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The generalized Kudryashov method for the nonlinear fractional partial differential equations with the beta-derivative

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In this article, we consider the exact solutions of the Hunter-Saxton and Schrödinger equations defined by Atangana's conformable derivative using the general Kudryashov method. Firstly, Atangana's conformable fractional derivative and its properties are included. Then, by introducing the generalized Kudryashov method, exact solutions of nonlinear fractional partial differential equations, which can be expressed with the conformable derivative of Atangana, are classified. Looking at the results obtained, it is understood that the generalized Kudryashov method can yield important results in obtaining the exact solutions of fractional partial differential equations containing beta-derivatives.

Keywords: The generalized Kudryashov method; Hunter-Saxton equation; Schrödinger equation; beta-derivative; wave solutions.

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1. Introduction

Recently, many articles have been made about obtaining analytical, numerical, exact solutions of the mathematical problems expressed by these events and some physical events that can be mathematically modeled and defined using fractional derivatives [1-5]. These and similar events are usually expressed in non-linear FPDEs. Besides the mentioned fractional differential equations have many application areas. Some of them are dynamics, engineering, physics, chemistry, biology, signal processing, continuum mechanics, control theory, respectively. Many different types of fractional derivative operators have been identified, some of which are as follows: Caputo derivative [6], Riemann-Liouville derivative [7], Caputo-Fabrizio [8], Jumarie's modified Riemann-Liouville derivative [9], Atangana-Baleanu derivative [10]. With the help of these derivative operators, various techniques have been developed that provide analytical, approximated, and exact solutions of nonlinear FPDEs such as the sub-equation method [11], the first integral method [12], the extended trial equation method [13], the modified trial equation method [14], the variational iteration method [15], local fractional Adomian decomposition method [16], Laplace transforms [17], local fractional Fourier series method [18], finite difference method [19], finite element method [20] and so on.

In [21], a new fractional derivative called conformable derivative has been defined, and then exact solutions of the time-heat differential equation obtained using this derivative have been obtained [22]. On the other hand, Atangana et al. have given some definitions, theorems, and features on the subject of conformable derivative [23]. Therefore, some applications have been made with the use of these features [24,25]. Finally, Atangana *et al.* gave a new definition of a fractional derivative called beta-derivative. In their articles, they solved the Hunter Saxton equation [26]. Respectively, fractional Hunter Saxton, fractional Sharma-Tasso-Olver, the space-time fractional modified Benjamin-Bona-Mahony, time fractional Schrödinger equations with Atangana's conformable derivative has been solved by the first integral method [27]. In [28,29], Martínez and Aguilar applied the fractional sub-equation method to construct exact solutions of the space-time conformable generalized Hirota-Satsuma-coupled KdV equation, coupled mKdV equation, the space-time resonant nonlinear Schrödinger equation with Atangana's conformable derivative. The authors in [30] consider the generalized exponential rational function method for the Radhakrishnan-Kundu-Lakshmanan equation with beta-conformable time derivative.

In this article, the effectiveness of the generalized Kudryashov method was investigated to determine the exact solutions of the FPDEs with Atangana's conformable derivatives. In some studies in the literature, this method has been applied to various nonlinear fractional problems [31-33].

The rest of the paper is organized as follows: In Sec. 2, some basic properties concerning the Atangana's conformable derivative are examined. Then the generalized Kudryashov method has been introduced in detail in Sec. 3. Section 4 includes some applications. This study was completed with a conclusion in Sec. 5.

2. Atangana's conformable derivatives (beta-derivatives)

Definition 1. In [21], a new fractional derivative called as conformable derivative is defined by Khalil *et al.* Let $f:[0,\infty)$ be a function α-th order, the conformable derivative of $f(t)$ for all $t>0$, $\alpha\in(0,1)$ is given as follows:

$$ {}_0D_t^{\alpha}\{f(t)\} = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}. \qquad (1)$$

Also, if f is α-differentiable in $(0,a)$, $a>0$, and

$\lim_{\varepsilon\to 0^+} f^{(\alpha)}(t)$ exists, then it can be written as $f^{(\alpha)}(0) = \lim_{\varepsilon\to 0^+} f^{(\alpha)}(t)$.

Definition 2. In [26], Atangana *et al.* gave the beta-derivative or Atangana's conformable as

$$ {}_0^A D_t^\alpha\{f(t)\} = \lim_{\varepsilon\to 0} \frac{f\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right)-f(t)}{\varepsilon}. \quad (2)$$

Although the conformable fractional derivative presented by Khalil *et al.* provides some fundamental features such as the chain rule, the Atangana's fractional derivative is preferred because it can provide the maximum properties of the fundamental derivatives. There are several important properties for the beta-derivatives [26]:

i) Taking that, $g \neq 0$ and f are two functions beta-differentiable with $\beta \in (0,1]$, then the following relation can be easily written and satisfied

$$ {}_0^A D_x^\alpha\{af(x)+bg(x)\} = a\,{}_0^A D_x^\alpha\{f(x)\} + b\,{}_0^A D_x^\alpha\{g(x)\}, \quad (3)$$

for all a and b real numbers.

ii) For c any constant, the following relation can be easily satisfied

$$ {}_0^A D_x^\alpha\{c\} = 0. \quad (4)$$

iii)

$$ {}_0^A D_x^\alpha\{c\}\{f(x)g(x)\} = g(x)\,{}_0^A D_x^\alpha\{f(x)\} + f(x)\,{}_0^A D_x^\alpha\{g(x)\}. \quad (5)$$

iv)

$$ {}_0^A D_x^\alpha\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\,{}_0^A D_x^\alpha\{f(x)\} - f(x)\,{}_0^A D_x^\alpha\{g(x)\}}{g^2(x)}. \quad (6)$$

