# Super dense coding in one-axis twisting model

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We study thermal superdense coding in a two-spin model under an external magnetic field. Its dependencies on magnetic field, strength of the spin squeezing and temperature are presented in detail. Our main goal now is to study how we can increase the thermal superdense coding capacity in the presence of magnetic field, strength of the spin squeezing and temperature. It shows that the dense coding tends to a valid value by setting the value of input quantum correlations. Our most important motivation for this study is to examine the relationship between the thermal properties of super quantum discord (SQD) and dense coding. The results show that the thermal properties of the SQD on our channel enable us to determine when and under what conditions the system is suitable for valid dense coding. Our proposals could be lead to that this scheme is efficient for quantum information processing.

*Keywords:* Super dense coding; one-axis twisting model; super quantum discord.

## 1. Introduction

Quantum information theory offers many methods on how information may be stored, manipulated, and transmitted [1]. The base of quantum information theory is well known to be the quantum entanglement. It is the most fascinating feature of quantum mechanics that plays an essential role in quantum cryptographic key distribution [2], quantum teleportation [3], and quantum dense coding [4]. For example, in the original dense coding protocol, the sender can transmit two bits of classical information to the receiver by sending a single qubit if they share a two-qubit maximally entangled state. Dense coding has been investigated both experimentally [5] and theoretically [6–8]. Recently, optimal dense coding have been extensively investigated in condensed matter physics due to their good integrability and scalability in the context of quantum information processing [9, 10]. Up to now, various systems have been studied in the field of quantum processes. The spin chains have been considered as one of the most suitable candidates for the study of entanglement and quantum correlations [11-13]. Recently, the cavity quantum electrodynamics system (CQES) has received much attentions since this system offer an ideal coupling between atoms and photons [14]. As a result, it has been widely applied for engineering quantum entanglement and quantum channel [15–21]. The common result of all models show that dynamical properties of the SQD on our channel enable us to determine when and under what conditions the system is suitable for dense coding capacity [22].

Quantum correlations have been comprehensively accepted as the main resource for different quantum information processing tasks. For a long time, the study of quantum correlations has been focused on entangelement [23]. However, recent studies indicated that entanglement does not reveal quantum correlation in separable states, therefore it cannot be considered as a complete measure of quantum correlation. The present work focuses on another concept of quantum correlation, namely quantum discord (QD) [24–27]. The QD is a measure of quantumness of correlations in a bipartite state, which was introduced as the difference between two natural quantum extension of the classical mutual information. Recently, numerous works have been made toward the significance and applications of QD [28, 29]. It should be noted that discord is captured by strong measurements (projective operators). As quantum states are fragile to quantum measurements, the measurement of an arbitrary quantum state in some orthogonal basis (projective measurement) leads to the loss of its coherence. While if the measurement be done weakly, it protects the coherence of the system when we perform a measurement which couples the system and the measuring device weakly. In 1988, Aharonov, Albert and Vaidman have proposed to use weak measurements [30]. Lately, it was shown that weak measurements can also protect QD from the decoherence [31]. It is known that the weak measurement captures more quantum correlation of a bipartite system than the strong (projective) one under certain situation. In technological applications, weak measurements are important for exploring the extra quantum correlation for information processing. Actually, replacing the projective measurement with weaker one in the definition of QD, gives rise to new correlation called super quantum discord (SQD) [33, 34]. Hence, a weak measurement enforced on one of the subsystems can lead to SQD, which is always larger than the normal QD captured by the strong measurement [32].

Our most important motivation for this study is to examine the relationship between thermal properties of SQD and superdense coding. We know that entangled states have the important role in dense coding. Here we propose a two-qubit spin squeezing model as quantum channels to study the thermal dense coding capacity $\chi$. Motivated from the recent study on the thermal of entanglement in the above model, we here devote to examining the quantum correlation properties

in terms of SQD with dense coding capacity. Our results suggest that the thermal properties of SQD in the channel can accurately determine the valid dense coding. Dense coding plays a major role in quantum communication, in such a way that valid dense coding detection using the thermal properties of SQD in the channel seems to play an important role in quantum information theory. Therefore, the notion of SQD can be a more useful resource for quantum information processing tasks, quantum communication and quantum computation than QD.

The remainder of this paper is organized as follows. In Sec. 2, we introduce the Hamiltonian of the systems that we want study dense coding and quantum correlations through them. In Sec. 3 firstly, we briefly review the dense coding scheme, then we obtain the thermal super dense coding and analyze its characters. Finally, the main results will be summarized in Sec. 4.

