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Arbitrary ℓ-solutions of the Schrödinger equation interacting with Hulthén-Hellmann potential model

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In this study, we obtain bound state solutions of the radial Schrödinger equation by the superposition of Hulthén and Hellmann potentials within the framework of Nikiforov-Uvarov (NU) method for arbitrary-state, with the Greene-Aldrich approximation for the centrifugal term. We also obtain the corresponding normalized wave functions expressed in terms of Jacobi polynomials for a particle exposed to this potential field. Explicitly, we have computed the numerical energy eigenvalues of different quantum states. Besides, we consider six exceptional cases of the potential and obtained their energy eigenvalues. Our results are in excellent agreement with the results in the literature. The behavior of the energy for the ground state and several excited states is illustrated graphically.

Keywords: Schrödinger equation; Nikiforov-Uvarov method; eigenvalues; eigenfunction; Hulthén-Hellmann potential.

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1. Introduction

In the earlier study of quantum mechanics, the Schrödinger equation was introduced as a second-order differential equation capable of describing the properties of a non-relativistic Quantum system[1-2]. Studies have revealed that exact solutions of the Schrödinger equation are available only for a limited set of physical and chemical quantum mechanical systems [3-4]. Hence, with an approximation to the centrifugal term, interest is geared towards arbitrary ℓ-solutions of the radial Schrödinger equation. Arbitrary ℓ-solutions play a dominant role in non-relativistic quantum mechanics since the wave function and associated eigenvalues contain all the necessary information for a full description of a quantum system [5-8]. With the experimental verification of the Schrödinger equation, researchers have devoted much interest in solving the radial Schrödinger equation to obtain bound state solutions with various methods for some potential models [9-25].

The Hulthén potential [26] is one of the essential short-range potentials in physics. Its application to diverse areas of physics, including nuclear and particle physics, atomic physics, condensed matter physics, and chemical physics, has been of great interest in recent times [27-29]. The study of this potential is essential in investigating the interaction existing between two particles [30]. The superposition of Yukawa and Coulomb potential, otherwise known as Hellmann potential [31], has been extensively used by many authors to obtain the energy of the bound state in atomic, nuclear, and particle physics [32-41]. The Hellmann potential is useful in the study of transitions of alkali metal molecules in condensed matter physics.

Recently, a lot of interest has been directed towards combining two or more potentials in relativistic and non-relativistic regimes [35]. The essence of combining two or more physical potential models is to have a broader range of applications [36]. Hence, in the present study, we attempt to study the radial Schrödinger equation with a newly proposed potential, obtained from a combination of Hulthén and Hellmann's potential (HHP). The potential is of the form:

$$V(r) = -\frac{\Lambda_1 e^{-\alpha r}}{1-e^{-\alpha r}} - \frac{\Lambda_2}{r} + \frac{\Lambda_3 e^{-\alpha r}}{r}, \qquad (1)$$

where r represents the inter-nuclear distance; Λ_1, Λ_2, and Λ_3, and are the strengths of Hulthén, Coulomb, and Yukawa potentials, respectively, and α is the screening parameter. The shape of this potential as a function of r for varying screening parameters is presented in Fig. 1. It is obvious that the

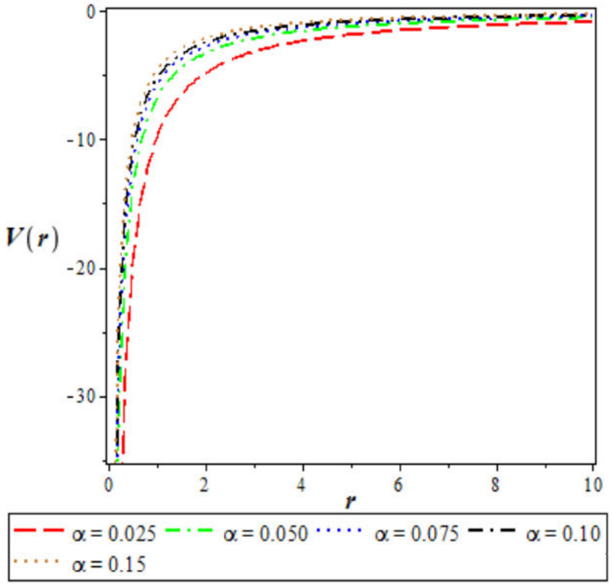

FIGURE 1. The shape of Hulthén-Hellmann's potential as a function of r for different screening parameters. We choose $\Lambda_1 = 0.015$, $\Lambda_2 = 2$ and $\Lambda_3 = -2$.

developed potential model works perfectly for both small and medium inter-nuclear distance.

2. Review of Nikiforov-Uvarov method

In this section, we briefly introduce the NU method. The details can be found in the Nikiforov-Uvarov handbook [42]. This method is useful to solve the second-order differential wave equation of the hypergeometric type:

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0, \tag{2}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials at most second-degree and $\tilde{\tau}(s)$ is a first-degree polynomial. The particular solution of Eq. (2) can be obtained by decomposing the wave function $\psi(s)$ as follows:

$$\psi(s)=\phi(s)y(s), \tag{3}$$

leading to a hypergeometric-type equation

$$\sigma(s)y''(s)+\tau(s)y'(s)+\lambda y(s)=0. \tag{4}$$

The first part of the wave functions $\psi(s)$ is the solution of the differential equation,

$$\sigma(s)\phi'(s)-\pi(s)\phi(s)=0, \tag{5}$$

where

$$\tau(s)=\tilde{\tau}(s)+2\pi(s), \tag{6}$$

and λ in Eq. (4) is the parameter defined as

$$\lambda=\lambda_n=-n\tau'(s)+\frac{n(n-1)}{2}\sigma''(s)=0,$$
$$(n=0,1,2,...). \tag{7}$$

The $\tau(s)$ is a polynomial function of the parameter s whose first derivative $\tau'(s)$ must be negative. This is the essential condition in choosing the proper solutions. That is:

$$\frac{d\tau(s)}{ds}<0. \tag{8}$$

The second part of the wave functions in Eq. (3) is a hypergeometric-type function obtained by Rodrigues relation:

$$y_n(s)=\frac{B_n(s)}{\rho(s)}\frac{d^n}{ds^n}[\sigma^n(s)\rho(s)], \tag{9}$$

where $B_n(s)$ is a constant related to normalization, and the weight function $\rho(s)$ can be found by [42]

$$\frac{d}{ds}[\sigma(s)\rho(s)]=\tau(s)\rho(s). \tag{10}$$

We define the function $\pi(s)$ and the parameter λ as

$$\pi(s)=\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\pm\sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2-\tilde{\tau}(s)+k\sigma(s)}, \tag{11}$$

and

$$\lambda=k+\pi'(s), \tag{12}$$

where $\pi(s)$ has to be a polynomial of degree at most one.