Taking into account Eq. (2),

$$ \varepsilon = \left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha-1} h,$$

and $h \to 0$, when $\varepsilon \to 0$, hence we get

$$ {}_0^A D_x^\alpha\{f(x)\} = \left(x+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha} \frac{df(x)}{dx}, \quad (7)$$

with

$$ \eta = \frac{\delta}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad (8)$$

where δ is a constant, and therefore the following relation can be given

$$ {}_0^A D_x^\alpha\{f(\eta)\} = \delta \frac{df(\eta)}{d\eta}. \quad (9)$$

3. The generalized Kudryashov method

In this section, the generalized Kudryashov method will be introduced in detail to obtain the exact solutions of FPDEs defined by Atangana's derivative [31-33].

Considering the following nonlinear FPDE with a beta-derivative for a function of of two real variables, space x, and time t:

$$ P(u, {}_0^A D_t^\alpha u, u_x, u_x x, ...) = 0. \quad (10)$$

The basic operation steps of the generalized Kudryashov method can be given as follows:

Step 1. First of all, to obtain the wave solution of Eq. (10), we should consider the traveling wave transformation as follows:

$$ u(x,t) = u(\eta), \quad \eta = kx - \frac{\delta}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \quad (11)$$

where k and δ are arbitrary constants. Then, by applying Eq. (11) to Eq. (10), a nonlinear ordinary differential equation can be found as

$$ N(u, u', u'', u''', ...) = 0, \quad (12)$$

where the prime indicates differentiation with respect to η.

Step 2. Suppose that the exact solutions of Eq. (12) can be investigated in the form

$$ u(\eta) = \frac{\sum_{i=0}^{N} a_i \psi^i(\eta)}{\sum_{j=0}^{M} b_j \psi^j(\eta)} = \frac{A[\psi(\eta)]}{B[\psi(\eta)]}, \quad (13)$$

where

$$ \psi(\eta) = \frac{1}{1 \pm e^{\eta}}.$$

We note that the function ψ is the solution of the equation:

$$ \psi_\eta = \psi' = \psi^2 - \psi. \quad (14)$$

Taking into consideration Eq. (11), we obtain

$$ u'(\eta) = \frac{A'\psi' B - AB'\psi'}{B^2} = \psi' \frac{A'B - AB'}{B^2} = (\psi^2 - \psi)\frac{A'B - AB'}{B^2}, \quad (15)$$

$$ u''(\eta) = \frac{\psi^2-\psi}{B^2}\left\{(2\psi-1)(A'B-AB') + \frac{\psi^2-\psi}{B} \times \left[B(A''B-AB'')-2A'BB'+2A(B')^2\right]\right\}. \quad (16)$$

Step 3. For the solutions of Eq. (10) or Eq. (12), the rational form of the two finite series defined using the solution function of Eq. (14) can be expanded as follows:

$$u(\eta)=\frac{a_0+a_1\psi+a_2\psi^2+...+a_N\psi^N}{b_0+b_1\psi+b_2\psi^2+...+b_M\psi^M}. \qquad (17)$$

To find the values of M and N in Eq. (13), that is the pole order for the general solution of Eq. (10). We progress as in the classical Kudryashov method on balancing the highest -order nonlinear terms in Eq. (12) and we can obtain a relation between M and N. Various solutions to the relevant differential equation can be calculated for some values of M and N.

Step 4. Replacing Eq. (11) into Eq. (10) provides a polynomial $R(\Omega)$ of Ω. Equating the coefficients of $R(\Omega)$ to zero, we get a system of algebraic equations. Solving this system, we can compute λ and the variable coefficients of $a_0,a_1,a_2,...,a_N,b_0,b_1,b_2,...,b_M$. With this approach, we get exact solutions to Eq. (10).

4. Applications to the time-fractional equations with beta-derivatives

In this section, we seek the exact solutions of the Hunter-Saxton and Schrödinger equations with Atangana's conformable derivative using the generalized Kudryashov method.

Example 1: We consider the Hunter-Saxton with Atangana's conformable derivatives [27]

$${}_0^A D_t^{\alpha}\{u_x\}+(u_x)^2+uu_x=\frac{1}{2}(u_x)^2, \quad 0<\alpha\leq 1, \qquad (18)$$

where x is the spatial variable and t represents the time. Also, it is said that Atangana's derivative is chosen in the way that we recover the traditional Hunter-Saxton equation in [27]. We handle the traveling wave solutions of Eq. (18) and we perform the transformation $u(x,t)=u(\xi)$ and

$$\xi=x-\frac{\lambda}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},$$

where λ is constant. Then, we reach

$$2\lambda u''-(u')^2-2uu'=0. \qquad (19)$$

Putting Eqs. (13) and (16) into Eq. (19) and balancing the highest order nonlinear terms of u'' and $(u')^2$ in Eq. (19), then the following formula is procured

$$N-M+2=2N-2M+2\Rightarrow N=M. \qquad (20)$$

If we choose $N=M=2$, then we get

$$U(\xi)=\frac{a_0+a_1\psi+a-2\psi^2}{b_0+B_1\psi+b_2\psi^2}, \qquad (21)$$

$$u'(\xi)=(\psi^2-\psi)\frac{(a_1+2a_2\psi)(b_0+b_1\psi+b_2\psi^2)-(b_1+2b_2\psi)(a_0+a_1\psi+a_2\psi^2)}{(b_0+b_1\psi+b_2\psi^2)^2}, \qquad (22)$$