## 2. Description of the model

The two-spin model under an external magnetic field and coupled to each other by a one-axis twisting spin squeezing interaction is given by [35, 36]:

$$\hat{H} = \mu S_x^2 + \Omega S_z, \tag{1}$$

where, transverse field term $S_z$ can control the strength of the external magnetic field $\Omega \geq 0$ in $z$ direction. The first term includes an ensemble of $N$ spin-$1/2$ particles with exchange symmetry that its dynamical properties can be described by collective operators $S_\alpha = (1/2)\Sigma_{i=1}^2 \sigma_\alpha^i, (\alpha \in x, y, z)$, and $\sigma_\alpha^i$ are the Pauli matrices for the $i^{th}$ spin. In the following, we consider N=2. $\mu \geq 0$ describes the strength of the spin squeezing interaction in $x$ direction which depends on the scattering lengths between particles and the condensate density. The interaction establishes pairwise correlations between all of individual spins in the collective spin system; in our case bipartite correlations are formed between the two spins. This model is considered as a simple one in solid state in order to generate and manipulate entangled states.

The eigenvalues $E_n$ and the corresponding eigenvectors $|\psi_n\rangle$ of Hamiltonian Eq. (1) in the standard basis of $|00\rangle, |10\rangle, |01\rangle, |11\rangle$ can be easily obtained as the following forms [38]:

$$E_{1,4} = \frac{\mu \pm \kappa}{2}, \qquad |\psi_{1,2}\rangle = \frac{1}{A_{\mp}}(\mu|00\rangle + (2\Omega \pm \kappa)|11\rangle),$$

$$E_2 = 0, \qquad |\psi_3\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |10\rangle),$$

$$E_3 = \mu, \qquad |\psi_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |01\rangle), \tag{2}$$

where $\kappa = \sqrt{\mu^2 + 4\Omega^2}$ and the normalization constant is $A_{\mp} = \sqrt{\mu^2 + (2\Omega \pm \kappa)^2}$. We can introduce thermal fluctuations for the system, the density matrix of this two spins at thermal equilibrium is $\rho = (1/Z)e^{-\beta H} =$

$\Sigma_{n=1}^4 P(n)|\psi_n\rangle\langle\psi_n|$, where $P(n) = (1/Z)e^{-\beta E_n}$ are thermally distributed populations in the quantum states, $\beta = (1/k)T(k = 1)$ and $T$ is the temperature. Also, $Z = \sum_n e^{-\beta E_n}$ is the partition function.

Equipped with these concepts and formulas, we are now in the status to obtain density matrix $\rho(T)$ in the standard basis $|00\rangle, |10\rangle, |01\rangle, |11\rangle$,

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{3}$$

with the element of density matrix

$$\rho_{11} = \frac{1}{z}\mu^2 e^{-\beta\mu/2}\left(\frac{e^{\frac{-\kappa\beta}{2}}}{A_+^2} - \frac{e^{\frac{\kappa\beta}{2}}}{A_-^2}\right),$$

$$\rho_{14} = \frac{1}{z}\mu e^{-\beta\mu/2}\left(\frac{e^{\frac{-\kappa\beta}{2}}(2\Omega + \kappa)}{A_+^2} - \frac{e^{\frac{\kappa\beta}{2}}(2\Omega - \kappa)}{A_-^2}\right),$$

$$\rho_{22} = \rho_{33} = \frac{1}{2z}\left(1 + e^{-\beta\mu}\right),$$

$$\rho_{23} = \rho_{32} = \frac{1}{2z}\left(-1 + e^{-\beta\mu}\right),$$

$$\rho_{44} = \frac{1}{z}e^{-\beta\mu/2}\left(\frac{e^{\frac{-\kappa\beta}{2}}(2\Omega + \kappa)^2}{A_+^2} + \frac{e^{\frac{\kappa\beta}{2}}(2\Omega - \kappa)^2}{A_-^2}\right),$$

$$\rho_{41} = \rho_{14}.$$

With knowing that density matrix is the X-state form, one can obtain the quantum correlation behaviors by using correlation measure. It is determined that the quantum correlations such as SQD and QD are more comprehensive than entanglement [37]. Recent studies indicated that at a certain temperature entanglement and quantum correlations can be increased with changing the value of the $\Omega$ and $\mu$. The results in Ref. [13, 38] indicate that thermal quantum correlations depends on the spin squeezing interaction $\mu$ and the external magnetic interaction $\Omega$ at the same of temperature. In following, we extend the quantum dense coding to spin channel. Their study show that quantum correlations can approach a maximum value by adjusting the external magnetic field, the spin squeezing interaction and temperature.