The discriminant under the square root sign in Eq. (10) must be set to zero and then has to be solved for k [42]. Finally, by solving Eqs. (7) and (12), the energy eigenvalue equation can be obtained.

2.1. Arbitrary ℓ-solution to the radial Schrödinger equation

To obtain the solution of the Schrödinger-like equation given in Eq. (2), we write the radial Schrödinger equation of the form [15,41,43]

$$\frac{d^2\psi_{n\ell}(r)}{dr^2}+\frac{2\mu}{\hbar^2}\times\left[E_{n\ell}-V(r)-\frac{\hbar^2\ell(\ell+1)}{2\mu r^2}\right]\psi_{n\ell}(r)=0, \tag{13}$$

where we have defined μ as the reduced mass, E_{nl} as the energy spectrum, $\hbar$ as the reduced Planck's constant, $V(r)$ as the interacting potential, $\ell(\ell+1)/r^2$ as the centrifugal term, n and l as the radial and orbital angular momentum quantum numbers, respectively. When $\ell=0$, Eq. (13) with HHP can be exactly solved, but for the case $\ell\neq 0$, Eq. (13) cannot be solved exactly with the proposed potential. So we introduce the Greene-Aldrich approximation scheme [4,15,41,44] to deal with the centrifugal barrier in solving the equation analytically. This approximation scheme is a good approximation to the centrifugal term which is valid for $\alpha\ll 1$, and it becomes

$$\frac{1}{r^2}\approx\frac{\alpha^2}{(1-e^{-\alpha r})^2}. \tag{14}$$

Inserting the potential in Eq. (1) and applying the approximation scheme in Eq. (14), one can obtain

$$\frac{d^2\psi_{nl}(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E_{nl}+\left(\frac{\Lambda_1 e^{-\alpha r}}{1-e^{-\alpha r}}+\frac{\Lambda_2\alpha}{1-e^{-\alpha r}}-\frac{\Lambda_3\alpha e^{-\alpha r}}{1-e^{-\alpha r}}\right)-\frac{\hbar^2}{2\mu}l(l+1)\frac{\alpha^2}{(1-e^{-\alpha r})^2}\right]\psi_{nl}(r)=0 \tag{15}$$

By using the change of variable from $r\rightarrow s$, our new coordinate becomes

$$s=e^{-\alpha r}. \tag{16}$$

We substitute Eq. (16) into Eq. (15) and after some simplifications, we have:

$$\frac{d^2\psi(s)}{ds^2}+\frac{1-s}{s(1-s)}\frac{d\psi(s)}{ds}+\frac{1}{[s(1-s)]^2}$$
$$\times\big[-(\varsigma+k_1-k_3)s^2+(2\varsigma+k_1-k_2-k_3)s$$
$$-(\varsigma-k_2+k-4)\big]\psi(s)=0, \tag{17}$$

where

$$-\zeta=\frac{2\mu E_{nl}}{\alpha^2\hbar^2},\quad k_1=\frac{2\mu\Lambda_1}{\alpha^2\hbar^2},\quad k_2=\frac{2\mu\Lambda_2}{\alpha\hbar^2},$$
$$\left.k_3=\frac{2\mu\Lambda_3}{\alpha\hbar^2},\quad k_4=l(l+1)\right\}. \tag{18}$$

Comparing Eqs. (17) and (2), we obtain the relevant polynomials as:

$$\left.\begin{array}{c}\tilde{\tau}=1-s;\quad \sigma(s)=s(1-s);\quad \sigma'(s)=1-2s;\quad \sigma''(s)=-2;\\ \sigma(s)=-(\varsigma+\Lambda_1-\Lambda_3)s^2+(2\varsigma+\Lambda_1-\Lambda_2-\Lambda_3)s-(\varsigma-\Lambda_2+k_4)\end{array}\right\} \tag{19}$$

Inserting the polynomials given by Eq. (19) into Eq. (11) gives the polynomial:

$$\pi(s)=-\frac{s}{2}\pm\sqrt{(\omega_1-K)s^2+(K-\omega_2)s+\omega_3}, \tag{20}$$

where

$$\omega_1=\left(\frac{1}{4}+\varsigma+k_1-k_3\right),$$
$$\left.\omega_2=(2\varsigma+\kappa_1-k_2-k_3),\quad \omega_3=(\varsigma-k_2+k_4)\right\}. \tag{21}$$

According to the NU method, the quadratic form under the square root sign of Eq. (20) must be solved by setting the discriminant of this quadratic equation equal to zero: $\Delta=b^2-4ac=0$. This discriminant gives a new quadratic equation, which can be solved for the constant k to get the two roots. Here, we take the negative root given as:

$$K=-(\omega_2+2\omega_3)-2\sqrt{\omega_3}\sqrt{\omega_3+\omega_2+\omega_1}. \tag{22}$$

