}{2}}L_n^{2s+1}. \qquad (78)$$

- For the case of subspace S_-, where $\lambda_2 < 0$ and $l_2 = -\lambda_2 + n'$, the total spinor is

$$\psi_{n\lambda_2 j_2} = e^{-\frac{iEt}{\hbar}}\begin{pmatrix} 0 \\ D_{-1} \\ 0 \\ -D_{+1} \\ \frac{-i}{Mc^2}\left(E-\frac{kq}{r}\right)D_{-1} \\ 0 \\ \frac{i}{Mc^2}\left(E-\frac{kq}{r}\right)D_{+1} \\ \frac{-i}{Mc^2}\left(\partial_r+\frac{1}{r}\right)D_{-1} \\ \frac{i}{Mc^2}\left(\frac{2i\nu}{r}\right)H_2 D_0 \\ \frac{-i}{Mc^2}\left(\partial_r+\frac{1}{r}\right)D_{+1} \end{pmatrix}\varPhi_1 \qquad (79)$$

with

$$\varPhi_1 = N_{nor}P_{j_2}^{\lambda_2}(\cos\theta)\,\rho^s e^{-\frac{\rho}{2}}L_n^{2s+1}. \qquad (80)$$

3.1.2. *Solutions with the parity $P = (-1)^j$*

By using the operator $\hat{\Pi}$, we extracted the following relations:

$$\Phi_3 = \Phi_1, E_3 = E_1, H_3 = H_1, H_2 = 0. \qquad (81)$$

Putting them into our system, we obtain

$$\left(\partial_r+\frac{2}{r}\right)E_2 + 2\frac{\nu}{r}E_1 + \frac{Mc}{\hbar}\varPhi_0 = 0, \qquad (82)$$

$$\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)E_1 + i\left(\partial_r+\frac{1}{r}\right)H_1 - \frac{Mc}{\hbar}\varPhi_1 = 0, \qquad (83)$$

$$\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)E_2 - 2\frac{i\nu}{r}H_1 - \frac{Mc}{\hbar}\varPhi_2 = 0, \qquad (84)$$

$$-\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)\varPhi_1 + \frac{\nu}{r}\varPhi_0 - \frac{Mc}{\hbar}E_1 = 0, \qquad (85)$$

$$\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)\varPhi_2 + \partial_r\varPhi_0 + \frac{Mc}{\hbar}E_2 = 0, \qquad (86)$$

$$i\left(\partial_r+\frac{1}{r}\right)\varPhi_1 + \frac{i\nu}{r}\varPhi_2 + \frac{Mc}{\hbar}H_1 = 0. \qquad (87)$$

As a first remark, no way to decouple this system. So, to solve them, we will focus only on the states with $j = 0$: this situation can be treated straightforwardly. Thus, we have:

$$\left(\partial_r + \frac{2}{r}\right) E_2 + \frac{Mc}{\hbar}\Phi_0 = 0, \quad (88)$$

$$\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right) E_1 + i\left(\partial_r + \frac{1}{r}\right) H_1 - \frac{Mc}{\hbar}\Phi_1 = 0, \quad (89)$$

$$\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right) E_2 - \frac{Mc}{\hbar}\Phi_2 = 0, \quad (90)$$

$$-\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right) \Phi_1 - \frac{Mc}{\hbar}E_1 = 0, \quad (91)$$

$$\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right) \Phi_2 + \partial_r\Phi_0 + \frac{Mc}{\hbar}E_2 = 0, \quad (92)$$

$$i\left(\partial_r + \frac{1}{r}\right) \Phi_1 + \frac{Mc}{\hbar}H_1 = 0, \quad (93)$$

or

$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2}{r^2} + \frac{1}{(c\hbar)^2}\left(E - \frac{kq}{r}\right)^2 - \left(\frac{Mc}{\hbar}\right)^2\right\}E_2 = 0. \quad (94)$$

Putting that

$$E_2 = \frac{F(r)}{r}\chi(\theta,\varphi), \quad (95)$$

and using the Eqs. (65) and (66), we obtain

$$\frac{d^2F(\rho)}{d\rho^2} + \left(-\frac{2-\gamma^2}{\rho^2} - \frac{\zeta}{\rho} - \frac{1}{4}\right)F(\rho) = 0. \quad (96)$$

Now, when we write that

$$F(\rho) = N\rho^{s+1}e^{-\frac{\rho}{2}}v(\rho) \quad (97)$$

Eq. (96) is transformed into

$$\rho^2\frac{d^2v(\rho)}{d\rho^2} + \left(2(s+1)\rho - \rho^2\right)\frac{dv(\rho)}{d\rho} + \left[\left(s(s+1) - \left(2-\gamma^2\right)\right) - (\zeta+s+1)\rho\right]v(\rho) = 0 \quad (98)$$

with

$$s = \frac{-1+\sqrt{9-4\gamma^2}}{2}. \quad (99)$$

Finally, the total eigensolutions are

$$E_n = \frac{Mc^2}{\sqrt{1 + \frac{\gamma^2}{\left(n+\frac{1}{2}+\frac{1}{2}(9-4\gamma^2)^{\frac{1}{2}}\right)^2}}}, \quad (100)$$

$$F_n(\rho) = \left(\frac{\sqrt{(Mc^2)^2 - E^2}n!}{(c\hbar)2(n+s+1)\Gamma(n+2s+2)}\right)^{\frac{1}{2}} \times \rho^{s+1}\exp\left(-\frac{\rho}{2}\right)L_n^{2a+1}. \quad (101)$$

We can see that the spectrum of the energy coincide with the habitual spectrum of energy of Coulomb potential, [59], and not depend on the geometry of space.