$$\begin{aligned} u''(\xi)=&\frac{(2\psi-1)(\psi^2-\psi)}{(b_0+b_1\psi+b_2\psi^2)^2}\left[(a_1+2a_2\psi)(b_0+b_1\psi+b_2\psi^2)-(b_1+2b_2\psi)(a_0+a_1\psi+a_2\psi^2)\right]\\ &+\frac{(\psi^2-\psi)^2}{(b_0+b_1\psi+b_2\psi^2)^3}\left[2a_2(b_0+b_1\psi+b_2\psi^2)^2-2b_2(a_1+2a_2\psi)(b_0+b_1\psi+b_2\psi^2)\right]\\ &+\frac{(\psi^2-\psi)^2}{(b_0+b_1\psi+b_2\psi^2)^3}\left[2(b_1+2b_2\psi)^2(a_0+a_1\psi+a_2\psi^2)\right]. \end{aligned} \qquad (23)$$

Therefore, the exact solutions of Eq. (18) are obtained as follows:

Case 1.

$$a_0=\frac{\lambda b_2}{4}, \quad a_1=b_0=0, \quad a_2=-\lambda b_2, \quad b_1=-b_2. \qquad (24)$$

When we substitute Eq. (24) into Eq. (21), we get the following solution of Eq. (18)

$$u_1(x,t)=\frac{\frac{\lambda}{4}\left\{1-4\left(1\pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^{\alpha})}\right)^{-2}\right\}}{\left(1\pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^{\alpha})}\right)^{-2}-\left(1\pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^{\alpha})}\right)^{-1}}. \qquad (25)$$

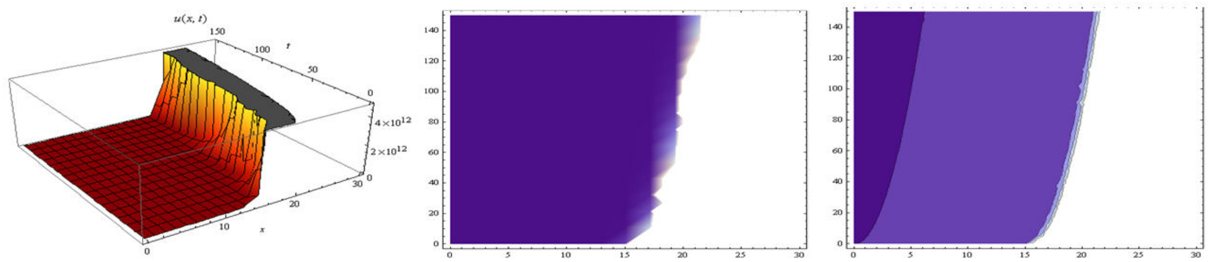

FIGURE 1. Three-dimensional, density and contour plots of the solution (26) for the values $\alpha = 0.5$ when $\lambda = 0.5$, $k = 2$.

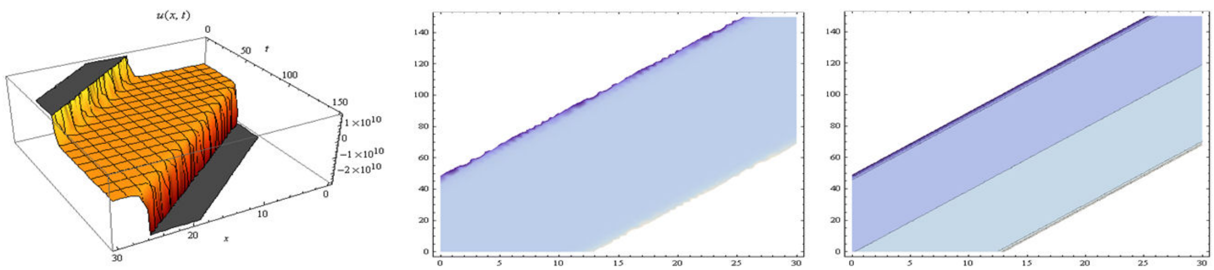

FIGURE 2. Three-dimensional, density and contour plots of the solution (26) for the values $\alpha = 1$ when $\lambda = 0.5$, $k = 2$.

Using several simple transformations to this solution, we get new exact solutions to Eq. (18),

$$u_{1,1}(x,t) = \frac{\lambda\left\{2\tanh\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right] - \tanh^2\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right]\right\}}{1 - \tanh^2\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right]}, \tag{26}$$

$$u_{1,2}(x,t) = \frac{\lambda\left\{2\coth\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right] - \coth^2\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right]\right\}}{1 - \coth^2\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right]}, \tag{27}$$

where $k_1 = 1/2$ and $\lambda_1 = \lambda/2$.

