

Double and dual numbers. $SU(2)$ groups, two-component spinors and generating functions

G.F. Torres del Castillo

Instituto de Ciencias, Universidad Autónoma de Puebla, 72570 Puebla, Pue., México

K.C. Gutiérrez-Herrera

Facultad de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla, 72570 Puebla, Pue., México.

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We explicitly show that the groups of 2×2 unitary matrices with determinant equal to 1 whose entries are double or dual numbers are homomorphic to $SO(2, 1)$ or to the group of rigid motions of the Euclidean plane, respectively, and we introduce the corresponding two-component spinors. We show that with the aid of the double numbers we can find generating functions for separable solutions of the Laplace equation in the $(2 + 1)$ Minkowski space, which contain special functions that also appear in the solution of the Laplace equation in the three-dimensional Euclidean space, in spheroidal and toroidal coordinates.

Keywords: Double numbers; dual numbers; unitary groups; spinors; Minkowski $(2 + 1)$ space; Laplace's equation; spheroidal coordinates; toroidal coordinates.

Mostramos explícitamente que los grupos de matrices unitarias 2×2 con determinante igual a 1 cuyas entradas son números dobles o números duales son homomorfos a $SO(2, 1)$ o al grupo de movimientos rígidos del plano euclidiano, respectivamente, e introducimos los espinores de dos componentes correspondientes. Mostramos que con la ayuda de los números dobles podemos hallar funciones generatrices para soluciones separables de la ecuación de Laplace en el espacio de Minkowski $(2 + 1)$, las cuales contienen funciones especiales que también aparecen en la solución de la ecuación de Laplace en el espacio euclidiano tridimensional, en coordenadas esféricas y toroidales.

Descriptores: Números dobles; números duales; grupos unitarios; espinores; espacio de Minkowski $(2 + 1)$; ecuación de Laplace; coordenadas esféricas; coordenadas toroidales.

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1. Introduction

Complex numbers are frequently employed in physics even in areas where the quantities of physical interest are real. For instance, the standard spherical harmonics, Y_{lm} , are complex-valued functions that are commonly used in electromagnetism, in spite of the fact that the potentials and electromagnetic fields are real. In some cases, the complex numbers appear in expressions for real-valued functions. Take, for example, the integral representations

$$P_l(\cos \theta) = \frac{1}{2\pi} \int_0^{2\pi} (\cos \theta + i \sin \theta \cos w)^l dw,$$

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(x \sin t - nt)} dt,$$

for the Legendre polynomials and the Bessel functions of integral order, respectively.

There exist other lesser known types of numbers, somewhat similar to the complex ones, called *double* and *dual* numbers (although other names are also employed), which are also useful in various applications (see, e.g., Refs. [1–10] and the references cited therein). The double and the dual numbers (introduced by Clifford in 1873 [5]) are characterized by the presence of “imaginary units” j and ε , respectively, such that $j^2 = 1$ and $\varepsilon^2 = 0$. Most of the existing

applications in physics for these numbers are restricted to the double numbers (see, e.g., [1–4, 6, 7]), in many cases looking for new theories that might substitute the established ones. As shown in Refs. [8–10], the double and the dual numbers are useful in the solution of well-established systems of real differential equations.

At first sight, it would seem that the use of a unit, j , whose square is $+1$ is hardly necessary or useful (for instance, it is not required in the solution of algebraic equations with real coefficients), but the examples given in Refs. [1–4, 6–10], together with those presented below, show that having a second unit, besides the real number 1, is indeed very useful.

It is well known that $SU(2)$, the group formed by the 2×2 complex unitary matrices with determinant equal to 1, is homomorphic to $SO(3)$, the group of rotations in the three-dimensional Euclidean space; this homomorphism is useful and physically relevant. For instance, under a rotation, the wavefunctions (spinors) of spin-1/2 particles in non-relativistic quantum mechanics, are transformed by means of a $SU(2)$ matrix. In this paper we explicitly show that the analogs of $SU(2)$, with double or dual numbers as entries, in place of complex numbers, are homomorphic to $SO(2, 1)$ (the group of Lorentz transformations in two spatial dimensions) or the group of rigid motions of the Euclidean plane, respectively. In fact, as shown below, the three cases (complex, double and dual) can be treated simultaneously and the

corresponding two-component spinors can be defined in a unified manner.