Since $\tau(s)$ given in Eq. (6) has a negative derivative in Eq. (8) for this value of $\pi(s)$ [42], $\pi(s)$ has the most suitable expression given as

$$\pi(s)=\frac{-s}{2}-[(\sqrt{\omega_3}+\sqrt{\omega_3+\omega_2+\omega_1})s-\sqrt{\omega_3}], \tag{23}$$

using Eq. (22). Therefore, we obtain $\tau(s)$ and $\tau'(s)$ as follows:

$$\tau(s)=1-2s-2\sqrt{\omega_3}\ s-2$$
$$\times\sqrt{\omega_3+\omega_2+\eta_1}\ s+2\sqrt{\omega_3}, \tag{24}$$

$$\tau'(s)=-2[1+\sqrt{\omega_3}+\sqrt{\omega_3+\omega_2+\omega_1}]. \tag{25}$$

where $\tau'(s)$ is the first derivative of $\tau(s)$. Using Eqs. (19), (23), (24) and (25), the following expressions for λ_n and λ are as follows:

$$\lambda_n=n^2+[1+2\sqrt{\omega_3}+2\sqrt{\omega_3+\omega_2+\omega_1}]n,$$
$$(n=0,1,2,...), \tag{26}$$

$$\lambda=-\frac{1}{2}-\sqrt{\omega_3}-\sqrt{\omega_3+\omega_2+\omega_1}-(\omega_2+2\omega_3)$$
$$-2\sqrt{\omega_3}\sqrt{\omega_3+\omega_2+\omega_1}, \tag{27}$$

where n is the number of nodes in the radial wavefunctions ψ_{nl}. When comparing Eqs. (26) and (27) with the help of Eq. (18), we obtain bound state energy eigenvalues of HHP as:

$$E_{nl}=\frac{\alpha^2\hbar^2 l(l+1)}{2\mu}-\Lambda_2\alpha-\frac{\alpha^2\hbar^2}{8\mu}$$
$$\times\left[\frac{(n{+}l{+}1)^2+\frac{2\mu\Lambda_3}{\alpha\hbar^2}-\frac{2\mu\Lambda_2}{\alpha\hbar^2}-\frac{2\mu\Lambda_1}{\alpha^2\hbar^2}+l(l{+}1)}{(n+l+1)}\right]^2. \tag{28}$$

We can obtain the other part of the wave function $\phi(s)$ and weight function $\rho(s)$ by inserting the values of $\sigma(s)$, $\pi(s)$ and $\tau(s)$ given in Eqs. (19), (23) and (34) respectively into Eqs. (5) and (10) as follows:

$$\phi(s)=s^{\sqrt{\omega_3}}(1-s)^{(1/2+\sqrt{\omega_3+\omega_2+\omega_1})}, \tag{29}$$

$$\rho(s)=s^{2\sqrt{\omega_3}}(1-s)^{(2+\sqrt{\omega_3+\omega_2+\omega_1})}. \tag{30}$$

Using Eq. (30), the Rodrigues relation in Eq. (9) is written as:

$$y_n=B_n s^{-2\sqrt{\omega_3}}(1-s)^{-2\sqrt{\omega_3+\omega_2+\omega_1}}\frac{d^n}{ds^n}$$
$$\times\left[s^n(1-s)^n s^{2\sqrt{\omega_3}}(1-s)^{-2\sqrt{\omega_3+\omega_2+\omega_1}}\right], \tag{31}$$

where B_n is the Jacobi polynomial; hence, the wave function in Eq. (3) becomes:

$$\psi_{nl}(s)=N_{nl}s^{\sqrt{\omega_3}}(1-s)^{((1/2)+\sqrt{\omega_3+\omega_2+\omega_1})}$$
$$\times P_n^{(2\sqrt{\omega_3},2\sqrt{\omega_3+\omega_2+\omega_1})} \tag{32}$$

where N_{nl} is the normalization constant. Using the normalization condition, we obtain the normalization constant as follows:

$$\frac{N_{nl}^2}{\alpha}\int_{-1}^{1}\left(\frac{1-\delta}{2}\right)^{2\sqrt{\omega_3}}\left(\frac{1+\delta}{2}\right)^{\beta}$$
$$\times\left[P_n^{(2\sqrt{\omega_3},2\beta-1)}(\delta)\right]^2dy=1, \tag{33}$$

TABLE I. Energy eigenvalues (eV) of the Hulthén-Hellmann potential with $\hbar = 2$, $\mu = 1$.

state	α	$\Lambda_1 = 0.025$, $\Lambda_2 = 1$, $\Lambda_3 = -1$	$\Lambda_1 = 0.050$, $\Lambda_2 = 2$, $\Lambda_3 = -2$	$\Lambda_1 = 0.015$, $\Lambda_2 = 4$, $\Lambda_3 = -4$
1s	0.025	-2.237656250	-8975156250	-48.92515625
	0.050	-1.550625000	-6.225625000	-30.17562500
	0.075	-1.350017362	-5.420850695	-2492640624
	0.10	1.255625000	-5.040000000	-22.49000000
	0.15	-1.166736110	-4.675069442	-20.18062500
2s	0.025	-0.5506250000	-2.225625000	-12.17562500
	0.050	-0.3806250000	-1.540000000	-7.49000000
	0.075	-0.3334027780	-1.341736111	-4333402775
	0.10	-0.3139062500	-1.250625000	-5.575625000
	0.15	-0.3034027778	-1.171111110	-5.010000000
2p	0.025	-0.5314062500	-2.187656250	-12.08765625
	0.050	-0.3475000000	-1.475625000	-7.350625000
	0.075	-0.2854340280	-1.250017360	-5.988906248
	0.10	-0.2501562500	-1.130625000	-5.330625000
	0.15	-0.2052777777	-0.9917361105	-4.655625000
3s	0.025	-0.2389062500	-0.9764062500	-5.370850695
	0.050	-0.1667361111	-0.67506494445	-3.291736110
	0.075	-0.1513908180	-0.5925945218	-2.715434026
	0.10	-0.1506250000	-0.5600000000	-2.454444445
	0.15	-0.1685262345	-0.5472299380	-2.225625000
3p	0.025	-0.2300173611	-0.9591840278	-5.331406250
	0.050	-0.1506250000	-0.645069445	-3.228402778
	0.075	-0.1269463735	-0.5487056328	-2.627100694
	0.10	-0.1167361111	-0.5011111110	-2.340000000
	0.15	-0.1124151234	-0.4550077159	-2.055625000
3d	0.025	-0.2126562500	-0.9251562500	-5.252934028
	0.050	-0.1200694444	-0.5867361110	-30103402778
	0.075	-0.08180748460	-0.4646778548	-2.454184026
	0.10	-0.05562500000	-0.3900000000	-2.117777778
	0.15	-0.01519290124	-0.2855632714	-1.730625000
4s	0.025	-0.13062500000	-0.5400000000	-2.990000000
	0.050	-0.09515625000	-0.3756250000	-1.825625000
	0.075	-0.09506944448	-0.3377777778	-1.509999999
	0.10	-0.1066015625	-0.3314062500	-1.375156250
	0.15	-0.1508506944	-0.3584027777	-1.280625000
4p	0.025	-0.1253515625	-0.5300390625	-2.967539062
	0.050	-0.08500000000	-0.3576562500	-1.788906250
	0.075	-0.07885850696	-0.3106293402	-1.457851562
	0.10	-0.08316406250	-0.2939062500	-1.306406250
	0.15	-0.1094444445	-0.2966840277	-1.175156250