Case 2.

$$a_0 = -\frac{a_2}{4}\left(\frac{3a_2}{\lambda b_2} + 10\right), \quad a_1 = -a_2, \quad b_0 = b_2, \quad b_1 = -2b_2. \tag{28}$$

When we substitute Eq. (28) into Eq. (21), we get the following solution of Eq. (18)

$$u_2(x,t) = \frac{\frac{a_2}{4b_2}\left\{-\frac{3a_2}{\lambda b_2} - 10 - 4\left(1 \pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^{\alpha})}\right)^{-1} + 4\left(1 \pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^{\alpha})}\right)^{-2}\right\}}{1 - 2\left(1 \pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^{\alpha})}\right)^{-1} + \left(1 \pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^{\alpha})}\right)^{-2}}. \tag{29}$$

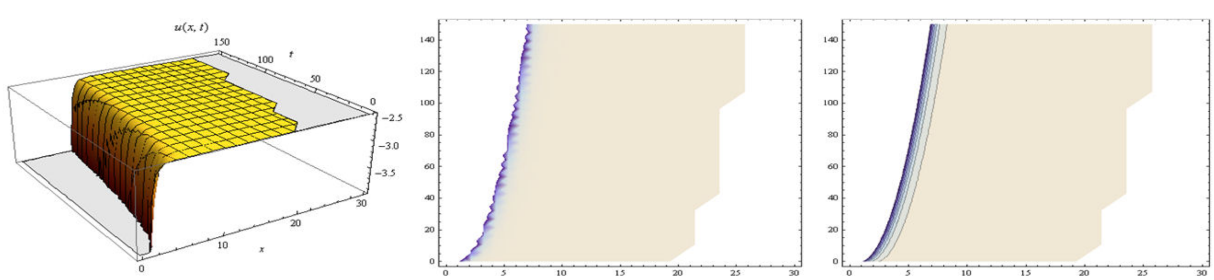

FIGURE 3. Three-dimensional, density and contour plots of the solution (30) for the values $\alpha = 0.5$ when $\lambda = 0.5$, $k = 2$, $a_2 = 1$, $b_2 = \sqrt{2}$.

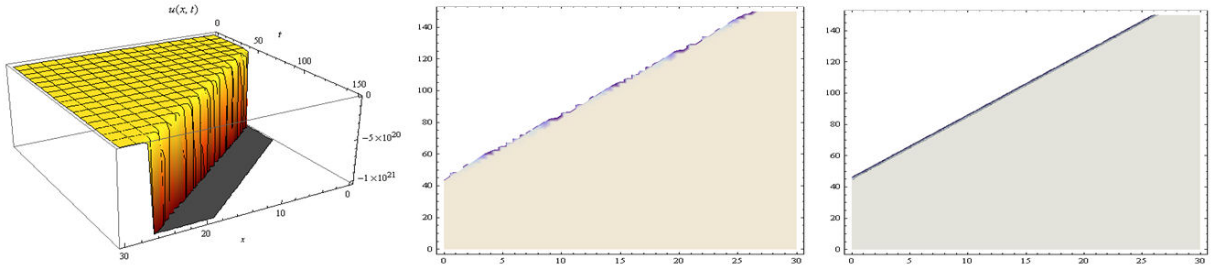

FIGURE 4. Three-dimensional, density and contour plots of the solution (30) for the values $\alpha = 1$ when $\lambda = 0.5$, $k = 2$, $a_2 = 1$, $b_2 = \sqrt{2}$.

Using several simple transformations to this solution, we get new exact solutions to Eq. (18),

$$u_{2,1}(x,t) = \frac{K\lambda b_2 \left\{\tanh^2\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right] - 11\right\} - 3Ka_2}{\left\{1 + \tanh\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]\right\}^2}, \tag{30}$$

$$u_{2,2}(x,t) = \frac{K\lambda b_2 \left\{\coth^2\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right] - 11\right\} - 3Ka_2}{\left\{1 + \coth\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]\right\}^2}, \tag{31}$$

where $K = a_2/(\lambda b_2^2)$.

Case 3.

$$a_0 = \frac{a_2}{4}\left(\frac{3a_2}{\lambda b_2} - 2\right), \quad a_1 = -a_2, \quad b_0 = b_1 = 0. \tag{32}$$

When we substitute Eq. (32) into Eq. (21), we get the following solution of Eq. (18)

$$u_3(x,t) = \frac{\frac{a_2}{4b_2}\left\{-\frac{3a_2}{\lambda b_2} - 2 - 4\left(1 \pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^\alpha)}\right)^{-1} + 4\left(1 \pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^\alpha)}\right)^{-2}\right\}}{\left(1 \pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^\alpha)}\right)^{-2}}. \tag{33}$$

Using several simple transformations to this solution, we can easily find new exact solutions to Eq. (18),

$$u_{3,1}(x,t) = \frac{K\lambda b_2 \left\{\tanh^2\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right] - 3\right\} + 3Ka_2}{\left\{1 - \tanh\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]\right\}^2}, \tag{34}$$

$$u_{3,2}(x,t) = \frac{K\lambda b_2 \left\{\coth^2\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right] - 3\right\} + 3Ka_2}{\left\{1 - \coth\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]\right\}^2}. \tag{35}$$

Case 4.

$$a_0 = -2\lambda b_2, \quad a_1 = 4\lambda b_2, \quad b_0 = b_2, \quad b_1 = -2b_2. \tag{36}$$

When we substitute Eq. (36) into Eq. (21), we get the following solution of Eq. (18)

$$u_4(x,t) = \frac{-2\lambda b_2 + 4\lambda b_2\left(\frac{1}{1 \pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^\alpha)}}\right) + a_2\left(\frac{1}{1 \pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^\alpha)}}\right)^2}{b_2 - 2b_2\left(\frac{1}{1 \pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^\alpha)}}\right) + b_2\left(\frac{1}{1 \pm e^{x-(\lambda/\alpha)(t+(1/\Gamma(\alpha))^\alpha)}}\right)^2}. \tag{37}$$