We also show that, making use of the double numbers, we can find a generating function for certain functions analogous to the associated Legendre functions, which arise in the solution of the Laplace equation in the three-dimensional Euclidean space, in spheroidal and toroidal coordinates.

In Sec. 2 we study the special unitary groups formed by 2×2 matrices whose entries are complex, double or dual numbers; we show that they are homomorphic to SO(3), SO(2, 1) or SE(2) (the group of rigid motions of the Euclidean plane that preserve the orientation), respectively, and we introduce the corresponding two-component spinors.

In Sec. 3 we study separable solutions of the Laplace equation in the Minkowski $(2 + 1)$ space, and we find that some of the separated equations coincide with some of the separated equations found in the solution by separation of variables of the Laplace equation in the three-dimensional Euclidean space in spheroidal and toroidal coordinates. In Sec. 4 we show that, making use of the double numbers, one obtains a generating function for the special functions analogous to the associated Legendre functions encountered at Sec. 3. The basic rules for the use of the double and the dual numbers can be found in Refs. [8, 10]; a rigorous and fairly complete discussion of their algebraic properties can be found, *e.g.*, in Refs. [4–7].

2. Special unitary groups

In Sec. 3.1 of Ref. [8] the Hamiltonian

$$H = \frac{1}{2m}(p_x^2 + p_y^2) - \frac{\hbar^2}{2}m\omega^2(x^2 + y^2), \quad (1)$$

was considered. Here m and ω are constants, and \hbar may be the usual imaginary unit, i ; the hypercomplex unit j (which satisfies the condition $j^2 = 1$); or the hypercomplex unit ε (which satisfies $\varepsilon^2 = 0$). (That is, \hbar^2 is equal to -1 , $+1$, or 0 , respectively, so that, in all cases, H is real.) This Hamiltonian is invariant under the group of 2×2 matrices of the form

$$U = \begin{pmatrix} a + \hbar b & c + \hbar d \\ -c + \hbar d & a - \hbar b \end{pmatrix}, \quad (2)$$

where a, b, c, d are real numbers satisfying the condition

$$a^2 + c^2 - \hbar^2(b^2 + d^2) = 1. \quad (3)$$

The matrices of the form (2), fulfilling Eq. (3), form a group with the usual matrix multiplication. In what follows, this group will be denoted by $SU(2)_\hbar$. When $\hbar = i$ this group is the usual SU(2) group which, as is well known, is homomorphic to the rotation group in three dimensions, SO(3).

The invariance of the Hamiltonian (1) under the group $SU(2)_\hbar$ is evident if one expresses H in the form

$$H = \frac{1}{2m}\Psi^\dagger\Psi, \quad (4)$$

where Ψ is the two-component spinor

$$\Psi \equiv \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \equiv \begin{pmatrix} p_x + \hbar m\omega x \\ p_y + \hbar m\omega y \end{pmatrix} \quad (5)$$

and the Hermitian adjoint of a matrix is defined as the transpose of the conjugate matrix (in all cases, the conjugate of $a + \hbar b$ is defined by $\overline{a + \hbar b} = a - \hbar b$). Indeed, as a consequence of (3), the matrix (2) satisfies $U^\dagger U = I$ and therefore (4) is invariant under the transformation $\Psi \mapsto U\Psi$.