where

$$\left.\begin{aligned} \beta &= 1 + \sqrt{4\omega_3 + 4\omega_2 + 4\omega_1}, \\ \beta - 1 &= \sqrt{4\omega_3 + 4\omega_2 + 4\omega_1} \end{aligned}\right\} \tag{34}$$

Comparing Eq. (33) with the standard integral of the form of Eq. (37) of [41],

$$\int_{-1}^{1}\left(\frac{1-p}{2}\right)^{x}\left(\frac{1+y}{2}\right)^{y}(P_n^{2x,2y-1}(p))^2dp=\frac{2\Gamma(x+n+1)\Gamma(y+n+1)}{n!x\Gamma(x+y+n+1)}. \tag{35}$$

Hence, we write the normalization constant as

$$N_{nl}=\left[\frac{n!2\sqrt{\omega_3}\alpha\Gamma(2\sqrt{\omega_3}+\sqrt{4\omega_3+4\omega_2+4\omega_1}+n+2)}{2\Gamma(2\sqrt{\omega_3}+n+1)\Gamma(\sqrt{4\omega_3+4\omega_2+4\omega_1}+n+2)}\right]^{1/2}. \tag{36}$$

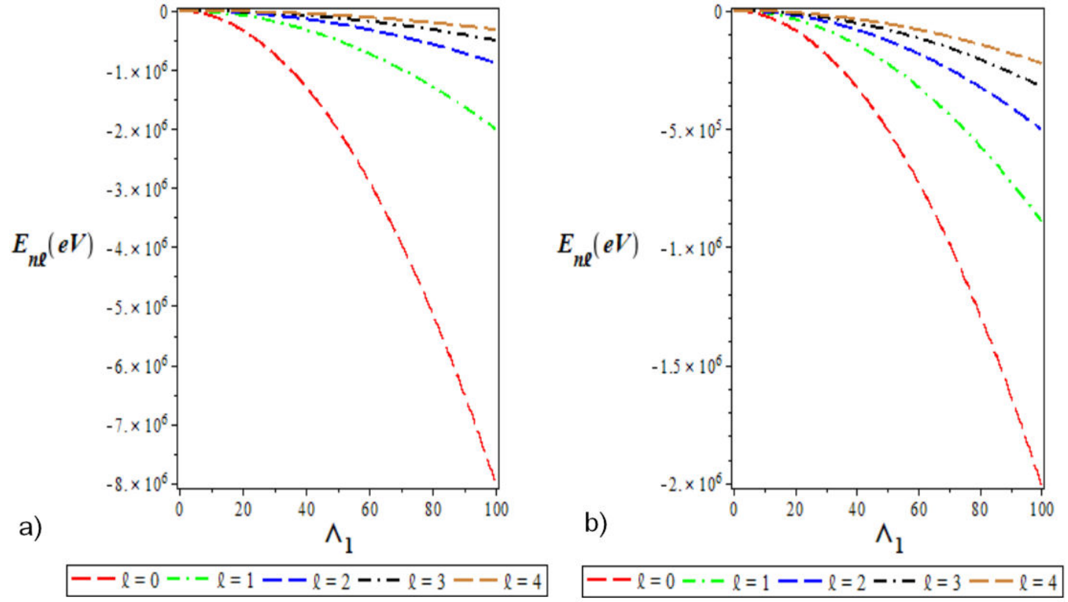

FIGURE 2. a). Variation of the ground state energy spectra for various l as a function of Λ_1. b). The plot of the first excited state energy spectra for different l as a function of Λ_1. We choose $\Lambda_1 = 0.10$, $\Lambda_2 = 4$, $\Lambda_3 = -4$, and $\alpha = 0.025$ for the ground and excited states.