Using several simple transformations to this solution, we get new exact solutions to Eq. (18),

$$u_{4,1}(x,t) = \frac{a_2\left\{1 - \tanh\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]\right\}^2 - 8\lambda b_2 \tanh\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}{b_2\left\{1 + \tanh\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]\right\}^2}, \tag{38}$$

$$u_{4,2}(x,t) = \frac{a_2\left\{1 - \coth\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]\right\}^2 - 8\lambda b_2 \coth\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}{b_2\left\{1 - \coth\left[k_1 x - \frac{\lambda_1}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]\right\}^2}. \tag{39}$$

Example 2: We consider the nonlinear Schrödinger equation [27] with Atangana's derivatives

$$i{}_0^A D_t^\alpha\{u\} + pu_{xx} + q|u|^2 u = 0, \quad 0 < \alpha \leq 1, \tag{40}$$

where u is a complex value function. We take the traveling wave solutions of Eq. (40) and we implement the transformation

$$u(x,t) = e^{i\theta}u(\eta), \quad \theta = \tau x + \frac{\lambda}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha$$
$$\eta = x - \frac{2r\lambda}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha, \tag{41}$$

where τ, r and λ are constants. Using Eqs. (7)-(9) and substituting Eq. (41) into Eq. (40), we obtain the following equation including the imaginary and real part

$$i\left[-2r\lambda\frac{du}{d\eta} + 2p\tau\frac{du}{d\eta}\right] + p\frac{d^2u}{d\eta^2}$$
$$-(\lambda + p\tau^2)u + qu^3 = 0. \tag{42}$$

From the imaginary part of Eq. (42), we have

$$r = \frac{p\tau}{\lambda}. \tag{43}$$

Also, the real part of Eq. (42) can be rewritten as

$$pu'' - (\lambda + p\tau^2)u + qu^3 = 0. \tag{44}$$

Putting Eqs. (13) and (16) into Eq. (44) and balancing the highest order nonlinear terms of u'' and u^3 in Eq. (44), then the following formula is found

$$N - M + 2 = 3N - 3M \Rightarrow N = M + 1. \tag{45}$$

If we choose $M = 1$ and $N = 2$, then we have

$$u(\eta) = \frac{a_0 + a_1\psi + a_2\psi^2}{b_0 + b_1\psi}, \tag{46}$$

$$u'(\eta) = (\psi^2 - \psi)\frac{(a_1 + 2a_2\psi)(b_0 + b_1\psi) - b_1(a_0 + a_1\psi + a_2\psi^2)}{(b_0 + b_1\psi)^2}, \tag{47}$$

$$u''(\eta) = \frac{(2\psi - 1)(\psi^2 - \psi)}{(b_0 + b_1\psi)^2}\left[(a_1 + 2a_2\psi)(b_0 + b_1\psi) - b_1(a_0 + a_1\psi + a_2\psi^2)\right]$$
$$+ \frac{(\psi^2 - \psi)^2}{(b_0 + b_1\psi)^3}\left[2a_2(b_0 + b_1\psi)^2 - 2b_1(a_1 + 2a_2\psi)(b_0 + b_1\psi) + 2b_1^2(a_0 + a_1\psi + a_2\psi^2)\right]. \tag{48}$$

The exact solutions of Eq. (40) are obtained as follows:
Case 1.

$$a_0 = -ib_0\sqrt{\frac{2p}{q}}, \quad a_1 = 2ib_0\sqrt{\frac{2p}{q}}, \quad a_2 = -2ib_0\sqrt{\frac{2p}{q}}, \quad \lambda = -p(2 + \tau^2), \quad r = -\frac{\tau}{(2 + \tau^2)}. \tag{49}$$

When we substitute Eq. (49) into Eq. (46), we get the following solution of Eq. (40)

$$u_1(x,t) = e^{i[\tau x - (p(2+\tau^2)/\alpha)(t + (1/\Gamma(\alpha)))^\alpha]}$$
$$\times \frac{ib_0\sqrt{\frac{2p}{q}}\left[-1 + 2\left(\frac{1}{1 \pm e^{x - (2p\tau/\alpha)(t + (1/\Gamma(\alpha)))^\alpha}}\right) - 2\left(\frac{1}{1 \pm e^{x - (2p\tau/\alpha)(t + (1/\Gamma(\alpha)))^\alpha}}\right)^2\right]}{b_0 + b_1\left(\frac{1}{1 \pm e^{x - (2p\tau/\alpha)(t + (1/\Gamma(\alpha)))^\alpha}}\right)}. \tag{50}$$

Using several simple transformations to this solution, we get new exact solutions to Eq. (40),

$$u_{1,1}(x,t) = Le^{i[\tau x + (\lambda_2/\alpha)(t + (1/\Gamma(\alpha)))^\alpha]}\frac{1 + \tanh^2\left[k_1x + \frac{\lambda_3}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}{\tanh\left[k_1x + \frac{\lambda_3}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}, \tag{51}$$

$$u_{1,2}(x,t) = Le^{i[\tau x + (\lambda_2/\alpha)(t + (1/\Gamma(\alpha)))^\alpha]}\frac{1 + \coth^2\left[k_1x + \frac{\lambda_3}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}{\coth\left[k_1x + \frac{\lambda_3}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}, \tag{52}$$

where $L = -i\sqrt{P/2q}$, $\lambda_2 = -p(2 + \tau^2)$ and $\lambda_3 = -2p\tau$.