The group $SU(2)_\hbar$ is a (real) Lie group of dimension three (the matrix (2) depends on four parameters, which are restricted by one equation) and a basis for its Lie algebra is given by the matrices

$$\begin{aligned} \sigma_1 &\equiv \begin{pmatrix} \hbar & 0 \\ 0 & -\hbar \end{pmatrix}, & \sigma_2 &\equiv \begin{pmatrix} 0 & \hbar \\ \hbar & 0 \end{pmatrix}, \\ \sigma_3 &\equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \quad (6)$$

The matrices σ_i are antihermitian because $\sigma_i^\dagger = -\sigma_i$. The trace of each matrix (6) is zero and these matrices form a basis for the *real* vector space formed by the 2×2 antihermitian matrices with trace equal to zero. The commutators of these matrices are given by $[\sigma_i, \sigma_j] = c_{ij}^k \sigma_k$, with sum over repeated indices, where the structure constants c_{ij}^k are determined by

$$c_{12}^3 = 2\hbar^2, \quad c_{23}^1 = -2, \quad c_{31}^2 = -2 \quad (7)$$

(note that all of them are real, despite the fact that the matrices (6) are not all real).

The products of the matrices (6) can be expressed in the compact form

$$\sigma_i \sigma_j = g_{ij} I + \frac{1}{2} c_{ij}^k \sigma_k, \quad (8)$$

where I is the unit 2×2 matrix and the g_{ij} are the entries of the diagonal 3×3 matrix

$$(g_{ij}) = \text{diag}(\hbar^2, \hbar^2, -1). \quad (9)$$

Thus, the matrix (g_{ij}) is singular only if $\hbar = \varepsilon$. With the aid of the matrices (6) we can construct a homomorphism between $SU(2)_\hbar$ and a subgroup of $SL(3, \mathbb{R})$ (the group of 3×3 real matrices with determinant equal to 1). In fact, if $g \in SU(2)_\hbar$, then the product $g\sigma_i g^{-1}$ is also a traceless antihermitian matrix. (In fact, $\text{tr}(g\sigma_i g^{-1}) = \text{tr}\sigma_i = 0$ and $(g\sigma_i g^{-1})^\dagger = (g^{-1})^\dagger \sigma_i^\dagger g^\dagger = g(-\sigma_i)g^{-1} = -g\sigma_i g^{-1}$, since the elements of $SU(2)_\hbar$ are unitary matrices.) Hence, there exist real numbers a_i^j such that

$$g\sigma_i g^{-1} = a_i^j \sigma_j. \quad (10)$$

The mapping $g \mapsto (a_i^j)$ given by Eq. (10) is a group homomorphism. In fact, if g' is another element of $SU(2)_\hbar$ then

there exists a matrix (b_j^i) such that $g'\sigma_i g'^{-1} = b_j^i \sigma_j$ and therefore,

$$\begin{aligned} (gg')\sigma_i(gg')^{-1} &= g(g'\sigma_i g'^{-1})g^{-1} = gb_j^i \sigma_j g^{-1} \\ &= b_j^i a_j^k \sigma_k = (a_j^k b_i^j) \sigma_k, \end{aligned}$$

thus showing that the product of the matrix (a_i^j) by (b_i^j) corresponds to the product gg' . (a_i^j is the entry at the j -th row and i -th column of the matrix (a_i^j) .)

In the cases where h is equal to i or to j , the matrices (a_j^i) preserve the metric tensor (g_{ij}) (see Eq. (11), below). Indeed, from Eqs. (8) and (10) we have

$$g(\sigma_i \sigma_j) g^{-1} = g_{ij} I + \frac{1}{2} c_{ij}^k a_k^m \sigma_m,$$

which must coincide with (using Eqs. (10) and (8) again)

$$\begin{aligned} g(\sigma_i \sigma_j) g^{-1} &= g \sigma_i g^{-1} g \sigma_j g^{-1} = a_i^k \sigma_k a_j^l \sigma_l \\ &= a_i^k a_j^l (g_{kl} I + \frac{1}{2} c_{kl}^m \sigma_m). \end{aligned}$$