3.2. The solutions in global monopoles

The metric of the space-time in this case is given by

$$ds^2 = dt^2 - dr^2 - b'^2r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad (102)$$

with the components of the 4-vector potential are

$$A_0 = \frac{kq}{r},\ \ A_r = A_\theta = 0,\ \ A_\varphi = \frac{\Phi}{2\pi b' r\sin\theta}, \quad (103)$$

where $b^2 = 1 - 8\pi G\eta^2$ and the parameter η being the energy scale of symmetry breaking. From Eq. (5), the DKP equation with the **AB** and Coulomb potentials is written by:

$$\left\{\frac{\beta^0}{\hbar c}\left(E - \frac{kq}{r}\right) + i\beta^1\partial_r + \frac{i\beta^2}{b'r}\left(\partial_\theta - b'\beta^2\beta^1\right)\right\}\psi + \left\{\frac{i\beta^3}{b'r\sin\theta}\Big((\partial_\varphi - i\alpha) - b'\sin\theta\beta^3\beta^1 - \cos\theta\beta^3\beta^2\Big) - \frac{Mc}{\hbar}\right\}\psi = 0, \quad (104)$$

By using Eqs. (13) and (14), Eq. (104) becomes

$$-\left(\frac{d}{dr} + \frac{2}{r}\right)E_2 - \frac{\nu}{b'r}(E_1 + E_3) = \frac{Mc}{\hbar}\Phi_0 \quad (105)$$

$$\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right)E_1 + i\left(\frac{d}{dr} + \frac{1}{r}\right)H_1 + \frac{i\nu}{b'r}H_2 = \frac{Mc}{\hbar}\Phi_1 \quad (106)$$

$$\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right)E_2 - \frac{i\nu}{b'r}(H_1 - H_3) = \frac{Mc}{\hbar}\Phi_2 \quad (107)$$

$$\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right)E_3 - i\left(\frac{d}{dr} + \frac{1}{r}\right)H_3 - \frac{i\nu}{b'r}H_2 = \frac{Mc}{\hbar}\Phi_3 \quad (108)$$

$$-\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right)\Phi_1 + \frac{\nu}{b'r}\Phi_0 = \frac{Mc}{\hbar}E_1 \quad (109)$$

$$-\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right)\Phi_2 - \frac{d}{dr}\Phi_0 = \frac{Mc}{\hbar}E_2 \quad (110)$$

$$-\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right)\Phi_3 + \frac{\nu}{b'r}\Phi_0 = \frac{Mc}{\hbar}E_3 \quad (111)$$

$$-i\left(\frac{d}{dr} + \frac{1}{r}\right)\Phi_1 - \frac{i\nu}{b'r}\Phi_2 = \frac{Mc}{\hbar}H_1 \quad (112)$$

$$\frac{i\nu}{b'r}(\Phi_1 - \Phi_3) = \frac{Mc}{\hbar}H_2 \quad (113)$$

$$i\left(\frac{d}{dr} + \frac{1}{r}\right)\Phi_3 + \frac{i\nu}{b'r}\Phi_2 = \frac{Mc}{\hbar}H_3 \quad (114)$$

As in the case of cosmic strings, we distinguished two cases.

3.2.1. Solutions with the parity $P=(-1)^{j+1}$

Using the operator $\hat{\Pi}$, we have

$$\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)E_1+i\left(\partial_r+\frac{1}{r}\right)H_1+\frac{i\nu}{b'r}H_2=\frac{Mc}{\hbar}\Phi_1, \quad (115)$$

$$-\frac{i}{c\hbar}\left(E-\frac{kq}{r}\right)\Phi_1=\frac{Mc}{\hbar}E_1, \quad (116)$$

$$-i\left(\partial_r+\frac{1}{r}\right)\Phi_1=\frac{Mc}{\hbar}H_1, \quad (117)$$

$$\frac{2i\nu}{b'r}\Phi_1=\frac{Mc}{\hbar}H_2, \quad (118)$$

After an algebraic calculation, and by putting (116), (117), and (118) in (115), we obtain

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{1}{(c\hbar)^2}\left(E-\frac{kq}{r}\right)^2-\left(\frac{Mc}{\hbar}\right)^2-\frac{\frac{j(j+1)}{b'^2}}{r^2}\right\}\Phi_1=0. \quad (119)$$

With the aid of the following relation

$$\Phi_1=\frac{R(r)}{r}, \quad (120)$$

and by inserting (65) (66) in (119), we have

$$\frac{d^2R(r)}{d\rho^2}+\left(-\frac{\frac{j(j+1)}{b'2}-\gamma^2}{\rho^2}-\frac{\zeta}{\rho}-\frac{1}{4}\right)R(r)=0. \quad (121)$$

Now let us make a change of variable

$$R(r)=N\rho^{s+1}\exp\left(-\frac{\rho}{2}\right)H(\rho). \quad (122)$$

In this case, (121) is transformed into

$$\rho^2\frac{d^2H(\rho)}{d\rho^2}+\left(2(s+1)\rho-\rho^2\right)\frac{dH(\rho)}{d\rho}+\left[\left(s(s+1)-\left(j(j+1)-\gamma^2\right)\right)-(\zeta+s+1)\rho\right]H(\rho)=0 \quad (123)$$

with

$$s=-\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\gamma^2} \quad (124)$$

is obtained by following the condition of quantification defined by

$$s+\varsigma+1=-n,\ (n=0,1,2,\dots). \quad (125)$$

So, the eigenvalues are given by

$$E_n=\frac{Mc^2}{\sqrt{1+\frac{\gamma^2}{\left(n+\frac{1}{2}-\left(\frac{j(j+1)}{b'^2}+\frac{1}{4}-\gamma^2\right)^{\frac{1}{2}}\right)^2}}} \quad (126)$$

$$\Phi_1=\left(\frac{\sqrt{(Mc^2)^2-E^2}n!}{(c\hbar)2(n+s+1)\Gamma(n+2s+2)}\right)^{\frac{1}{2}}\times\rho^{s+1}\exp\left(-\frac{\rho}{2}\right)L_n^{2s+1} \quad (127)$$

with $j=|(m-\alpha)/b'|+n'$.