## 3. Superdense Coding

Now, we carry out the thermal optimal dense coding in one-axis twisting model as a quantum channel. For this purpose, the set of mutually orthogonal unitary transformations is necessary to be made. The set of mutually orthogonal unitary transformations for two-qubit are given as follows [39]:

$$U_{00}|j\rangle = |j\rangle$$

$$U_{01}|j\rangle = |j + 1(mod2)\rangle$$

$$U_{10}|j\rangle = e^{\sqrt{-1}(2\pi/2)j}|j\rangle$$

$$U_{11}|j\rangle = e^{\sqrt{-1}(2\pi/2)j}|j + 1(mod2)\rangle \qquad (4)$$

where $|j\rangle$ is the single qubit computational basis ($|j\rangle = |0\rangle, |1\rangle$). The average state of the ensemble of signal states generated by the unitary transformations Eq. (4) given by:

$$\overline{\rho^*} = \frac{1}{4}\sum_{i=0}^{3}(U_i \otimes I_2)\rho(U_i^\dagger \otimes I_2) \qquad (5)$$

where 0 stands for 00, 1 for 01, 2 for 10, 3 for 11, and $\rho$ is the density matrix of the quantum channel. Eq. (5) represents the
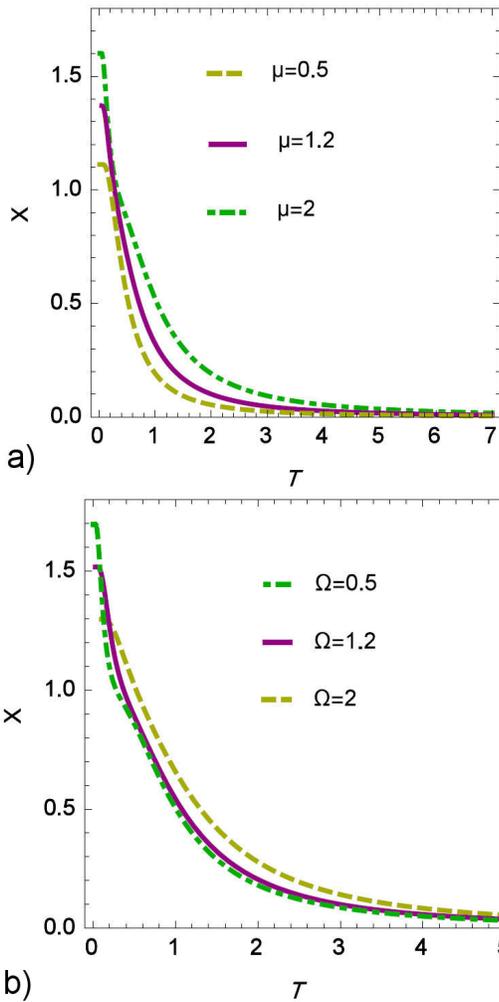


a)



b)

FIGURE 1. The super dense coding capacity as a function of temperature, (a) with external field interaction $\Omega = 1$ and different of value for spin squeezing interaction $\mu$, (b) with spin squeezing interaction $\mu = 2$ and different of value for external field interaction for the spin channel.

operations that Alice(sender) performs on the shared entangled state $\rho$. If the sender does the set of mutually orthogonal unitary transformations, the maximum dense coding capacity $\chi$ can be obtained by

$$\chi = S(\overline{\rho^*}) - S(\rho) \qquad (6)$$

where $S(\overline{\rho^*})$ is an von Neumann entropy for the average state of ensemble of signal states $\overline{\rho^*}$, and $S(\rho)$ is the von Neumann entropy of the quantum channel. If $\chi > 1$ dense coding is valid, and for optimal dense coding $\chi$ must be the maximum, i.e. $\chi_{max} = 2$. In the following, we will use one-axis twisting spin squeezing interaction as a quantum channel to study the optimal dense coding.