By virtue of the linear independence of the set $\{I, \sigma_1, \sigma_2, \sigma_3\}$, this amounts to

$$a_i^k a_j^l g_{kl} = g_{ij} \quad (11)$$

and

$$a_i^k a_j^l c_{kl}^m = c_{ij}^k a_k^m. \quad (12)$$

Multiplying both sides of Eq. (12) by $a_s^n g_{mn}$ we obtain

$$a_s^n g_{mn} a_i^k a_j^l c_{kl}^m = c_{ij}^k a_k^m a_s^n g_{mn}$$

or, using Eq. (11),

$$g_{mn} c_{kl}^m a_s^n a_i^k a_j^l = g_{ks} c_{ij}^k. \quad (13)$$

We introduce the real constants

$$c_{sij} \equiv g_{ks} c_{ij}^k$$

and from Eqs. (7) and (9) we find that $c_{sij} = -2h^2 \varepsilon_{sij}$. Therefore, if $h \neq \varepsilon$, Eq. (13) is equivalent to $\varepsilon_{nkl} a_s^n a_i^k a_j^l = \varepsilon_{sij}$, which means that $\det(a_i^j) = 1$. An explicit computation shows that also in the case where $h = \varepsilon$, $\det(a_i^j) = 1$ (see Eq. (14) below).

Thus, if $h = i$ or $h = j$, the matrix (a_i^j) is orthogonal or pseudo-orthogonal; respectively, that is, (a_i^j) belongs to $SO(3)$ or to $SO(2, 1)$. In the remaining case, where $h = \varepsilon$, condition (3) reads $a^2 + c^2 = 1$, and therefore we can parameterize a and c in the form $a = \cos \theta/2$, $c = \sin \theta/2$, then, a straightforward computation shows that if $g \in SU(2)_h$ has the form (2), making use of (6) and (10),

$$(a_i^j) = \begin{pmatrix} \cos \theta & \sin \theta & 2(b \sin \theta/2 - d \cos \theta/2) \\ -\sin \theta & \cos \theta & 2(b \cos \theta/2 + d \sin \theta/2) \\ 0 & 0 & 1 \end{pmatrix}, \quad (14)$$

which represents an orientation-preserving rigid motion of the Euclidean plane. (If x, y are the coordinates of a point

of the Euclidean plane with respect to a set of Cartesian axes, then x', y' given by

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta & x_0 \\ -\sin \theta & \cos \theta & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

are the coordinates with respect to the same axes of the point obtained by rotating the plane through an angle θ in the clockwise direction about the origin and then translating the points of the plane by the vector (x_0, y_0) .)

2.1. Two-component spinors

Traditionally, spinors are associated with the orthogonal or pseudo-orthogonal groups, $SO(n)$ or $SO(p, q)$, and the most important examples are related to the $SO(3)$ group (*e.g.*, in the description of the spin for a spin-1/2 particle) and the $SO(3, 1)$ group (*e.g.*, in the spinor formalism employed in special or general relativity and in the Dirac equation for the electron). The basic (one-index) spinors form representation spaces for the spin groups, which are covering groups of the orthogonal or pseudo-orthogonal groups. In the standard approach, the spin groups are represented by complex matrices (belonging, *e.g.*, to $SU(2)$ or to $SL(2, \mathbb{C})$) and the one-index spinors have complex components.

As we shall show below, this can be modified in two ways: instead of spinors with complex components, we can consider spinors whose components are double or dual numbers and, instead of orthogonal or pseudo-orthogonal groups, we can consider “inhomogeneous” groups, specifically, the group of rigid motions of the Euclidean plane (formed by rotations and translations on the Euclidean plane).

A two-component spinor will be represented by a column of the form

$$\Psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \equiv \begin{pmatrix} a + hb \\ c + hd \end{pmatrix}, \quad (15)$$

just as in Eq. (5), where a, b, c, d are real numbers. Under the change of spinor frame given by $g \in SU(2)_h$, the components (15) transform according to

$$\Psi \mapsto g\Psi. \quad (16)$$

Hence, $\Psi^\dagger \Psi$ is invariant under these transformations. (Note that $\Psi^\dagger \Psi = \overline{\psi^1} \psi^1 + \overline{\psi^2} \psi^2$ is always real, but only in the case where $h = i$ it is positive definite.)