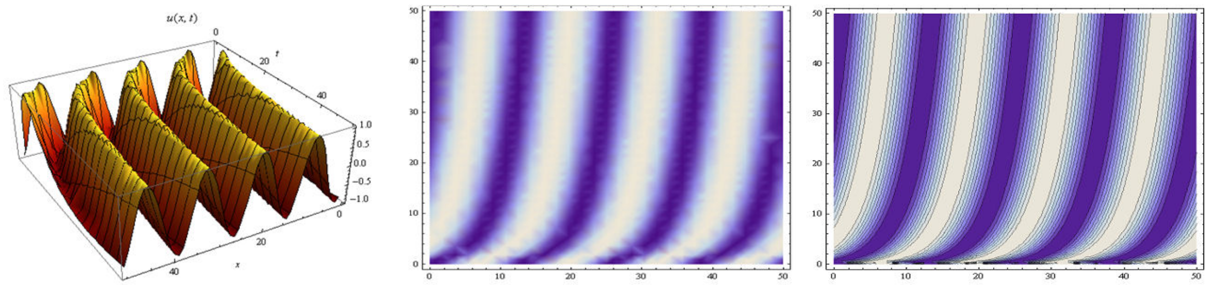

FIGURE 5. Three-dimensional, density and contour plots of the solution (55) for the values $\alpha = 0.001$, when $p = k = 2$, $q = 1$ $\tau = 0.5$.

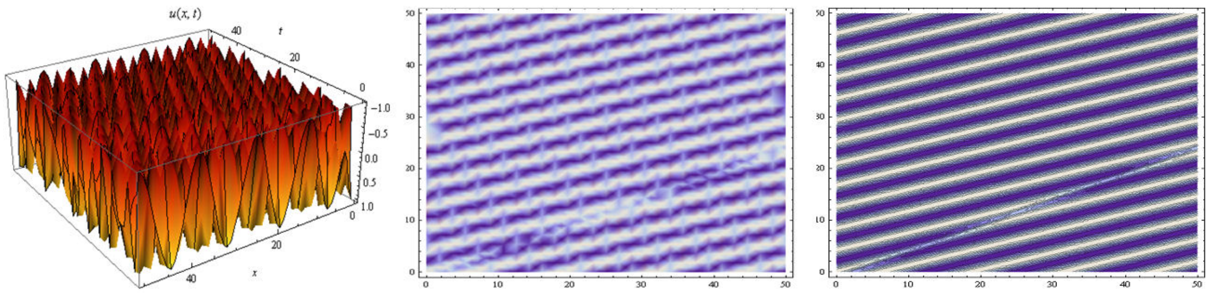

FIGURE 6. Three-dimensional, density and contour plots of the solution (55) for the values $\alpha = 1$ when $p = k = 2$, $q = 1$ $\tau = 0.5$.

Case 2.

$$a_0 = -ib_0\sqrt{\frac{p}{2q}}, \quad a_1 = 0, \quad a_2 = 2ib_0\sqrt{\frac{2p}{q}}, \quad b_1 = 2b_0, \quad \lambda = -\frac{p}{2}(1+2\tau^2), \quad r = -\frac{2\tau}{1+2\tau^2}. \tag{53}$$

When we substitute Eq. (53) into Eq. (46), we get the following solution of Eq. (40)

$$u_2(x,t) = e^{i[\tau x-(p(1+2\tau^2)/2\alpha)(t+(1/\Gamma(\alpha)))^\alpha]}\frac{i\sqrt{\frac{p}{2q}}\left[-1+2\left(\frac{1}{1\pm e^{x-(2p\tau/\alpha)(t+(1/\Gamma(\alpha)))^\alpha}}\right)^2\right]}{1+2\left(\frac{1}{1\pm e^{x-(2p\tau/\alpha)(t+(1/\Gamma(\alpha)))^\alpha}}\right)}. \tag{54}$$

Applying simple transformations to this solution, we gain new exact solutions to Eq. (40),

$$u_{2,1}(x,t) = Le^{i[\tau x+(\lambda_4/\alpha)(t+(1/\Gamma(\alpha)))^\alpha]}\tanh\left[k_1 x + \frac{\lambda_3}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha\right], \tag{55}$$

$$u_{2,2}(x,t) = Le^{i[\tau x+(\lambda_4/\alpha)(t+(1/\Gamma(\alpha)))^\alpha]}\coth\left[k_1 x + \frac{\lambda_3}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha\right], \tag{56}$$

where $L = -i\sqrt{P/2q}$ and $\lambda_4 = -(p(1+2\tau^2)/2)$.

Case 3 .

$$a_0 = -ib_0\sqrt{\frac{p}{2q}}, \quad a_1 = a_2 = 0, \quad b_1 = -2b_0, \quad \lambda = -\frac{p}{2}(1+2\tau^2), \quad r = -\frac{2\tau}{1+2\tau^2}. \tag{57}$$

When we replace Eq. (57) into Eq. (46), we obtain the following solution of Eq. (40)

$$u_3(x,t) = e^{i[\tau x-(p(1+2\tau^2)/2\alpha)(t+(1/\Gamma(\alpha)))^\alpha]}\frac{-i\sqrt{\frac{p}{2q}}}{1-2\left(\frac{1}{1\pm e^{x-(2p\tau/\alpha)(t+(1/\Gamma(\alpha)))^\alpha}}\right)}. \tag{58}$$