Figure 1(a) presents the effects of temperature on dense coding described by Eq. (4) for $\Omega = 1$ and different values of $\mu$. We found that valid dense coding exists at T = 0, whereas with the enhancement of temperature the dense coding capacity decreases quickly to the minimum value. From Fig. 1(a) one can see that valid dense coding vanishes with increasing $\mu$ in the limit of low temperatures. In order to transmit $\rho(T)$ with valid dense coding, we require the value of $\chi$ to be strictly greater than 1. In Fig. 1(b), we consider the strength of the external magnetic field $\mu = 2$ with various values of spin squeezing interaction $\Omega$. It can be said that valid dense coding happens with decreasing $\Omega$ at $\mu = 2$ for the smaller values of temperature. In general from Figs. 1, the influence of $\Omega$ and $\mu$ on the valid dense coding can be obvious when the temperature is increasing from zero. The optimal dense coding vanishing occurs by increasing temperature in the a short range. By choosing the appropriate value for $\Omega$ and $\mu$, we can preserve valid dense coding at the short range of temperature, for which exists valid dense coding by increasing $\Omega$ and $\mu$.

In the following, we tend to study the effects of $\Omega$ and $\mu$ on dense coding capacity at low temperature. We can identify valid dense coding with the help of calculating SQD in the
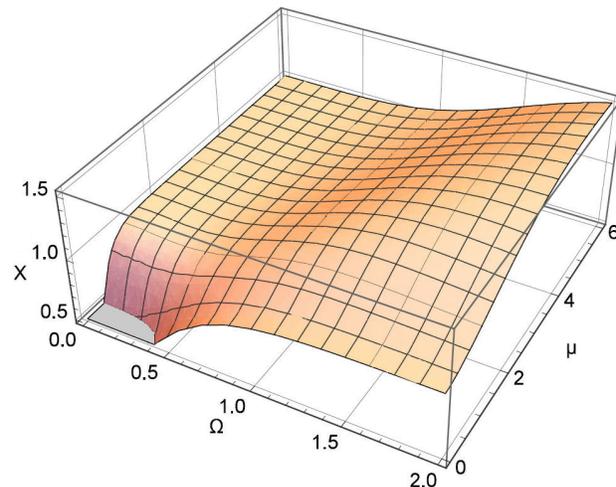


FIGURE 2. The super dense coding capacity as a function of external magnetic interaction $\Omega$, spin squeezing interaction $\mu$ for the spin channel in T = 0.5.
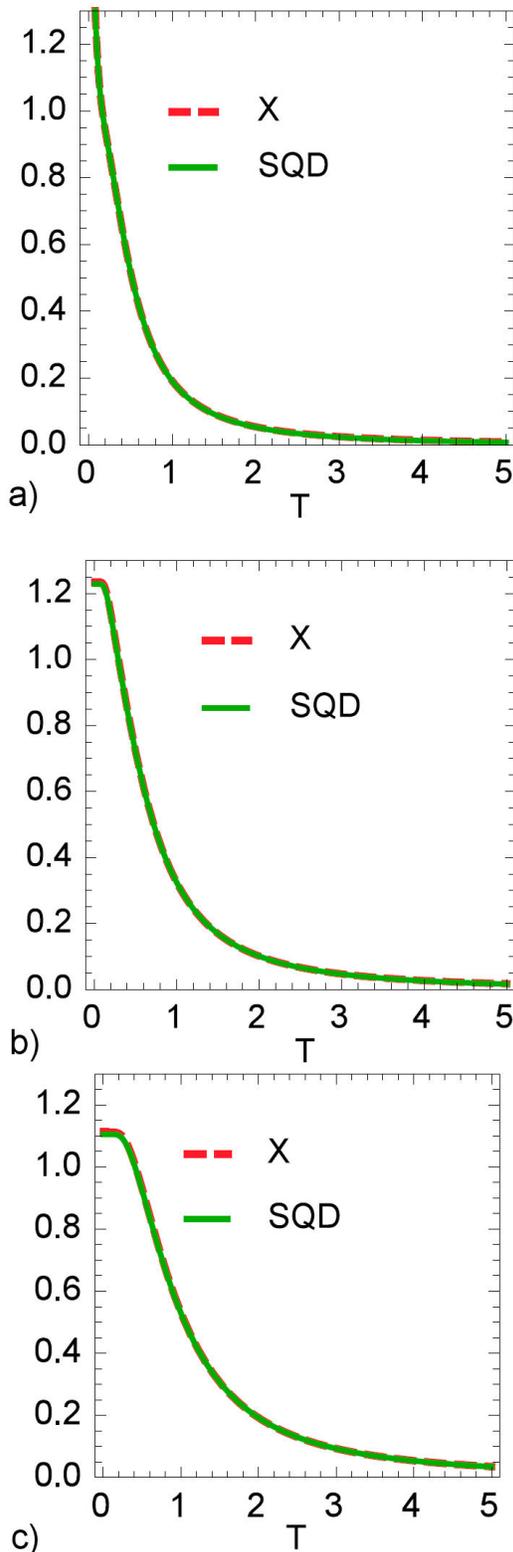
FIGURE 3. The super dense coding capacity and SQD as a function of temperature with spin squeezing interaction $\mu = 1$ with different of values external field (a) $\Omega = 0.5$ (b) $\Omega = 1.2$ (c) $\Omega = 2$.