A non-zero spinor Ψ defines a vector belonging to a *real* vector space of dimension three with components R_i given by

$$hR_i = \Psi^\dagger \sigma_i \Psi \quad i = 1, 2, 3. \quad (17)$$

In fact, since $\Psi^\dagger \sigma_i \Psi$ is a 1×1 matrix, the conjugate of the right-hand side of Eq. (17) is equal to

$$(\Psi^\dagger \sigma_i \Psi)^\dagger = \Psi^\dagger \sigma_i^\dagger \Psi = -\Psi^\dagger \sigma_i \Psi,$$

which shows that the numbers R_i are indeed real. Making use of the explicit expression of the matrices σ_i [see Eqs. (6)], one finds that

$$R_1 = \overline{\psi^1}\psi^1 - \overline{\psi^2}\psi^2, \quad R_2 + hR_3 = 2\overline{\psi^1}\psi^2. \quad (18)$$

According to Eqs. (17) and (10), under (16) the components R_i transform as

$$hR_i \mapsto (g\Psi)^\dagger \sigma_i (g\Psi) = \Psi^\dagger g^{-1} \sigma_i g \Psi = \Psi^\dagger \tilde{a}_i^j \sigma_j \Psi = h\tilde{a}_i^j R_j,$$

where (\tilde{a}_i^j) is the inverse of the matrix (a_i^j) . Then, denoting by (g^{ij}) the inverse of the matrix (g_{ij}) (which exists only when h is i or j), from Eq. (11) it follows that $g^{ij} R_i R_j$ is invariant. In fact, when $h = i$ we find that $g^{ij} R_i R_j = -(\Psi^\dagger \Psi)^2$, while in the case where $h = j$, $g^{ij} R_i R_j = (\Psi^\dagger \Psi)^2$.

In the case where $h = \varepsilon$, from Eq. (14) we see that the 2×2 block at the upper left corner of the matrix (a_i^j) represents an ordinary rotation about the origin on the plane $R_3 = 0$, hence, under these transformations $(R_1)^2 + (R_2)^2$ is invariant and one finds that $(R_1)^2 + (R_2)^2 = (\Psi^\dagger \Psi)^2$.

Owing to the specific form of the matrices (2) (the entries at the second row are, up to a sign, the conjugates of those at the first row) one readily finds that the components of

$$\hat{\Psi} \equiv \begin{pmatrix} -\overline{\psi^2} \\ \psi^1 \end{pmatrix} \quad (19)$$

transform in the same manner as the components of Ψ . The two-component spinor $\hat{\Psi}$ is the *mate* of Ψ as defined in Ref. [11] for the case where $h = i$, and differs by a constant factor from the definition of the mate of a spinor in a space with indefinite metric given there. In the applications of the spinors in quantum mechanics (where $h = i$), the spinor $\hat{\Psi}$ represents a state with the spin in the opposite direction to that corresponding to Ψ . (In the Bloch sphere, Ψ and $\hat{\Psi}$ correspond to diametrically opposite points.)

As an illustration of the differences between the three types of numbers considered here, we look for spinors Ψ which are proportional to their mates (with the proportionality factor being a complex, double, or dual number according to the case at hand): $\hat{\Psi} = \lambda\Psi$. According to Eq. (19), we have $-\overline{\psi^2} = \lambda\psi^1$, $\overline{\psi^1} = \lambda\psi^2$, which leads to $\overline{\lambda\lambda} = -1$. This condition cannot be satisfied in the case of the complex or the dual numbers; however, in the case of the double numbers it has the general solution $\lambda = \pm j e^{j\alpha}$, with $\alpha \in \mathbb{R}$ arbitrary (note that $j\bar{j} = (-j)j = -1$), furthermore, in that case, $\Psi^\dagger \Psi = 0$.