Equation (129) can be rewritten into another form as

$$E_{nl}=\frac{Mc^2}{\sqrt{1+\gamma^2\kappa'^{-2}}}, \quad (128)$$

with

$$\kappa'=n+\frac{1}{2}+\left\{\frac{j(j+1)}{b'^2}+\frac{1}{4}-\gamma^2\right\}^{\frac{1}{2}}$$

and $j=|(m-\alpha)/b'|+n'$. As in the case of the cosmic string, we can make the following remarks: (i) for very small values of γ, the energy spectrum can be expanding in a power series in γ. This expansion gives

$$E_{nl}\simeq Mc^2\left[1-\frac{\gamma^2}{2N'^2}-\frac{\gamma^4}{2N'^3\left\{\frac{j(j+1)}{b'^2}+\frac{1}{4}\right\}^{\frac{1}{2}}}+\dots\right], \quad (129)$$

with

$$N'=\left[n+\underbrace{n'+\left|\frac{m-\alpha}{b'}\right|}_{l}+1\right]$$

is the principal quantum number, and $[N']$ means the biggest integer inferior to N' : the different terms in (129) can be interpreted as follow: the first term corresponds to the rest energy of the particle. The second term is the same as the energy of a particle of mass M in a Coulomb field in the non-relativistic approximation. This term depends on the geometrical parameter of space-time b'. The third term determines the relativistic correction to the energy. As in the case of the cosmic string, we see also that the correction to the energy depends on the quantum number n, $j=|(m-\alpha)/b'|+n'$, and the geometric parameter of space-time b'.

In both limits $\alpha\rightarrow 0$ (annihilation of the Aharonov-Bohm potential) and $b'\rightarrow 1$ (flat space), we obtain

$$E_{nl}=\frac{Mc^2}{\sqrt{1+\gamma^2\kappa^{-2}}}, \quad (130)$$

with

$$\kappa'=n+\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\gamma^2},$$

and $j=|m|+n'$. Thus, we recover the habitual spectrum of the energy of Coulomb potential [59].

Now, concerning the total wave function, we consider two subspace S_+ and S_-

- For the case of subspace S_+, where $\lambda_1 > 0$ and $j_1 = \lambda_1 + n$, the total spinor is

$$\psi_{n\lambda_1 j_1} = e^{-\frac{iEt}{\hbar}} \begin{pmatrix} 0 \\ D_{-1} \\ 0 \\ -D_{+1} \\ \frac{-i}{Mc^2}\left(E - \frac{kq}{r}\right) D_{-1} \\ 0 \\ \frac{i}{Mc^2}\left(E - \frac{kq}{r}\right) D_{+1} \\ \frac{-i}{Mc^2}\left(\partial_r + \frac{1}{r}\right) D_{-1} \\ \frac{i}{Mc^2}\left(\frac{2i\nu}{b'r}\right) H_2 D_0 \\ \frac{-i}{Mc^2}\left(\partial_r + \frac{1}{r}\right) D_{+1} \end{pmatrix} \varPhi_1 \qquad (131)$$

with

$$\varPhi_1 = N_{nor} P_{j_1}^{-\lambda_1}(\cos\theta)\, \rho^s e^{-\frac{\rho}{2}} L_n^{2s+1}. \qquad (132)$$

- For the case of subspace S_-, where $\lambda_2 < 0$ and $j_2 = -\lambda_2 + n'$, the total spinor is

$$\psi_{n\lambda_1 j_2} = e^{-\frac{iEt}{\hbar}} \begin{pmatrix} 0 \\ D_{-1} \\ 0 \\ -D_{+1} \\ \frac{-i}{Mc^2}\left(E - \frac{kq}{r}\right) D_{-1} \\ 0 \\ \frac{i}{Mc^2}\left(E - \frac{kq}{r}\right) D_{+1} \\ \frac{-i}{Mc^2}\left(\partial_r + \frac{1}{r}\right) D_{-1} \\ \frac{i}{Mc^2}\left(\frac{2i\nu}{b'r}\right) H_2 D_0 \\ \frac{-i}{Mc^2}\left(\partial_r + \frac{1}{r}\right) D_{+1} \end{pmatrix} \varPhi_1 \qquad (133)$$

with

$$\varPhi_1 = N_{nor} P_{j_2}^{\lambda_2}(\cos\theta)\, \rho^s e^{-\frac{\rho}{2}} L_n^{2s+1}. \qquad (134)$$

3.2.2. *Solutions with the parity $P = (-1)^j$*

Following the definition of the operator $\hat{\Pi}$, we have

$$\left(\partial_r + \frac{2}{r}\right) E_2 + 2\frac{\nu}{b'r} E_1 + \frac{Mc}{\hbar}\varPhi_0 = 0 \quad (135)$$

$$\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right) E_1 + i\left(\partial_r + \frac{1}{r}\right) H_1 - \frac{Mc}{\hbar}\varPhi_1 = 0 \quad (136)$$

$$\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right) E_2 - 2\frac{i\nu}{b'r} H_1 - \frac{Mc}{\hbar}\varPhi_2 = 0 \quad (137)$$

$$-\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right) \varPhi_1 + \frac{\nu}{b'r}\varPhi_0 - \frac{Mc}{\hbar} E_1 = 0 \quad (138)$$

$$\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right) \varPhi_2 + \partial_r \varPhi_0 + \frac{Mc}{\hbar} E_2 = 0 \quad (139)$$

$$i\left(\partial_r + \frac{1}{r}\right)\varPhi_1 + \frac{i\nu}{b'r}\varPhi_2 + \frac{Mc}{\hbar} H_1 = 0 \quad (140)$$