the quantum channel. Figure 2 indicates that the dense coding capacity is plotted as a function of $\Omega$ and $\mu$ at same tempera-

ture $T = 0.5$. It indicates that thermal dense coding capacity depends on the spin squeezing interaction $\mu$ and the external magnetic interaction $\Omega$ at the same of temperature. One can see clearly that there is no thermal valid dense coding when the strength of the spin squeezing or the strength of the external magnetic filed is zero. Hence, we can say that weaker or stronger external magnetic field will change the valid dense coding for the smaller value of T. We are looking for whether we can identify valid dense coding with the help of calculating the quantum correlation in the quantum channel. Formalism for a special subset of two qubit density operators named X-state is provided to calculate SQD in Refs. [32, 47, 56] or see appendix of this paper. In order to see the relationship between dynamical properties of quantum correlations and thermal dense coding capacity, in Fig. 3 we plot thermal SQD and dense coding capacity for various values of external field $\Omega$ with spin squeezing interaction $\mu$. They show that dense coding capacity is similar to the behavior of SQD. Therefore, the prediction of dense coding behavior is feasible through the thermal properties of SQD in the quantum channel. As a result through the thermal properties of SQD, we can determine when and under what conditions the channel is suitable for dense coding. Moreover, we can see the value of valid dense coding with the dencreasing of external field in $\mu = 1$.

## 4. Conclution

In this work, we have studied the thermal valid dense coding in a system of two spins under an external magnetic field and coupled to each other by a one-axis twisting spin squeezing. We investigated that the thermal properties of SQD how can have important role in predictng valid dense coding. We turn to the main goal of our study, first we investigated the role of squeezing interaction and the external magnetic interaction on dense coding capacity. The results showed that dense coding capacity decreases by increase of temperature. Also, the spin squeezing interaction and the external magnetic interaction have an important role in thermal valid dense coding. Comparing thermal properties of the quantum channel and dense coding capacity leads to an interesting outcome. That we can identify valid dense coding with the help of calculating the SQD in the quantum channel.

## Appendix

### A.

Here, we briefly explain QD and SQD as the measures of quantum correlations used in our investigation. QD was originally defined as the difference between total correlation and the classical correlation and it measures all nonclassical correlations. It prepares us information on the quantum nature of the correlations in a bipartite system without emphasis that it is entangled or separable [40–42]. For a bipartite system $AB$ quantum discord is given by [24]:

$$QD(\rho_{AB}) = I(\rho_{AB}) - C(\rho_{AB}) \qquad (A.1)$$

the quantity $C(\rho_{AB})$ is defined as a measure of classical correlation [43]:

$$C(\rho_{AB}) = S(\rho_A) - S(\rho_{A|B}) \qquad (A.2)$$

Where $\rho_{AB}$ denotes the bipartite density matrix of a composite system $AB$, $\rho_A$ and $\rho_B$ represent the density matrices of parts $A$ and $B$. The quantity $S(\rho) = -tr\rho \log \rho$ refers to the Neumann entropy and $\rho_A = tr_B \rho_{AB}$ represent the reduced states for subsystem $A(B)$, where $tr$ stands for the trace of matrix [44–46]. The quantum conditional entropy is given by

$$S(\rho_{A|B}) = \min_{\Pi_i^B} S(\rho_{A|\Pi_i^B}) = \min_{\Pi_i^B} \sum_i S(\rho_{A|i}) \qquad (A.3)$$