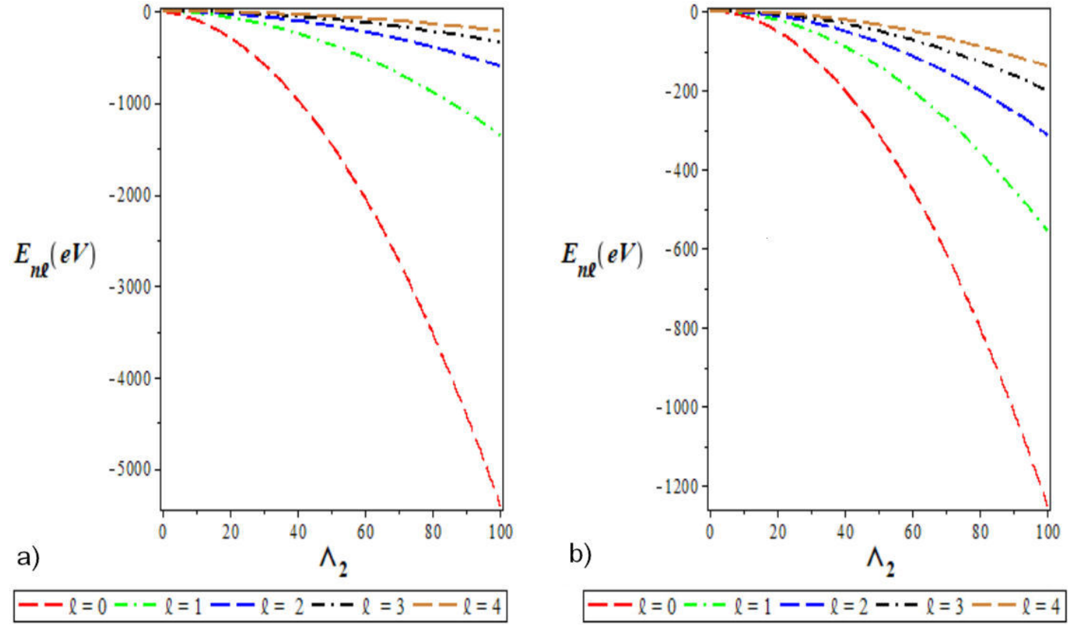

FIGURE 3. a) Variation of the ground state energy spectra for various l as a function of Λ_2. b) A plot of the first excited state energy spectra for various l as a function of Λ_2. We choose $\Lambda_1 = 0.10$, $\Lambda_2 = 4$, $\Lambda_3 = -4$, and $\alpha = 0.025$ for the ground and excited states.

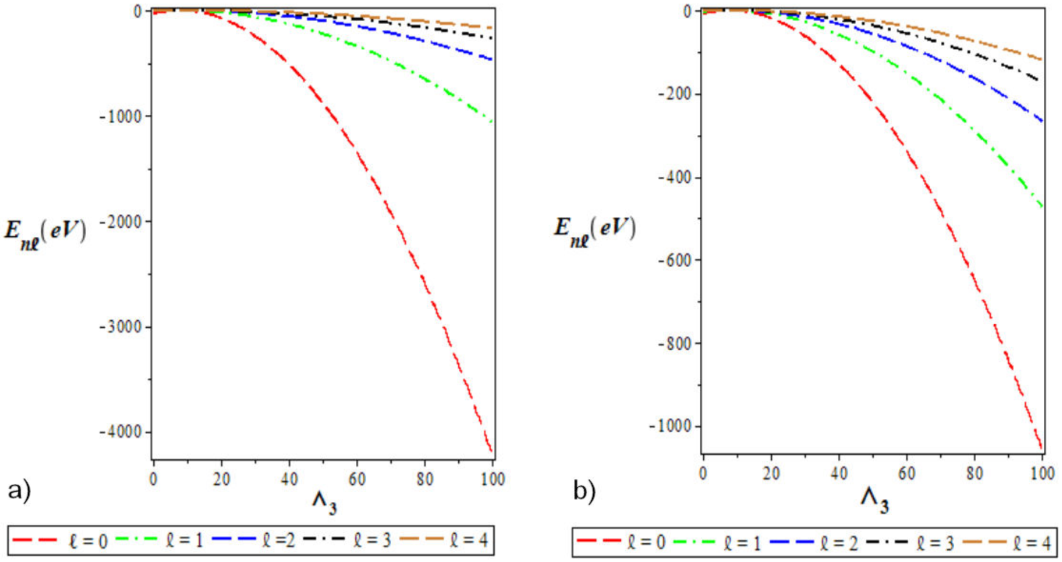

FIGURE 4. a) Variation of the ground state energy spectra for various l as a function of Λ_3. b) The plot of the first excited state energy spectra for various l as a function of Λ_3. We choose $\Lambda_1 = 0.10$, $\Lambda_2 = 4$, $\Lambda_3 = -4$, and $\alpha = 0.025$ for the ground and excited states.

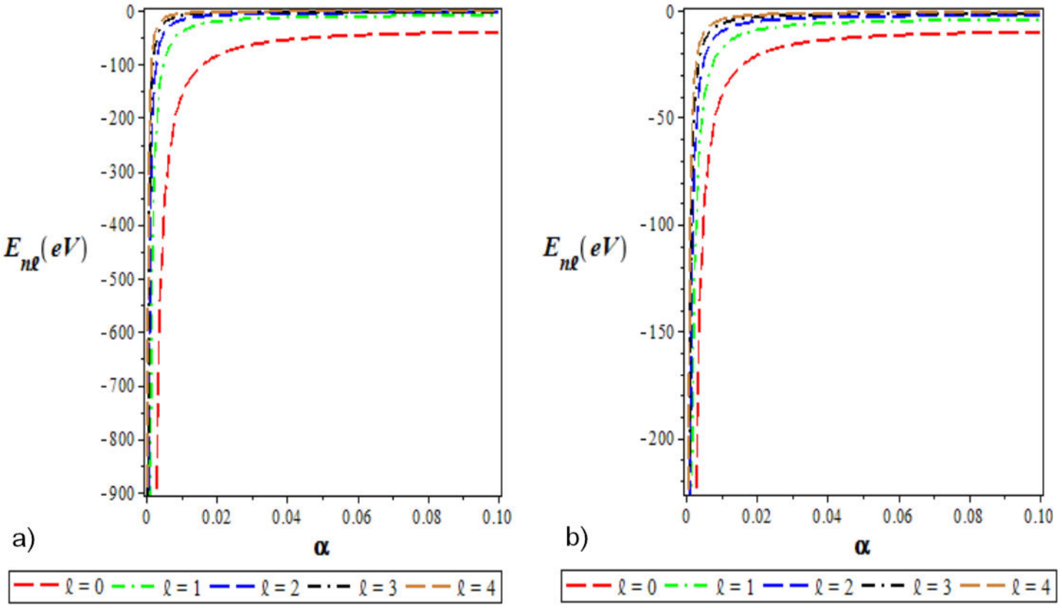

FIGURE 5. a) Variation of the ground state energy spectra for various l as a function of the screening parameter (α). b) A plot of the first excited state energy spectra for different l as a function of the screening parameter (α). We choose $\Lambda_1 = 0.10$, $\Lambda_2 = 4$, $\Lambda_3 = -4$, and $\alpha = 0.025$ for the ground and excited states.