As described above, we focus only on the special case where $j = 0$: following the same procedure we obtain

$$\left(\partial_r + \frac{2}{r}\right) E_2 + \frac{Mc}{\hbar}\varPhi_0 = 0 \quad (141)$$

$$\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right) E_1 + i\left(\partial_r + \frac{1}{r}\right) H_1 - \frac{Mc}{\hbar}\varPhi_1 = 0 \quad (142)$$

$$\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right) E_2 - \frac{Mc}{\hbar}\varPhi_2 = 0 \quad (143)$$

$$-\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right) \varPhi_1 - \frac{Mc}{\hbar} E_1 = 0 \quad (144)$$

$$\frac{i}{c\hbar}\left(E - \frac{kq}{r}\right) \varPhi_2 + \partial_r\varPhi_0 + \frac{Mc}{\hbar} E_2 = 0 \quad (145)$$

$$i\left(\partial_r + \frac{1}{r}\right)\varPhi_1 + \frac{Mc}{\hbar} H_1 = 0 \quad (146)$$

or with the component E_2, we obtain

$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2}{r^2} + \frac{1}{(c\hbar)^2}\left(E - \frac{kq}{r}\right)^2 - \left(\frac{Mc}{\hbar}\right)^2\right\} E_2 = 0 \qquad (147)$$

Using Eqs. (65), (66), and putting that

$$E_2 = \frac{F(r)}{r}\chi(\theta,\varphi), \qquad (148)$$

and by expanding Eq. (147), we have

$$\frac{d^2F(\rho)}{d\rho^2} + \left(-\frac{2-\gamma^2}{\rho^2} - \frac{\zeta}{\rho} - \frac{1}{4}\right) F(\rho) = 0. \qquad (149)$$

Now, let's make that

$$F(\rho) = N\rho^{s+1} e^{-\frac{\rho}{2}} v(\rho), \qquad (150)$$

Eq. (149) is transformed into

$$\rho^2\frac{d^2v(\rho)}{d\rho^2} + \left(2(s+1)\rho - \rho^2\right)\frac{dv(\rho)}{d\rho} + \left[\left(s(s+1) - \left(2-\gamma^2\right)\right) - (\zeta + s + 1)\rho\right] v(\rho) = 0 \qquad (151)$$

with

$$s = \frac{-1 + \sqrt{9 - 4\gamma^2}}{2}. \qquad (152)$$

Finally, the eigensolutions are

$$E_n = \frac{Mc^2}{\sqrt{1+\frac{\gamma^2}{\left(n+\frac{1}{2}+\frac{1}{2}(9-4\gamma^2)^{\frac{1}{2}}\right)^2}}} \tag{153}$$

$$F_n(\rho) = \left(\frac{\sqrt{\left((Mc^2)^2-E^2\right)}n!}{(c\hbar)\,2\,(n+s+1)\,\Gamma(n+2s+2)}\right)^{\frac{1}{2}} \times \rho^{s+1}\exp\left(-\frac{\rho}{2}\right)L_n^{2a+1}. \tag{154}$$

4. The vector bosons with the Aharonov-bohm and Coulomb potentials in the presence of topological defects in non-commutative space

4.1. The solutions in non commutative cosmic strings

As described above, according to the parity P, we distinguished two cases

4.1.1. *Case of the parity $P=(-1)^{j+1}$*

The **AB** vector potential in the background of a cosmic string takes the form

$$A_0 = \frac{kq}{r}, \quad A_r = A_\theta = 0, \quad A_\varphi = \frac{\Phi}{2\pi a' r\sin\theta}. \tag{155}$$

In non-commutative cosmic strings, by putting (120) in (119), we have

$$\left\{\frac{d^2}{dr^2}-\frac{(Mc^2)^2-E^2}{(c\hbar)^2}-\frac{2(kq)E}{(c\hbar)^2 r}-\frac{j(j+1)-\left(\frac{kq}{c\hbar}\right)^2}{r^2}\right\}R(r) - \frac{\boldsymbol{L\Theta}}{2\hbar}\left[\left(\frac{kqE}{(c\hbar)^2}\right)\frac{1}{r^3}+\left(j(j+1) - \left(\frac{kq}{c\hbar}\right)^2\right)\frac{1}{r^4}\right]R(r)=0, \tag{156}$$

where

$$j = |\lambda| + n', \ (n'=0,1,2,\ldots) \tag{157}$$

$$\lambda = \frac{m-\alpha}{a'}. \tag{158}$$

The quantum number j are the eigenvalues of J_z and J^2, respectively: we can see that these eigenvalues depend on the magnetic flux α and the geometric parameter of space a'.

Using the perturbation technique, Eq. (156), for $\Theta=0$, takes the form

$$\left\{\frac{d^2}{dr^2}-\frac{(Mc^2)^2-\left(E^{(0)}\right)^2}{(c\hbar)^2}-\frac{2(kq)E^{(0)}}{(c\hbar)^2 r} - \frac{j(j+1)-\left(\frac{kq}{c\hbar}\right)^2}{r^2}\right\}R^{(0)}(r)=0. \tag{159}$$

Now, using the following substitutions

$$\rho = \xi r,\ \xi^2 = \frac{4\left(M^2c^4-\left(E^{(0)}\right)^2\right)}{\hbar^2c^2}, \tag{160}$$

$$\gamma = \frac{kq}{\hbar c},\ \varsigma = \frac{2\gamma E^{(0)}}{\hbar c\xi}, \tag{161}$$

$$\left\{\frac{d^2}{d\rho^2}+\left(-\frac{l(l+1)-\gamma^2}{\rho^2}-\frac{\zeta}{\rho}-\frac{1}{4}\right)\right\} \times R^{(0)}(\rho)=0, \tag{162}$$

and putting that

$$R^{(0)}(\rho) = N\rho^{s+1}e^{-\frac{\rho}{2}}H(\rho), \tag{163}$$

Eq. (162) is transformed into

$$\rho^2\frac{d^2H(\rho)}{d\rho^2}+\left(2(s+1)\rho-\rho^2\right)\frac{dH(\rho)}{d\rho} + \left[\left(s(s+1)-\left(l(l+1)-\gamma^2\right)\right) - (\zeta+s+1)\rho\right]H(\rho)=0, \tag{164}$$