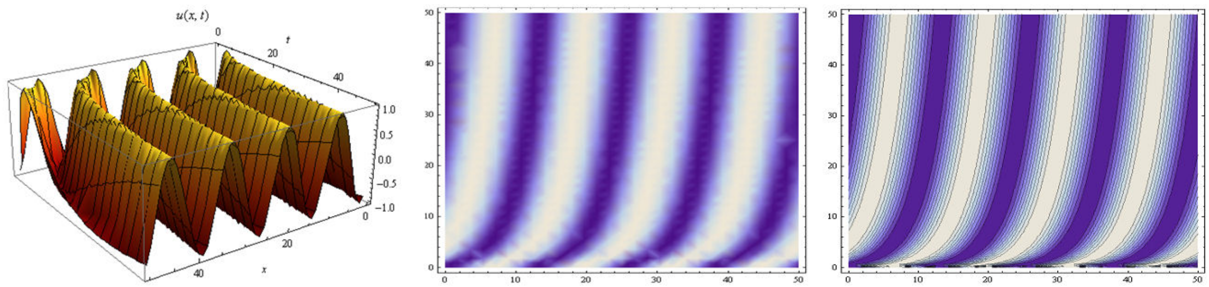

FIGURE 7. Three-dimensional, density and contour plots of the solution (59) for the values $\alpha = 0.01$ when $p = k = 2$, $q = 1$ $\tau = 0.5$.

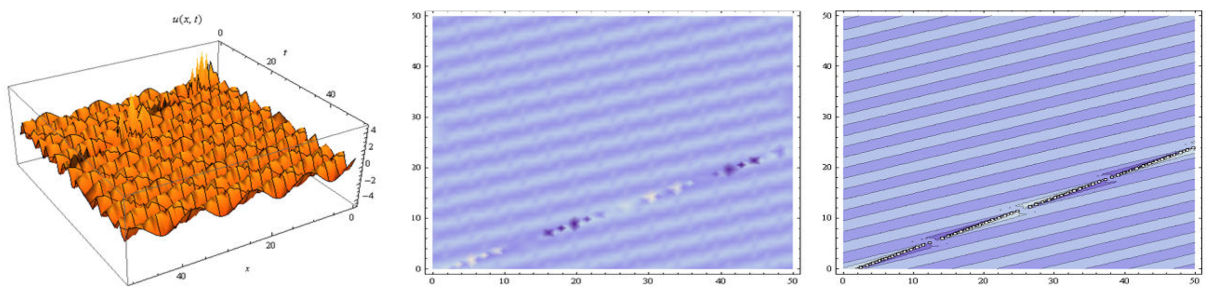

FIGURE 8. Three-dimensional, density and contour plots of the solution (59) for the values $\alpha = 1$ when $p = k = 2$, $q = 1$ $\tau = 0.5$.

Fulfilling several transformations to this solution, we gain new exact solutions to Eq. (40),

$$u_{3,1}(x,t) = Le^{i[\tau x + (\lambda_4/\alpha)(t+(1/\Gamma(\alpha)))^\alpha]} \frac{1}{\tanh\left[k_1 x + \frac{\lambda_3}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}, \tag{59}$$

$$u_{3,2}(x,t) = Le^{i[\tau x + (\lambda_4/\alpha)(t+(1/\Gamma(\alpha)))^\alpha]} \frac{1}{\coth\left[k_1 x + \frac{\lambda_3}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}, \tag{60}$$

Case 4 .

$$a_0 = 0, \quad a_1 = -ib_1\sqrt{\frac{2p}{q}}, \quad a_2 = ib_1\sqrt{\frac{2p}{q}}, \quad b_0 = -\frac{b_1}{2}, \quad \lambda = p(1-\tau^2), \quad r = \frac{\tau}{1-\tau^2}. \tag{61}$$

When we replace Eq. (61) into Eq. (46), we obtain the following solution of Eq. (40)

$$u_4(x,t) = e^{i[\tau x + (p(1-\tau^2)/\alpha)(t+(1/\Gamma(\alpha)))^\alpha]} \frac{2i\sqrt{\frac{p}{2q}}\left[\left(\frac{1}{1 \pm e^{x-(2p\tau/\alpha)(t+(1/\Gamma(\alpha)))^\alpha}}\right) - \left(\frac{1}{1 \pm e^{x-(2p\tau/\alpha)(t+(1/\Gamma(\alpha)))^\alpha}}\right)^2\right]}{1 - 2\left(\frac{1}{1 \pm e^{x-(2p\tau/\alpha)(t+(1/\Gamma(\alpha)))^\alpha}}\right)}. \tag{62}$$

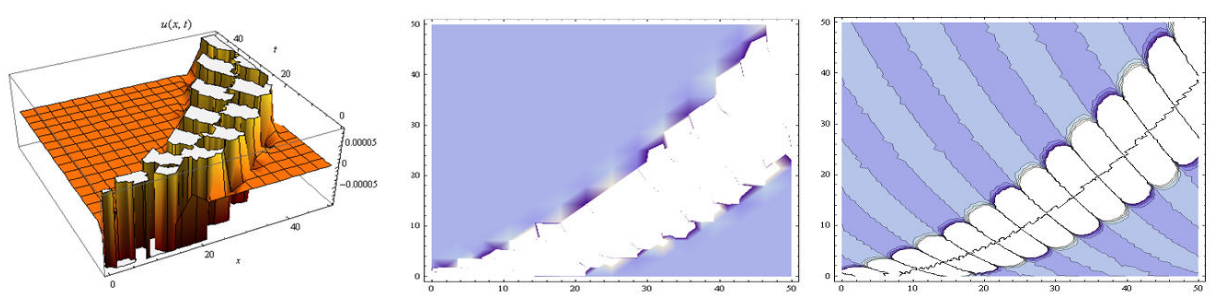

FIGURE 9. Three-dimensional, density and contour plots of the solution (63) for the values $\alpha = 0.5$ when $p = 2$, $q = k = 1$, $\tau = 0.5$.