3. Discussion

In Table I, we have reported some numerical results of the energy (eV) levels of Hulthén-Hellmann potential for various values of the potential strength: $\Lambda_1 = 0.025$, $\Lambda_2 = 2$, $\Lambda_3 = 1$; $\Lambda_1 = 0.050$, $\Lambda_2 = 2$, $\Lambda_3 = -1$ and $\Lambda_1 = 0.15$, $\Lambda_2 = 2$, $V_2 = -2$, respectively, as a function of the screening parameter for $\hbar = 2\mu = 1$. The results show that the energy decreases as the screening parameter, and the potential strength increase. We have plotted the graph of energy eigenvalues as a function of the potential strength, screening parameter, and the reduced mass in the ground and excited states. From Figs. 2a) and b) - Figs. 4a) and b) respectively, we plotted the ground and excited states energy eigenvalues of the different quantum states as a function of the Hulthen, Coulomb, and Yukawa potential strengths, respectively. We observed that there is a decrease in energy in both the ground and excited states as the potential strength, Λ_1, Λ_2, and Λ_3, increases respectively. This is well distinguished, especially in excited states.

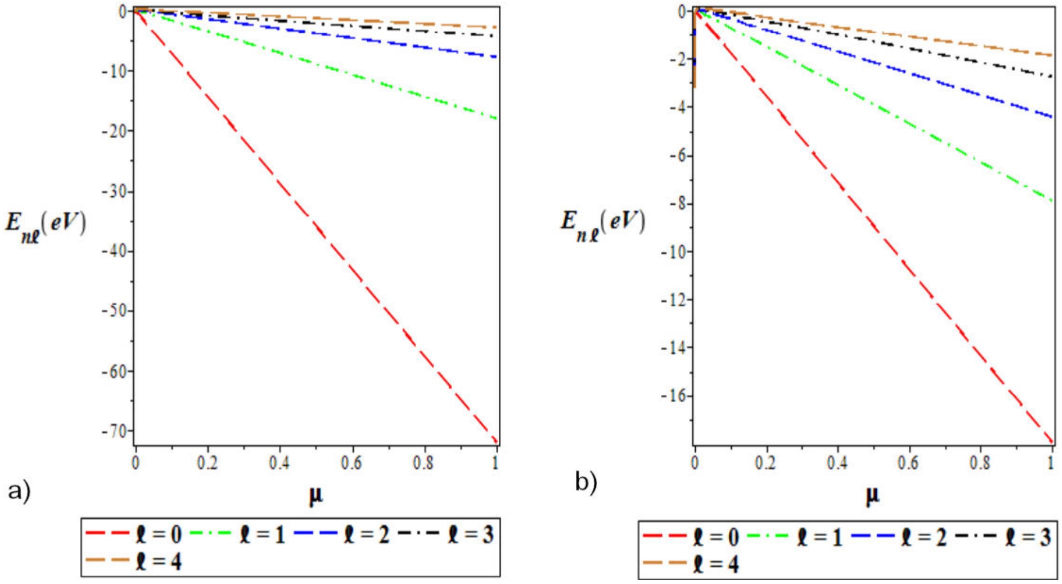

FIGURE 6. a) Variation of the ground state energy spectra for various l as a function of the screening parameter (μ). b) A plot of the first excited state energy spectra for different l as a function of the reduced mass (μ). We choose $\Lambda_1 = 0.10$, $\Lambda_2 = 4$, $\Lambda_3 = -4$, and $\alpha = 0.025$ for the ground and excited states.

In Fig. 5a) and 5b), we graphically show the variation of energy eigenvalues of HHP as a function of the screening parameter. Here, the energy eigenvalue increases with a small screening parameter ($\alpha \ll 1$, the limit of our screening parameter) and is invariant as the screening parameter becomes larger in the ground and excited states. In Fig. 6a) and 6b), there is a decrease in the ground and excited states energy eigenvalues for different quantum states as the reduced mass, μ increases.

4. Special cases

4.1. Hellmann potential

If we set the parameter $\Lambda_1 = 0$, the Hulthén-Hellmann potential in Eq. (1) reduces to Hellmann potential of the form

$$V(r) = -\frac{\Lambda_2}{r} + \frac{\Lambda_3 e^{-\alpha r}}{r}, \tag{37}$$

hence, its corresponding energy eigenvalue is of the form

$$E_{nl} = \frac{\alpha^2\hbar^2 l(l+1)}{2\mu} - \Lambda_2\alpha - \frac{\alpha^2\hbar^2}{8\mu} \times \left[\frac{(n+l+1)^2 + \frac{2\mu}{\alpha\hbar^2}(\Lambda_3 - \Lambda_2) + l(l+1)}{(n+l+1)}\right]^2. \tag{38}$$

Equation (38) is in agreement with Eq. (39) of [15]. In Table II and Table III, we present the numerical energy eigenvalues for Hellman potential for ($\Lambda_1 = 0$, $\Lambda_2 = 2$, $\Lambda_3 = 1$) and ($\Lambda_1 = 0$, $\Lambda_2 = 5$, $\Lambda_3 = -1$), respectively. The results showed good agreement with the earlier results of [19] with NU, AP, and SUSY methods of [45] and PT method of [46].

4.2. Hulthén potential

If we set $\Lambda_2 = \Lambda_3 = 0$, the Hulthén-Hellmann potential in Eq. (1) reduces to Hulthén potential

$$V(r) = -\frac{\Lambda_1 e^{-\alpha r}}{1 - e^{-\alpha r}}, \tag{39}$$

where $\Lambda_1 = Ze^2\alpha$ is the strength of Hulthén potential, Z is the atomic number, e is the electronic charge, and α is the screening parameter [5]. Making the same substitution into Eq. (27), the energy eigenvalues of Hulthén potential becomes:

$$E_{nl} = \frac{\alpha^2\hbar^2 l(l+1)}{2\mu} - \frac{\alpha^2\hbar^2}{8\mu} \times \left[\frac{(n+l+1)^2 + \frac{2\mu\Lambda_1}{\alpha^2\hbar^2} + l(l+1)}{(n+l+1)}\right]^2. \tag{40}$$

Equation (40) is in agreement with Eq. (41) of [47], Eq. (32) of [48]; Eq. (24) of [49]; Eq. (28) of [32]; Eq. (37) of [50] and Eq. (20) of [16]. Table IV is the numerical energy eigenvalues of Hulthén potential for 2p, 3p, 3d, and 4p with $\hbar = \mu = e = 1$ and for $Z = 1$. We compare the results with those obtained with AIM [43], EQR [51], and SUSY [52]. Our results obtained with the Greene-Aldrich approximation scheme for small screening parameters using the NU method are in excellent agreement with those obtained by other methods.