The spinor formalism can be employed in the study of differential geometry (see, e.g., Ref. [11]) and, according to the results of this section, it is possible to develop a unified formalism applicable to three-dimensional Riemannian manifolds of any signature.

The one-index spinors are the basic objects from which vectors and tensors of any rank can be constructed; a field of an arbitrary spin can be expressed in terms of its spinor

components; the basic transformation rule (16) determines the transformation of a spinor with any number of indices or, equivalently, of any spin (see, e.g., Ref. [11]).

We close this section with some remarks of a more formal nature. By contrast with the real and the complex numbers, which are fields with the usual operations of sum and product, the double and the dual numbers are only commutative rings with identity. The standard definition of a vector space makes use of a field of scalars; its analog in the case of a ring is called a module (see, e.g., Ref. [12]).

3. The Laplace equation in the Minkowski (2+1) space

In this section we shall consider the Laplace equation for the Minkowski (2+1) space, which is a three-dimensional space with a metric tensor given, e.g., by

$$ds^2 = -dx^2 + dy^2 + dz^2, \quad (20)$$

in terms of an appropriate coordinate system (similar to the Cartesian coordinate systems of the Euclidean space). Instead of the coordinates (x, y, z) appearing in Eq. (20), we can make use of the *local* coordinates (r, θ, ϕ) defined by

$$\begin{aligned} x &= r \sinh \theta \cosh \phi, \\ y &= r \sinh \theta \sinh \phi, \\ z &= r \cosh \theta, \end{aligned} \quad (21)$$

in terms of which the metric tensor (20) takes the form

$$ds^2 = -r^2 d\theta^2 + dr^2 + r^2 \sinh^2 \theta d\phi^2. \quad (22)$$

This last expression shows that the coordinates (r, θ, ϕ) can be regarded as orthogonal, so that the standard formula for the Laplace operator in orthogonal coordinates is applicable, taking care of the minus signs. The Laplace equation in this case (which is just the wave equation in two spatial dimensions) is given by

$$-\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (23)$$

or, equivalently,

$$\begin{aligned} -\frac{1}{r^2 \sinh \theta} \frac{\partial}{\partial \theta} \left(\sinh \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) \\ + \frac{1}{r^2 \sinh^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0. \end{aligned} \quad (24)$$

Equation (24) admits separable solutions $R(r)\Theta(\theta)\Phi(\phi)$, where $\Theta(\theta)$ has to satisfy the equation

$$\begin{aligned} \frac{1}{\sinh \theta} \frac{d}{d\theta} \left(\sinh \theta \frac{d\Theta}{d\theta} \right) \\ - \left[l(l+1) + \frac{m^2}{\sinh^2 \theta} \right] \Theta = 0, \end{aligned} \quad (25)$$

and l, m are separation constants. This equation appears in the solution by separation of variables of the Laplace equation in the three-dimensional Euclidean space in prolate spheroidal equations (with l being an integer) (see, e.g., Ref. [13], Eq. (8.6.7) or Ref. [14], Table 1.06) and in toroidal coordinates (where l is a half-integer) (see, e.g., Ref. [13], Eq. (8.10.11) or Ref. [14], Sec. IV).

Now, in place of (21), we define the local coordinates (r, θ, ϕ) by

$$\begin{aligned} x &= r \cosh \theta \cosh \phi, \\ y &= r \cosh \theta \sinh \phi, \\ z &= r \sinh \theta, \end{aligned} \tag{26}$$

and we find that the metric tensor (20) takes the form

$$ds^2 = -dr^2 + r^2 d\theta^2 + r^2 \cosh^2 \theta d\phi^2. \tag{27}$$

Hence, the Laplace equation (23) is now given by

$$\begin{aligned} -\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \cosh \theta} \frac{\partial}{\partial \theta} \left(\cosh \theta \frac{\partial u}{\partial \theta} \right) \\ + \frac{1}{r^2 \cosh^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0. \end{aligned} \tag{28}$$