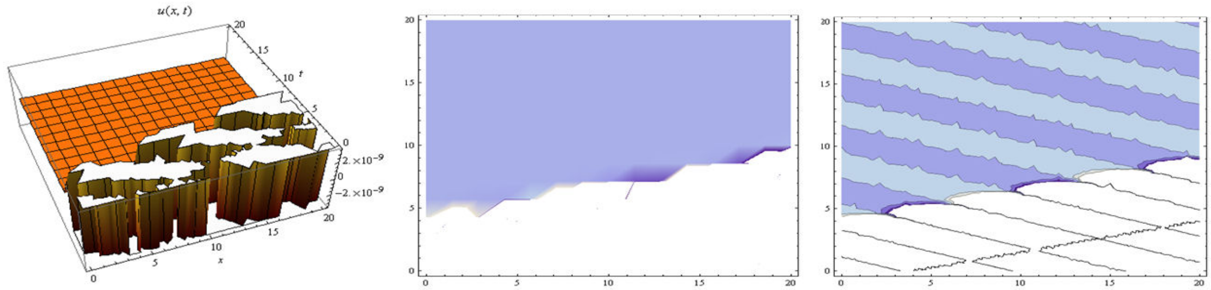

FIGURE 10. Three-dimensional, density and contour plots of the solution (63) for the values $\alpha = 1$ when $p = 2$, $q = k = 1$, $\tau = 0.5$.

Using several transformations to this solution, we procure new exact solutions to Eq. (40),

$$u_4(x,t) = Me^{i[\tau x+(\lambda_5/\alpha)(t+(1/\Gamma(\alpha)))^\alpha]}\frac{1}{\cosh\left[k_1x+\frac{\lambda_3}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha\right]\sinh\left[k_1x+\frac{\lambda_3}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}, \tag{63}$$

where $M = \mp i\sqrt{p/2q}$ and $\lambda_5 = p(1-\tau^2)$.

Case 5 .

$$a_0 = ib_0\sqrt{\frac{p}{2q}}, \quad a_1 = i(2b_0+b_1)\sqrt{\frac{p}{2q}}, \quad a_2 = 0, \quad \lambda = -\frac{p}{2}p(1+2\tau^2), \quad r = -\frac{2}{1+2\tau^2}. \tag{64}$$

If we embed Eq. (64) into Eq. (46), we compute the following solution of Eq. (40)

$$u_5(x,t) = e^{i[\tau x-(p(1+2\tau^2)/2\alpha)(t+(1/\Gamma(\alpha)))^\alpha]}\frac{i\sqrt{\frac{p}{2q}}\left[b_0+(2b_0+b_1)\left(\frac{1}{1\pm e^{x-(2p\tau/\alpha)(t+(1/\Gamma(\alpha)))^\alpha}}\right)\right]}{b_0+b_1\left(\frac{1}{1\pm e^{x-(2p\tau/\alpha)(t+(1/\Gamma(\alpha)))^\alpha}}\right)}. \tag{65}$$

From this solution where $N = i\sqrt{p/2q}$, we have new exact solutions to Eq. (40),

$$u_{5,1}(x,t) = Ne^{i[\tau x+(\lambda_4/\alpha)(t+(1/\Gamma(\alpha)))^\alpha]}\frac{b_1-(2b_0+b_1)\tanh\left[k_1x+\frac{\lambda_3}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}{b_1-2b_0-b_1\tanh\left[k_1x+\frac{\lambda_3}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}, \tag{66}$$

$$u_{5,2}(x,t) = Ne^{i[\tau x+(\lambda_4/\alpha)(t+(1/\Gamma(\alpha)))^\alpha]}\frac{b_1-(2b_0+b_1)\coth\left[k_1x+\frac{\lambda_3}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}{b_1-2b_0-b_1\coth\left[k_1x+\frac{\lambda_3}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha\right]}, \tag{67}$$

Remark. The solutions of Eqs. (18) and (40) were found by using the generalized Kudryashov method, have been checked using Mathematica Release 9. To our knowledge, these solutions that we obtained in this paper, are new and are not shown in the previous literature.

5. Conclusions

In this study, the generalized Kudryashov method was applied to find new exact solutions of the Hunter-Saxton and Schrödinger equations defined by Atangana's conformable derivative. This method is defined by the rational form of finite series, which includes the solution function of the Riccati equation. The number of terms of the finite series is determined by the balance principle. The balance relation obtained by the application of the balance principle shows us that the related problem can be solved for different values of the finite series. In this study, different solution classes are classified for the upper values of finite series calculated for Hunter Saxton and Schrödinger equations defined by Atangana's conformable derivative. By applying this method to the determined problems, rational hyperbolic function solutions were found. For some values of the parameters that are included in the solution functions, physical behaviors on three-dimensional, density, and contour graphics were examined. Thus, it has been observed that the generalized Kudryashov method gives very effective results in constructing the exact solutions of nonlinear FPDEs defined with Atangana's derivative. In our future studies, we will apply the generalized Kudryashov method to some other nonlinear fractional problems defined with Atangana's derivative.

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