TABLE II. Comparison of energy eigenvalues (eV) for a particular case of Hellmann potential as a function of the screening parameter α with $\hbar = 2\mu = 1$ for $\Lambda_1 = 0$, $\Lambda_2 = 2$, and $\Lambda_3 = 1$.

State	α	Present method	(SUSY) [45]	(NU) [19]	(AP) [19]	(PT) [46]
1S	0.001	-0.2515002500	-0.251500	-0.251500	-0.250969	-0.250999
	0.005	-0.2575062500	-0.257506	-0.257506	-0.254933	-0.254963
	0.01	-0.2650250000	-0.265025	-0.265025	-0.259823	-0.289852
2S	0.001	-0.06400100000	-0.064001	-0.064001	-0.063243	-0.063494
	0.005	-0.07002500000	-0.070025	-0.070025	-0.067106	-0.067353
	0.01	-0.07760000000	-0.077600	-0.077600	-0.071689	-0.071928
2P	0.001	-0.06375025000	-0.063750	-0.064000	-0.063495	-0.063495
	0.005	-0.06875625000	-0.068756	-0.070000	-0.067377	-0.067377
	0.01	-0.07502500000	-0.075025	-0.077500	-0.072020	-0.072020
3S	0.001	-0.02928002778	-0.029280	-0.029280	-0.028283	-0.028764
	0.005	-0.03533402778	-0.035334	-0.035334	-0.031993	-0.032457
	0.01	-0.04300277778	-0.043003	-0.043003	-0.036142	-0.036557
3P	0.001	-0.02916802778	-0.029169	-0.029279	-0.028765	-0.028765
	0.005	-0.03475625000	-0.034756	-0.035309	-0.032480	-0.032480
	0.01	-0.04180277778	-0.041803	-0.042903	-0.036648	-0.036644
3d	0.001	-0.02894469445	-0.028945	-0.029388	-0.028767	-0.250833
	0.005	-0.03361736111	-0.033617	-0.035817	-0.032526	-0.254151
	0.01	-0.03946944445	-0.039469	-0.043825	-0.036813	-0.258269
4S	0.001	-0.01712900000	-0.017129	-0.029280	-0.016130	-0.016601
	0.005	-0.02322500000	-0.023225	-0.035334	-0.019646	-0.020077
	0.01	-0.03102500000	-0.031025	-0.043003	-0.023289	-0.023551
4p	0.001	-0.01706556250	-0.017066	-0.017128	-0.016602	-0.016602
	0.005	-0.02288906250	-0.022889	-0.023200	-0.020100	-0.020098
	0.01	-0.03030625000	-0.030306	-0.030925	-0.023711	-0.023641
4d	0.001	-0.01693906250	-0.016939	-0.017189	-0.016604	-0.016604
	0.005	-0.02222656250	-0.022227	-0.023464	-0.020142	-0.020142
	0.01	-0.02890625000	-0.028906	-0.031356	-0.023857	-0.023814
4f	0.001	-0.01675025000	-0.016750	-0.017311	-0.016607	-0.016607
	0.005	-0.02125625000	-0.021257	-0.024027	-0.020206	-0.020206
	0.01	-0.02690000000	-0.026900	-0.032356	-0.024072	-0.024056

4.3. Hulthén-Yukawa potential

If we set $\Lambda_2 = 0$, the Hulthén-Hellmann potential in Eq. (1) reduces to Hulthén-Yukawa potential

$$V(r) = -\frac{\Lambda_1 e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{\Lambda_3 e^{-\alpha r}}{r}. \tag{41}$$

Similarly, Eq. (27) reduces to the energy equation for the Hulthén-Yukawa potential

$$E_{nl} = \frac{\alpha^2 \hbar^2 l(l+1)}{2\mu} - \frac{\alpha^2 \hbar^2}{8\mu} \times \left[\frac{(n+l+1)^2 + \frac{2\mu\Lambda_3}{\alpha\hbar^2} - \frac{2\mu\Lambda_1}{\alpha^2\hbar^2} + l(l+1)}{(n+l+1)}\right]^2. \tag{42}$$

Equation (42) is in agreement with Eq. (24) of [43].

TABLE III. Comparison of energy eigenvalues (eV) for a particular case of Hellmann potential as a function of the screening parameter α with $\hbar = 2\mu = 1$ for $\Lambda_1 = 0$, $\Lambda_2 = 2$, and $\Lambda_3 = -1$.

State	α	Present method	(NU) [19]	(AP) [19]	(PT) [46]
1S	0.001	-2.250500250	-2.250500	- 2.248981	- 2.24900
	0.005	-2.252506250	-2.252506	- 2.244993	- 2.24501
	0.01	-2.255025000	-2.255025	- 2.240030	- 2.24005
2S	0.001	-0.5630010000	- 0.563001	- 0.561502	- 0.561502
	0.005	-0.5650250000	- 0.565025	- 0.557549	- 0.557550
	0.01	-0.5676000000	- 0.567600	- 0.552697	- 0.552697
2P	0.001	-0.5622502500	- 0.563000	- 0.561502	- 0.561502
	0.005	-0.5612562500	- 0.565000	- 0.557541	- 0.557541
	0.01	-0.5600250000	- 0.567500	- 0.552664	-0.552664
3S	0.001	-0.2505022500	- 0.250502	- 0.249004	- 0.249004
	0.005	-0.2525562500	- 0.252556	-0245110	- 0.245111
	0.01	-0.2552250000	- 0.255225	-0.240435	- 0.240435
3p	0.001	-0.2501680278	-0.250501	- 0.249004	- 0.249004
	0.005	-0.2508673611	-0.252531	-0.245102	-0.245103
	0.01	-0.2518027778	-0.255125	-0.240404	-0:240404
3d	0.001	-0.2495002500	-0.250833	-0.249003	-0.249003
	0.005	-0.2475062500	-0.254151	-0.245086	-0.245086
	0.01	-0.2450250000	-0.258269	-0.240341	-0.240341
4S	0.001	-0.1411290000	-0.141129	-0.139633	-0.139633
	0.005	-0.1432250000	-0.143225	-0.135819	-0.135819
	0.01	-0.1460250000	-0.146025	-0.131380	-0.131381
4p	0.001	-0.1409405625	-0.141128	-0.139632	0.139633
	0.005	-0.1422640625	-0.143200	-0.135811	0.135811
	0.01	-0.1440562500	-0.145925	-0.131350	-0.131351
4d	0.001	-0.1405640625	-0.141314	-0.139632	-0.139632
	0.005	-0.1403515625	-0.144089	-0.135795	-0.135796
	0.01	-0.1401562500	-0.147606	-0.131290	-0.131290
4f	0.001	-0.1400002500	-0.141686	-0.139631	-0.139631
	0.005	-0.1375062500	-0.145902	-0.135772	-0.135772
	0.01	-0.1344000000	-0.151106	-0.131200	-0.131200