with the minimization being over all projection-valued measurements, $\Pi_i^B$, performed on the subsystem B. Where $\{\Pi_j^B\}$ denotes a complete set of positive operator-valued measure (POVM) performed on the subsystem $B$, in such a way that $\sum_j \Pi_j^B = 1$. The probability for obtaining outcome $i$ is $p_i = tr_{AB}[(I_A \otimes \Pi_i^B)\rho_{AB}(I_A \otimes \Pi_i^B)]$, and the corresponding post measurement state for the subsystem A is

$$\rho_{A|i} = \frac{(I_A \otimes \Pi_i^B)\rho_{AB}(I_A \otimes \Pi_i^B)}{p_i} \qquad (A.4)$$

where $I_A$ is the identity operator on the Hilbert space $H_A$. The total correlation is quantified by the quantum mutual information $I(\rho_{AB})$:

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) + \Sigma_{i=1}^4 \epsilon_i \log_2 \epsilon_i \qquad (A.5)$$

For $X$ state, described by the density matrix Eq. 3, the reduced matrix of $\rho_A$ and $\rho_B$ is given by:

$$S(\rho_A) = -(\rho_{11} + \rho_{22}) \log_2 (\rho_{11} + \rho_{22})$$
$$- (\rho_{33} + \rho_{44}) \log_2 (\rho_{33} + \rho_{44}) \qquad (A.6)$$

$$S(\rho_B) = -(\rho_{11} + \rho_{33}) \log_2 (\rho_{11} + \rho_{33})$$
$$- (\rho_{22} + \rho_{44}) \log_2 (\rho_{22} + \rho_{44}) \qquad (A.7)$$

The eigenvalues of the density matrix $\rho_{AB}$, $\epsilon_i$, are given by:

$$\epsilon_1 = \frac{1}{2} \left[ (\rho_{11} + \rho_{44}) + \sqrt{(\rho_{11} - \rho_{44})^2 + 4|\rho_{14}|^2} \right] \quad (A.8)$$

$$\epsilon_2 = \frac{1}{2} \left[ (\rho_{11} + \rho_{44}) - \sqrt{(\rho_{11} - \rho_{44})^2 + 4|\rho_{14}|^2} \right] \quad (A.9)$$

$$\epsilon_3 = \frac{1}{2} \left[ (\rho_{22} + \rho_{33}) + \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{23}|^2} \right] \quad (A.10)$$

$$\epsilon_4 = \frac{1}{2} \left[ (\rho_{22} + \rho_{33}) - \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{23}|^2} \right] \quad (A.11)$$

The calculation of QD is a complex function due to the optimization over all possible measurements, at present there exists no general analytical expression of discord even for the simplest case of two-qubit state. Hopefully, for the a special bipartite quantum X-state described by the density matrix $\rho$ Eq. (3), the analytical expression of the $QD$ is specified by [47]. Despite some counterexamples have been given in [48, 49], Huang confirmed numerically, in a recent article [50], that the Ali and et al. algorithm is valid with worst case absolute error 0.0021 for two qubit X states. According to proposed scheme proposed in Ref. [47, 51]:

$$QD(\rho_{AB}) = \min(Q_1, Q_2), \qquad (A.12)$$

where,

$$Q_i = H(\rho_{11} + \rho_{33}) + \sum_{i=1}^4 \epsilon_i \log_2 \epsilon_i + D_j,$$

$$D_1 = H\left( \frac{1 + \sqrt{[1 - 2(\rho_{33} + \rho_{44})]^2 + 4(|\rho_{14}| + |\rho_{23}|)^2}}{2} \right),$$

$$D_2 = -\sum_i \rho_{ii} \log_2 \rho_{ii} - H(\rho_{11} + \rho_{33}),$$

$$H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x).$$

It is necessary to say a few important feature to discord. Firstly discord is always non-negative [52]. Secondly, discord is asymmetric under the interchange of $A$ and $B$. The main idea of calculation discord is to extract some information about $A$ by reading the state of $B$ without disturbing of state $A$ in any way. When discord is minimized, we disturb these correlation at least and hence we extract maximum information [53, 54].