To solve (164), we use the Frobenius method [60–62]. This can be written as a power series expansion around the origin:

$$H(\rho) = \sum_{k=0}^{\infty} c_k\rho^k. \tag{165}$$

Putting (165) into (164), we obtain the following recurrence relations:

$$c_{k+1} = \frac{k+(\zeta+s+1)}{k(k+1)+2(s+1)(k+1)}a_k \tag{166}$$

By starting with $c_0=1$, we have

$$c_1 = \frac{(\zeta+s+1)}{2(s+1)}, \tag{167}$$

$$c_2 = \frac{1+(\zeta+s+1)}{2+4(s+1)}a_1. \tag{168}$$

Now, imposing that $c_{k+1}=0$, we obtain

$$\zeta+s+1=-n, \tag{169}$$

where

$$s = -\frac{1}{2}+\left(\left(j+\frac{1}{2}\right)^2-\gamma^2\right)^{\frac{1}{2}}.$$

Finally, the eigensolutions are

$$R_n^{(0)}(r)=\left(\frac{\sqrt{\left((Mc^2)^2-\left(E^{(0)}\right)^2\right)}n!}{(c\hbar)\,2\,(n+s+1)\,\Gamma\,(n+2s+2)}\right)^{\frac{1}{2}}\times\rho^{s+1}\exp\left(-\frac{\rho}{2}\right)L_n^{2s+1}, \tag{170}$$

$$E_{nl}^{(0)}=\frac{Mc^2}{\sqrt{1+\frac{\gamma^2}{\left(n+\frac{1}{2}+\sqrt{\left(\left|\frac{m-\alpha}{a'}\right|+n'+\frac{1}{2}\right)^2-\gamma^2}\right)^2}}}=\frac{Mc^2}{\sqrt{1+\frac{\gamma^2}{\left(n+\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\gamma^2}\right)^2}}}. \tag{171}$$

with $j=|(m-\alpha)/a'|+n'$.

The last equation can be also put into another form as

$$E_{nl}^{(0)}=\frac{Mc^2}{\sqrt{1+\gamma^2\kappa^{-2}}}, \tag{172}$$

where

$$\kappa=n+\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\gamma^2}.$$

In the non-relativistic approximation: for very small values of the constant γ, the energy spectrum can be expanding in a power series in γ as follows [59]

$$E_{nl}^{(0)}\simeq Mc^2\left[1-\frac{\gamma^2}{2N^2}-\frac{\gamma^4}{2N^4}\left(\frac{N}{j+\frac{1}{2}}-\frac{3}{4}\right)+\dots\right], \tag{173}$$

where

$$N=\left[n+\underbrace{n'+\left|\frac{m-\alpha}{a'}\right|}_{j}+1\right]$$

is the principal quantum number, and $[N]$ means the biggest integer inferior to N : the different terms in (173) can be interpreted as follows: the first term corresponds to the rest energy of the particle. The second term is the same as the energy of a particle of mass M in a Coulomb field in the non-relativistic approximation. This term depends on geometrical parameters of space a'. The third term determines the relativistic correction to the energy. We see that the correction to the energy depends on the quantum number n, $j=n'+|(m-\alpha)/a'|$, and with the geometric parameter of space-time a'. Finally, in the case of a commutative space ($\Theta=0$), in both limits $\alpha\to 0$ (annihilation of the Aharonov-Bohm potential) and $a'\to 1$ (flat space), we obtain

$$E_{nl}^{(0)}=\frac{Mc^2}{\sqrt{1+\gamma^2\kappa^{-2}}}, \tag{174}$$

with

$$\kappa=n+\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\gamma^2}$$

and $j=|m|+n'$. Eq. (174) coincide with the habitual spectrum of energy of Coulomb potential [59].

For the expectation value of r^{-k} [39,63,64], we have

$$\left\langle r^{-k}\right\rangle=\int_0^\infty r^{-k}\left|\phi_n^{(0)}(r)\right|^2 dr\delta_{mm'}. \tag{175}$$

Putting (170) into (175), we have

$$\left\langle r^{-k}\right\rangle=\left(\frac{2^k\left(\sqrt{\left((Mc^2)^2-\left(E^{(0)}\right)^2\right)}\right)^k n!}{(c\hbar)\,2\,(n+s+1)\,\Gamma\,(n+2s+2)}\right)\times\int_0^\infty\rho^{2s+2-k}e^{-\rho}\left(L_n^{2s+1}\right)^2 d\rho. \tag{176}$$

By using the following relation [63, 64]

$$\int_0^\infty e^{-x}x^{\varrho+s}L_n^{\varrho}(x)\,L_m^{\beta}(x)\,dx=(-1)^{n-m}\times\frac{\Gamma(\varrho+s+1)\,\Gamma(\beta+m+1)\,\Gamma(s+1)}{m!\,(n-m)!\Gamma(\beta+1)\,\Gamma(s-n+m+1)}\times{}_3F_2\left(\begin{matrix}-m,s+1,\beta-\varrho-s\\ \beta+1,n-m+1\end{matrix}\right) \tag{177}$$

we obtain [52, 63, 64]-

$$\left\langle r^{-3}\right\rangle=\frac{4\left\{\sqrt{(Mc^2)^2-\left(E^{(0)}\right)^2}\right\}^3}{(2s+1)\,(2s)\,(n+s+1)}\left[1+\frac{n}{s+1}\right], \tag{178}$$

$$\left\langle r^{-4}\right\rangle=\frac{4\left\{\sqrt{(Mc^2)^2-\left(E^{(0)}\right)^2}\right\}^4}{(2s-1)\,s\,(2s+1)\,(n+s+1)}\times\left[1+\frac{3n}{s+1}+\frac{3n\,(n-1)}{(s+1)\,(2s+3)}\right], \tag{179}$$