4.4. Hulthén-Coulomb potential

If we set $\Lambda_3 = 0$, the Hulthén-Hellmann potential in Eq. (1) reduces to Hulthén-Coulomb potential

$$V(r) = -\frac{\Lambda_1 e^{-\alpha r}}{1 - e^{-\alpha r}} - \frac{\Lambda_2}{r}. \tag{43}$$

Equation (27) reduces to energy eigenvalues of Hulthén-Coulomb potential.

$$E_{nl} = \frac{\alpha^2 \hbar^2 l(l+1)}{2\mu} - \Lambda_2 \alpha - \frac{\alpha^2 \hbar^2}{8\mu} \times \left[\frac{(n+l+1)^2 - \frac{2\mu\Lambda_2}{\alpha\hbar^2} - \frac{2\mu\Lambda_1}{\alpha^2\hbar^2} + l(l+1)}{(n+l+1)} \right]^2. \tag{44}$$

4.5. Yukawa Potential

When the potential strength of Hulthén and Coulomb potentials are equal to zero, *i.e.*, $\Lambda_1 = \Lambda_2 = 0$, the Hulthén-Hellmann potential in Eq. (1) reduces to the Yukawa potential

TABLE IV. Comparison of energy eigenvalues (eV) of the Hulthén potential as a function of the screening Parameters α for 2p, 3p, 3d, and 4p states and for $Z = 1$ in atomic units ($\hbar = \mu = e = 1$).

State	α	Present (NU)	AIM [43]	EQR [51]	SUSY [52]
2p	0.025	-0.1128125000	-0.1128125	-0.1128125	-0.1127605
	0.050	-0.1012500000	-0.1012500	-0.1012500	-0.1010425
	0.075	-0.09031249994	-0.0903125	-0.0903125	-0.0898478
	0.10	-0.08000000000	-0.0800000	-0.0800000	-0.0791794
	0.15	-0.06124999998	-0.0612500	-0.0612500	-0.0594415
3p	0.025	-0.04070312500	-0.0437590	-0.0437590	-0.0437068
	0.050	-0.03336810000	-0.0333681	-0.0333681	-0.0331632
	0.075	-0.02438370000	-0.0243837	-0.0243837	-0.0239331
	0.10	-0.01680560000	-0.0168056	-0.0168056	-0.0160326
	0.15	-0.00586810000	-0.0058681	-0.0058681	-0.0043599
3d	0.025	-0.04360440000	-0.0437587	-0.0437587	-0.0436030
	0.050	-0.03275080000	-0.0333681	-0.0333681	-0.0327532
	0.075	-0.02299480000	-0.0243837	-0.0243837	-0.0230306
	0.10	-0.01433640000	-0.0162600	-0.0162600	-0.0144832
	0.15	-0.00031240000	-0.0058681	-0.0058681	-0.0132820
4p	0.025	-0.01994860000	-0.0200000	-0.0200000	-0.0199480
	0.050	-0.01104420000	-0.0112500	-0.0112500	-0.0110430
	0.075	-0.00453700000	-0.0050000	-0.0050000	-0.0045385
	0.10	-0.00042690000	-0.0012500	-0.0012500	-0.0004434

$$V(r) = -\frac{\Lambda_3 e^{-\alpha r}}{r}, \tag{45}$$

with the corresponding energy eigenvalues obtained from Eq. (27) as

$$E_{nl} = \frac{\alpha^2\hbar^2 l(l+1)}{2\mu} - \frac{\alpha^2\hbar^2}{8\mu} \times \left[\frac{(n+l+1)^2 - \frac{2\mu\Lambda_3}{\alpha\hbar^2} + l(l+1)}{(n+l+1)}\right]^2. \tag{46}$$

Equation (46) is in agreement with Eq. (34) of [16].

4.6. Coulomb potential

If we set $\Lambda_1 = \Lambda_2 = \alpha = 0$, the Hulthén-Hellmann potential in Eq. (1) reduces to the familiar Coulomb potential

$$V(r) = -\frac{\Lambda_2}{r}, \tag{47}$$

and we write the energy eigenvalues as

$$E_{n_p} = \frac{Z^2 e^4 \mu}{2\hbar^2 n_p^2}, \tag{48}$$

where the principal quantum number, $n_p = (n + l + 1)$, and $\Lambda_2 = Ze^2$ is the nuclear charge. Equation (48) agrees with Eq. (39) of [16] and Eq. (16) of [51].

5. Conclusion

In this work, we have studied the bound state solutions of the Schrödinger equation for any arbitrary ℓ-state by the superposition of Hulthén and Hellmann potentials using the Nikiforov-Uvarov method. After making appropriate approximation to the centrifugal barrier, we obtained the energy eigenvalues and the corresponding normalized wave functions. We also considered six special cases of the potential and obtained their energy eigenvalues. Our results are in excellent agreement with the results in the literature. Also, we have presented plots of energy eigenvalues for various values of the potential parameters in the ground and excited states.

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