Equation (28) admits separable solutions $R(r)\Theta(\theta)\Phi(\phi)$, where $\Theta(\theta)$ has to satisfy the equation

$$\frac{1}{\cosh \theta} \frac{d}{d\theta} \left(\cosh \theta \frac{d\Theta}{d\theta} \right) - \left[l(l+1) - \frac{m^2}{\cosh^2 \theta} \right] \Theta = 0, \tag{29}$$

and l, m are separation constants. Equation (29) coincides with one of the separated equations obtained in the solution by separation of variables of the Laplace equation in the three-dimensional Euclidean space, in oblate spheroidal coordinates (see, e.g., Ref. [13], Eq. (8.6.13) or Ref. [14], Table 1.07).

Thus, even though the Minkowski $(2 + 1)$ space may not seem as interesting as the three-dimensional Euclidean space or the standard Minkowski $((3+1))$ space, as we have shown, the solution of the differential equation (23) is relevant to the solution of the Laplace equation in the three-dimensional Euclidean space.

4. Generating functions

In this section we shall show that one can obtain solutions of Eqs. (25) and (29) by means of generating functions. The starting point is the one employed in Ref. [15]: if the metric tensor of the space has the form $ds^2 = g_{ij} dx^i dx^j$, with sum over repeated indices, $i, j, \dots = 1, 2, \dots, p$, and the components g_{ij} are constant, then the function $(k_1 x^1 + k_2 x^2 + \dots + k_p x^p)^l$ is a solution of the Laplace equation if and only if the constants k_1, k_2, \dots, k_p satisfy the condition

$$g^{ij} k_i k_j = 0, \tag{30}$$

where (g^{ij}) is the inverse of the matrix (g_{ij}) . The proof is a straightforward computation, taking into account that, in these coordinates, the Laplacian is given by $\nabla^2 u = g^{ij} (\partial/\partial x^i) (\partial u/\partial x^j)$.

4.1. Generating solutions of Eq. (25)

In the case of the three-dimensional Euclidean space, with $g_{ij} = \delta_{ij}$, condition (30) reads $(k_1)^2 + (k_2)^2 + (k_3)^2 = 0$, and therefore one is led to make use of complex numbers [15], but in the case of the Minkowski $(2 + 1)$ space, in the coordinates (x, y, z) appearing in Eq. (20), condition (30) takes the form

$$-(k_1)^2 + (k_2)^2 + (k_3)^2 = 0, \tag{31}$$

which can be satisfied by nonzero real numbers k_1, k_2, k_3 . However, as we shall see, it is convenient to make use of double numbers:

$$(k_1, k_2, k_3) = (j \cosh v, -j \sinh v, 1), \tag{32}$$

where v is an auxiliary parameter. Since, by definition, $j^2 = 1$, the components (32) satisfy condition (31) for all values of v . Then, making use of Eqs. (21) and the fact that $e^{jv} = \cosh v + j \sinh v$, we have

$$\begin{aligned} k_1 x + k_2 y + k_3 z &= r (j \cosh v \sinh \theta \cosh \phi \\ &\quad - j \sinh v \sinh \theta \sinh \phi + \cosh \theta) \\ &= r [\cosh \theta + j \sinh \theta (\cosh \phi \cosh v - \sinh \theta \sinh v)] \\ &= r [\cosh \theta + j \sinh \theta \cosh(\phi - v)] \\ &= r [\cosh \theta + \frac{1}{2} j \sinh \theta e^{-j(\phi-v)} \\ &\quad + \frac{1}{2} j \sinh \theta e^{j(\phi-v)}], \end{aligned} \tag{33}$$

which shows that $k_1 x + k_2 y + k_3 z$ depends on ϕ and v only through the difference $\phi - v$.