Now, with the aid of the following relations,

$$\boldsymbol{\Theta}.\boldsymbol{L}=\Theta L_z, \tag{180}$$

$$\hat{r}=r-\frac{L.\Theta}{4\hbar r}+0\left(\Theta^2\right), \tag{181}$$

$$\frac{1}{\hat{r}}=\frac{1}{r}+\frac{L.\Theta}{4\hbar r^3}+0\left(\Theta^2\right), \tag{182}$$

we obtain

$$E^{(NC)} = -\frac{\Theta}{\hbar}\left(\frac{m-\alpha}{a'}\right)\left(j\left(j+1\right)-\left(\frac{kq}{\hbar c}\right)^2\right) \times\left[\frac{2\left(\left(Mc^2\right)^2-\left(E^{(0)}\right)^2\right)^4}{\left(2s-1\right)s\left(2s+1\right)\left(n+s+1\right)} \times\left[1+\frac{3n}{s+1}+\frac{3n\left(n-1\right)}{\left(s+1\right)\left(2s+3\right)}\right]\right] - \frac{\Theta}{\hbar}\left(\frac{m-\alpha}{a'}\right)\left[\left(\frac{\left(kq\right)E}{\left(c\hbar\right)^2}\right) \times\frac{2\left(\left(Mc^2\right)^2-\left(E^{(0)}\right)^2\right)^3}{\left(2s+1\right)\left(2s\right)\left(n+s+1\right)}\left\{1+\frac{n}{s+1}\right\}\right]. \tag{183}$$

Now, concerning the total wave function, we have two cases according to the subspace S_+ and S_-

- For the case of subspace S_+, where $\lambda_1 > 0$ and $j_1 = \lambda_1 + n'$, the total spinor is

$$\psi^{(0)}_{n\lambda_1 j_1} = e^{-\frac{iEt}{\hbar}}\begin{pmatrix} 0 \\ D_{-1} \\ 0 \\ -D_{+1} \\ \frac{-i}{Mc^2}\left(E-\frac{kq}{r}\right)D_{-1} \\ 0 \\ \frac{i}{Mc^2}\left(E-\frac{kq}{r}\right)D_{+1} \\ \frac{-i}{Mc^2}\left(\partial_r+\frac{1}{r}\right)D_{-1} \\ \frac{i}{Mc^2}\left(\frac{2i\nu}{r}\right)H_2D_0 \\ \frac{-i}{Mc^2}\left(\partial_r+\frac{1}{r}\right)D_{+1}\end{pmatrix}\Phi^{(0)}_1 \tag{184}$$

with

$$\Phi^{(0)}_1 = N_{nor}P^{-\lambda_1}_{j_1}\left(\cos\theta\right)\rho^s e^{-\frac{\rho}{2}}L^{2s+1}_n. \tag{185}$$

- For the case of subspace S_-, where $\lambda_2 < 0$ and $j_2 = -\lambda_2 + n'$, the total spinor is

$$\psi^{(0)}_{n\lambda_2 j_2} = e^{-\frac{iEt}{\hbar}}\begin{pmatrix} 0 \\ D_{-1} \\ 0 \\ -D_{+1} \\ \frac{-i}{Mc^2}\left(E-\frac{kq}{r}\right)D_{-1} \\ 0 \\ \frac{i}{Mc^2}\left(E-\frac{kq}{r}\right)D_{+1} \\ \frac{-i}{Mc^2}\left(\partial_r+\frac{1}{r}\right)D_{-1} \\ \frac{i}{Mc^2}\left(\frac{2i\nu}{r}\right)H_2D_0 \\ \frac{-i}{Mc^2}\left(\partial_r+\frac{1}{r}\right)D_{+1}\end{pmatrix}\Phi^{(0)}_1 \tag{186}$$

with

$$\Phi^{(0)}_j = N_{nor}P^{\lambda_2}_{j_2}\left(\cos\theta\right)\rho^s e^{-\frac{\rho}{2}}L^{2s+1}_n. \tag{187}$$

4.1.2. *Case of the parity $P = (-1)^j$*

As in the case of commutative space, we only consider the case where $j = 0$: consequently, we want to solve the differential equations for the components E_2 or $F(r)$. Thus, in non-commutative cosmic strings, the differential equations for the component $F(r)$ is given by

$$\left\{\frac{d^2}{dr^2}+\frac{E^2-M^2c^4}{\hbar^2c^2}-\frac{2\left(\frac{kq}{\hbar c}\right)E}{\left(c\hbar\right)^2 r}-\frac{2-\left(\frac{kq}{\hbar c}\right)^2}{r^2}\right\}F\left(r\right) - \frac{\boldsymbol{L}.\boldsymbol{\Theta}}{2\hbar}\left[\left(\frac{\left(\frac{kq}{\hbar c}\right)E}{\left(c\hbar\right)^2}\right)\frac{1}{r^3} + \left(2-\left(\frac{kq}{\hbar c}\right)^2\right)\frac{1}{r^4}\right]F\left(r\right)=0. \tag{188}$$

Using the perturbation technique, (188), for $\Theta = 0$, takes the form

$$\left\{\frac{d^2}{dr^2}+\frac{\left(E^{(0)}_n\right)^2-\left(Mc^2\right)^2}{\hbar^2c^2}-\frac{2\left(\frac{kq}{\hbar c}\right)E^{(0)}_n}{\left(c\hbar\right)^2 r} - \frac{2-\left(\frac{kq}{\hbar c}\right)^2}{r^2}\right\}F^{\Theta=0}_n\left(r\right)=0. \tag{189}$$

As in the case of cosmic strings, the eigensolutions are

$$E^{(0)}_n = \frac{Mc^2}{\sqrt{1+\frac{\gamma^2}{\left(n+\frac{1}{2}+\frac{1}{2}\left(9-4\gamma^2\right)^{\frac{1}{2}}\right)^2}}}. \tag{190}$$

$$F^{(0)}_n\left(\rho\right) = \left(\frac{\sqrt{\left(\left(Mc^2\right)^2-\left(E^{(0)}_n\right)^2\right)}n!}{\left(c\hbar\right)2\left(n+s+1\right)\Gamma\left(n+2s+2\right)}\right)^{\frac{1}{2}} \times\rho^{s+1}\exp\left(-\frac{\rho}{2}\right)L^{2s+1}_n, \tag{191}$$