Now, let us to define what we call to the SQD [55]. A kind of quantum correlation in quantum information processing is SQD with weak measurement which is always larger than the QD captured by the strong measurement. These are called weak measurement operators because they do not cause complete collapse. The weak measurement operators are given as:

$$P(\mp x) = \sqrt{\frac{1 \pm \tanh x}{2}} \Pi_0 + \sqrt{\frac{1 \mp \tanh x}{2}} \Pi_1 \quad (A.13)$$

where $x$ is the strength parameter of measurement, $\Pi_0$ and $\Pi_1$ are orthogonal projectors that satisfy $\Pi_0 + \Pi_1 = I$. In addition, in the strong measurement limit we have the projective measurement operators, *i.e.*, $\lim_{x\to\infty} P(+x) = \Pi_0$ and $\lim_{x\to\infty} P(-x) = \Pi_1$. For small x, the distance between the initial state and state after the measurement is close to zero, *i.e.*, action of $P(\mp x)$ does not alter the state of the system much. If we replace all projection measurements with weaker ones in classical correlation and quantum discord, it leads to a new type of quantum correlations called SQD. The classical correlation represents the information gained about the subsystem $A$ after performing the measurements $P^B(x) = P(x)$ on subsystem $B$ [52]. The SQD denoted by $D_w(\rho_{AB})$ is defined as:

$$D_w(\rho_{AB}) = S(\rho_B) - S(\rho_{AB})$$
$$+ \min_{\{P^B(x)\}} S_w(A|P^B(x)) \tag{A.14}$$

This is a positive quantity which follows from monotonicity of the mutual information, where the weak quantum conditional entropy is given by:

$$S_w(A|P^B(x)) = p(+x)S(\rho_{A|P^B(+x)})$$
$$+ p(-x)S(\rho_{A|P^B(-x)}), \tag{A.15}$$

with

$$p(\mp x) = tr_{AB}[(I_A \otimes P^B(\pm x))\rho_{AB}(I_A \otimes P^B(\pm x))],$$

and

$$\rho_{A|P^B(\pm x)} = \frac{tr_B[(I_A \otimes P^B(\pm x))\rho_{AB}(I_A \otimes P^B(\pm x))]}{tr_{AB}[(I_A \otimes P^B(\pm x))\rho_{AB}(I_A \otimes P^B(\pm x))]}.$$

where $I_A$ is the identity operator on the Hilbert space $H_A$. Also, $P^B(\pm x)$ is the weak measurement operator performed on subsystem $B$ and $\rho_{A|P^B(\pm x)}$ is the post measurement state for the subsystem $A$ after the weak measurement. Unluckily, $SQD$ is also difficult to calculate. At recent years, some researchers have tried to calculate it. They were able to give only few explicit formulae for general X-type two-qubit states [32,56]. In this article, we will use the method of reference [32]. From Eq. (A.14) the SQD is given by

$$D_w(\rho_{AB}) = p(+x)[\lambda_+ \log_2^{\lambda_+} + \lambda_- \log_2^{\lambda_-}]$$
$$+ p(-x)[\kappa_+ \log_2^{\kappa_+} + \kappa_- \log_2^{\kappa_-}]$$
$$+ S(\rho_B) - S(\rho_{AB})]$$

that

$$\kappa_\pm = \frac{1}{2}\left(1 \pm \frac{\tanh[x]\sqrt{a_1^2 - a_2^2 + (a_3 - c_3 z_3)^2}}{1 - b_3 z_3}\right)$$

$$\lambda_\pm = \frac{1}{2}\left(1 \pm \frac{\tanh[x]\sqrt{a_1^2 + a_2^2 + (a_3 + c_3 z_3)^2}}{1 + b_3 z_3}\right)$$

and

$$p(+x) = \frac{1}{2}(1 + b_3 z_3 \tanh[x])$$

$$p(-x) = \frac{1}{2}(1 - b_3 z_3 \tanh[x])$$

These parameters are determined by the entries of the density matrix,

$$a_3 = \rho_{11} - \rho_{44} - \rho_{22} - \rho_{33}$$
$$b_3 = \rho_{11} - \rho_{44} - \rho_{22} + \rho_{33}$$
$$c_3 = \rho_{11} + \rho_{44} - \rho_{22} - \rho_{33}$$
$$c_1 = 2(\rho_{23} + \rho_{14})$$
$$c_2 = 2(\rho_{23} - \rho_{14})$$
$$a_1 = z_1 Re[c_1] + z_2 Im[c_2]$$
$$a_2 = z_2 Re[c_2] - z_1 Im[c_1]$$

Now, we can compute the minimum value of $D_w(\rho_{AB})$. In order to avoid redundant narrating, we only state that the minimum value lies at $z_1 = z_2 = 0$ and $z_3 = 1$ in our work.

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