With the aid of (33), one can convince oneself that, for $l = 0, 1, 2, \dots$, the expression $(k_1 x + k_2 y + k_3 z)^l$ must be of the form

$$(k_1 x + k_2 y + k_3 z)^l = \sum_{m=-l}^l j^{|m|} r^l f_{lm}(\theta) e^{jm\phi} e^{-jmv}. \tag{34}$$

Since $(k_1 x + k_2 y + k_3 z)^l$ must be a solution of the Laplace equation in the Minkowski $(2 + 1)$ space and the parameter v is arbitrary, it follows that each term of (34) is a separable solution of the Laplace equation in the Minkowski $(2 + 1)$ space. In particular, this means that $f_{lm}(\theta)$ is a solution of Eq. (25). (The factor $j^{|m|}$ is included in order for $f_{lm}(\theta)$ to be a real-valued function.)

Thus, setting $v = 0$ in Eq. (34), we obtain the generating function

$$\begin{aligned} (\cosh \theta + \frac{1}{2} j \sinh \theta e^{-j\phi} + \frac{1}{2} j \sinh \theta e^{j\phi})^l \\ = \sum_{m=-l}^l j^{|m|} f_{lm}(\theta) e^{jm\phi}. \end{aligned} \tag{35}$$

(In the case of the spherical harmonics, considered in Ref. [15], it was useful to keep the factor analogous to e^{-jmv} , in order to obtain integral expressions for the spherical harmonics and the Legendre polynomials.) As a simple example, we have

$$\begin{aligned} \left(\cosh \theta + \frac{1}{2} j \sinh \theta e^{-j\phi} + \frac{1}{2} j \sinh \theta e^{j\phi} \right)^2 &= \frac{1}{4} \sinh^2 \theta e^{-2j\phi} \\ &+ j \sinh \theta \cosh \theta e^{-j\phi} + \cosh^2 \theta + \frac{1}{2} \sinh^2 \theta \\ &+ j \sinh \theta \cosh \theta e^{j\phi} + \frac{1}{4} \sinh^2 \theta e^{2j\phi}, \end{aligned}$$

which shows that the functions $f_{2,\pm 2}(\theta)$, $f_{2,\pm 1}(\theta)$, and $f_{20}(\theta)$, are proportional to $\sinh^2 \theta$, $\sinh \theta \cosh \theta$, and $2 \cosh^2 \theta + \sinh^2 \theta$, respectively.

4.2. Generating solutions of Eq. (29)

Following essentially the same steps as in the preceding subsection, with (k_1, k_2, k_3) given again by (32), but using now the coordinates (26) we have

$$\begin{aligned} k_1 x + k_2 y + k_3 z &= r \left[\sinh \theta + \frac{1}{2} j \cosh \theta e^{-j(\phi-v)} \right. \\ &\left. + \frac{1}{2} j \cosh \theta e^{j(\phi-v)} \right] \end{aligned} \quad (36)$$

and, therefore, for $l = 0, 1, 2, \dots$, we have

$$\begin{aligned} r^l \left(\sinh \theta + \frac{1}{2} j \cosh \theta e^{-j\phi} + \frac{1}{2} j \cosh \theta e^{j\phi} \right)^l \\ = \sum_{m=-l}^l j^{|m|} r^l h_{lm}(\theta) e^{jm\phi}, \end{aligned} \quad (37)$$

where each term of the sum on the right-hand side is a separable solution of the Laplace equation in the Minkowski $(2+1)$ space and the functions $h_{lm}(\theta)$ are solutions of Eq. (29).

5. Final remarks

The real groups found in Sec. 2 are related by means of a contraction in the sense defined in Ref. [16]; however, as we have shown, the use of the complex, double and dual numbers allows us to study these groups simultaneously, without having to take limits.

An advantage of the use double and the dual numbers is that their basic algebraic rules are the same as those of the real or the complex numbers. This means that we can do many computations that are equally applicable to complex, double or dual quantities.

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