In the general case, the spectrum of energy is written as

$$E^{NC} = -\frac{\Theta}{\hbar}\left(\frac{m-\alpha}{a'}\right)\left(2-\left(\frac{kq}{\hbar c}\right)^2\right) \times\left[\frac{2\left(\left(Mc^2\right)^2-\left(E^{(0)}_n\right)^2\right)^4}{\left(2s-1\right)a\left(2s+1\right)\left(n+s+1\right)} \times\left[1+\frac{3n}{s+1}+\frac{3n\left(n-1\right)}{\left(s+1\right)\left(2s+3\right)}\right]\right] - \frac{\Theta}{\hbar}\left(\frac{m-\alpha}{a'}\right)\left[\left(\frac{\left(kq\right)E}{\left(c\hbar\right)^2}\right)\right.$$

$$\times\frac{2\left(\left(Mc^2\right)^2-\left(E_n^{(0)}\right)^2\right)^3}{(n+s+1)\,\Gamma\left(n+2s+2\right)}\left[1+\frac{n}{s+1}\right]\bigg]. \quad (192)$$

4.2. The solutions in non-commutative global monopoles

By using the same procedure as in the case of non-commutative cosmic strings, the eigensolutions are recapitulated as follows:

4.2.1. For case of the parity $P=(-1)^{j+1}$

$$R_n^{(0)}(r)=\left(\frac{\sqrt{\left(\left(Mc^2\right)^2-\left(E_n^{(0)}\right)^2\right)}n!}{(c\hbar)\,2\left(n+s+1\right)\Gamma\left(n+2s+2\right)}\right)^{\frac{1}{2}}$$

$$\times\,\rho^{s+1}\exp\left(-\frac{\rho}{2}\right)L_n^{2s+1}, \quad (193)$$

$$E_{nl}^{(0)}=\frac{Mc^2}{\sqrt{1+\frac{\gamma^2}{\left\{n+\frac{1}{2}+\left(\frac{j(j+1)}{b'^2}+\frac{1}{4}-\gamma^2\right)^{\frac{1}{2}}\right\}^2}}}. \quad (194)$$

with $j=|(m-\alpha)/b'|+n'$. Eq. (194) can be rewritten into another form as

$$E_{nl}^{(0)}=\frac{Mc^2}{\sqrt{1+\gamma^2\kappa'^{-2}}}, \quad (195)$$

with

$$\kappa'=n+\frac{1}{2}+\left\{\frac{j\left(j+1\right)}{b'^2}+\frac{1}{4}-\gamma^2\right\}^{\frac{1}{2}}$$

and $j=|(m-\alpha)/b'|+n'$. As in the case of cosmic strings, we can make the following remarks: (i) for very small values of γ, the energy spectrum can be expanding in a power series in γ. This expansion gives

$$E_{nl}^{(0)}\simeq Mc^2\left[1-\frac{\gamma^2}{2N'^2}-\frac{\gamma^4}{2N'^3\left\{\frac{j(j+1)}{b'^2}+\frac{1}{4}\right\}^{\frac{1}{2}}}+\ldots\right], \quad (196)$$

with

$$N'=\left[n+\underbrace{n'+|(m-\alpha)/b'|}_{j}+1\right]$$

is the principal quantum number, and $[N']$ means the biggest integer inferior to N' : the different terms in (196) can be interpreted as follow: the first term corresponds to the rest energy of the particle. The second term is the same as the energy of a particle of mass M in a Coulomb field in the non-relativistic approximation. This term depends on the geometrical parameter of space-time b'. The third term determines the relativistic correction to the energy. As in the case of the cosmic string, we see also that the correction to the energy depends on the quantum number n, $j=|(m-\alpha)/b'|+n'$, and the geometric parameter of space-time b'. In the case of a commutative space ($\Theta=0$), and in both limits $\alpha\to 0$ (annihilation of the **Aharonov-Bohm** potential) and $b'\to 1$ (flat space), we obtain

$$E_{nl}^{(0)}=\frac{Mc^2}{\sqrt{1+\gamma^2\kappa^{-2}}}, \quad (197)$$

with

$$\kappa'=n+\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-\gamma^2},$$

and $j=|m|+n'$. Thus, as in the case of cosmic string, we recover the habitual spectrum of the energy of Coulomb potential.

Finally, we have

$$E^{(NC)}=-\frac{\Theta}{\hbar}\left(m-\alpha\right)\left(\frac{j\left(j+1\right)}{b'^2}-\left(\frac{kq}{\hbar c}\right)^2\right)$$

$$\times\left[\frac{2\left(\left(Mc^2\right)^2-\left(E^{(0)}\right)^2\right)^4}{(2s-1)\,s\,(2s+1)\,(n+s+1)}\right.$$

$$\left.\times\left[1+\frac{3n}{s+1}+\frac{3n\left(n-1\right)}{(s+1)\left(2s+3\right)}\right]\right]$$

$$-\frac{\Theta}{\hbar}\left(m-\alpha\right)\left[\left(\frac{(kq)\,E}{(c\hbar)^2}\right)\right.$$

$$\left.\times\frac{2\left(\left(Mc^2\right)^2-\left(E^{(0)}\right)^2\right)^3}{(2s+1)\,(2s)\,(n+s+1)}\left[1+\frac{n}{s+1}\right]\right]. \quad (198)$$

Now, concerning the total wave function, we have two cases:

- For the case of subspace S_+, where $\lambda_1>0$ and $j_1=\lambda_1+n'$, the total spinor is

$$\psi_{n\lambda_1 j_1}^{(0)}=e^{-\frac{iEt}{\hbar}}\begin{pmatrix}0\\ D_{-1}\\ 0\\ -D_{+1}\\ \frac{-i}{Mc^2}\left(E-\frac{kq}{r}\right)D_{-1}\\ 0\\ \frac{i}{Mc^2}\left(E-\frac{kq}{r}\right)D_{+1}\\ \frac{-i}{Mc^2}\left(\partial_r+\frac{1}{r}\right)D_{-1}\\ \frac{i}{Mc^2}\left(\frac{2i\nu}{b'r}\right)H_2D_0\\ \frac{-i}{Mc^2}\left(\partial_r+\frac{1}{r}\right)D_{+1}\end{pmatrix}\varPhi_1^{(0)} \quad (199)$$

with

$$\varPhi_1^{(0)}=N_{nor}P_{j_1}^{-\lambda_1}\left(\cos\theta\right)\rho^s e^{-\frac{\rho}{2}}L_n^{2s+1}. \quad (200)$$

For the case of subspace S_-, where $\lambda_2<0$ and $j_2=-\lambda_2+n'$, the total spinor is

$$\psi^{(0)}_{n\lambda_1 j_2} = e^{-\frac{iEt}{\hbar}} \begin{pmatrix} 0 \\ D_{-1} \\ 0 \\ -D_{+1} \\ \frac{-i}{Mc^2}\left(E - \frac{kq}{r}\right) D_{-1} \\ 0 \\ \frac{i}{Mc^2}\left(E - \frac{kq}{r}\right) D_{+1} \\ \frac{-i}{Mc^2}\left(\partial_r + \frac{1}{r}\right) D_{-1} \\ \frac{i}{Mc^2}\left(\frac{2i\nu}{b'r}\right) H_2 D_0 \\ \frac{-i}{Mc^2}\left(\partial_r + \frac{1}{r}\right) D_{+1} \end{pmatrix} \Phi^{(0)}_1 \quad (201)$$

with

$$\Phi^{(0)}_1 = N_{nor} P^{\lambda_2}_{j_2}(\cos\theta)\, \rho^s e^{-\frac{\rho}{2}} L^{2s+1}_n. \quad (202)$$

4.2.2. For case of the parity $P = (-1)^j$

The eigensolutions are

$$F^{(0)}_n(r) = \left(\frac{\sqrt{\left((Mc^2)^2 - \left(E^{(0)}_n\right)^2\right) n!}}{(c\hbar)\, 2\,(n+s+1)\,\Gamma(n+2s+2)} \right)^{\frac{1}{2}}$$

$$\times\, \rho^{s+1} \exp\left(-\frac{\rho}{2}\right) L^{2s+1}_n, \quad (203)$$

$$E^{(0)}_n = \frac{Mc^2}{\sqrt{1 + \frac{\gamma^2}{\left(n + \frac{1}{2} + \frac{1}{2}(9-4\gamma^2)^{\frac{1}{2}}\right)^2}}}. \quad (204)$$

we have

$$E^{(NC)} = -\frac{\Theta}{\hbar}(m-\alpha)\left(2 - \left(\frac{kq}{\hbar c}\right)^2\right)$$

$$\times \left[\frac{2\left(\left(Mc^2\right)^2 - \left(E^{(0)}\right)^2\right)^4}{(2s-1)\, s\,(2s+1)(n+s+1)} \right.$$

$$\left. \times \left[1 + \frac{3n}{s+1} + \frac{3n(n-1)}{(s+1)(2s+3)}\right] \right]$$

$$- \frac{\Theta}{\hbar}(m-\alpha)\left[\left(\frac{(kq)\,E}{(c\hbar)^2} \right) \right.$$

$$\left. \times \frac{2\left(\left(Mc^2\right)^2 - \left(E^{(0)}\right)^2\right)^3}{(2s+1)(2s)(n+s+1)} \left[1 + \frac{n}{s+1}\right] \right]. \quad (205)$$

5. Conclusion

This paper is devoted to studying the solutions of the relativistic quantum motion of a charged vector particles in the presence of an Aharonov-Bohm and Coulomb potentials in the space-times produced by idealized cosmic strings and global monopoles in non-commutative space-time. These solutions have been obtained, and the influence of the parameter of the geometry of both topological defects has been discussed. Also the remarks, which Cheng [24] has been proposed concerning the **AB** effect, have been extended in our case: thus, the presence of **AB** potential changes completely the fundamental commutation relations of the angular momentum. Following the works of [28, 29, 33], we note that (i) the KAM relations are not satisfied, even when the particle is restricted to the doubly connected space where it does not touch the magnetic field on the z-axis. Besides, (ii) the region where the magnetic field exists and where it is inaccessible to the electron should be taken into account in the physical commutation relations; finally (iii), the Pauli criterion which said that "the appropriate eigenfunctions are those which are square-integrable and are closed under the operation of ladder operators" is inapplicable to the vector **AB**. The existence of the magnetic field on the z-axis is the principal cause of breaking down the symmetry of the particle's motion around the z-axis. The eigenfunctions and eigenvalues of J_z and J^2 have been presented under the following boundary condition $\psi(r,\theta,\varphi)|_{\theta=0,\pi} = 0$, the space S is split into two subspaces, $S_+$is spanned by all the wave function $\psi_{j_1\lambda_1}(\theta,\varphi)$, and $S_-$is spanned by all the wave function $\psi_{j_2\lambda_2}(\theta,\varphi)$. By applying the perturbative approach, we studied the vector bosons in the NC space: the spectrum of energy in the gravitational field of cosmic strings and global monopoles are different. It is explicitly shown that (i) the KAM relations are not affected by the parameter Θ of the NC space, and (ii) the degeneracy of the initial spectral line is broken in the transition from commutative space-time into the non-commutative space-time.

Appendix

A. The matrices β

The matrices β used in this paper are [36, 37]

$$\beta^0 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&+i&0&0&0&0&0 \\ 0&0&0&0&0&+i&0&0&0&0 \\ 0&0&0&0&0&0&+i&0&0&0 \\ 0&-i&0&0&0&0&0&0&0&0 \\ 0&0&-i&0&0&0&0&0&0&0 \\ i&0&0&-i&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\beta^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & -i & 0 & +i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\beta^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +i & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +i & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & +i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \beta^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & +i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & +i & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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