The investigation of a classical particle in the presence of fractional calculus

Won Sang Chung[a], S. Zare[b], H. Hassanabadi[*,b,c], J. Kříž[c] and E. Maghsoodi[d]

[a] *Department of Physics and Research Institute of Natural Science, College of Natural Science, Gyeongsang National University, Jinju 660-701, Korea.*

[b] *Faculty of Physics, Shahrood University of Technology, Shahrood, Iran.*

[c] *Department of Physics, University of Hradec Králové, Rokitanského 62, 500 03 Hradec Králové, Czechi*

[d] *Department of Physics, Faculty of Science, Lorestan University, Khoramabad, Iran.*

[*] *e-mail: h.hasanabadi@shahroodut.ac.ir*

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In this article, by applying a preliminary and comprehensive definition of the fractional calculus, its effect on different aspects of physics is specified, as in the case of Laplace transforms, Riemann-Liouville, and Caputo derivatives. Applications of the fractional calculus in studying the dynamics of particle motion in classical mechanics are investigated analytically. Furthermore, we compare our results with those obtained from the usual methods and we show that both solutions coincide provided the fractional effects are removed.

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1. Introduction

The calculus of differentiation and integration is known as the fractional calculus. The fractional derivatives for the first time were proposed by Gottfried Wilhelm Leibniz in (1695) [1]. Now the fractional calculus has been considered as a new tool for modeling the complex systems [2-16]. Since then, the fractional derivative was examined for various functions. The fractional derivative of the exponential function and the power function, respectively, are obtained by Liouville in (1832) and Riemann in (1847) [1]. Many researchers consider an integral form for the fractional derivative and the two most popular types of fractional derivatives are Riemann-Louville and Caputo. The fractional derivative has many interesting and unexpected properties; for example, under special conditions, the derivative of a constant can be nonzero, such as the case of the Riemann-Liouville fractional derivative. On the other hand, the Caputo derivative of a constant, as the ordinary derivative, vanishes. For further information on fractional calculus, the interested reader is referred to Refs. [17-21].

Different definitions of fractional derivatives can be proposed, each with remarkable properties [22-26], all of them valid and mathematically acceptable.

In Ref. [37], the authors proposed a new fractional differential equation to describe the mechanical oscillations of a simple system and they analyzed the systems mass-spring and spring-damper. In Ref. [38], the authors proposed a fractional differential equation to describe the vertical motion of a body through the air. Two-dimensional projectile motion in a free and in a resistive medium were investigated using the so-called conformable derivative in Ref. [39]. The motion of a projectile by using the Riemann-Liouville fractional derivative and the Caputo approach is studied in Refs. [40,41].

Recently, using fractional calculus, the dynamics of a particle have been studied for resisted horizontal motion within a viscoelastic medium and in the presence of a uniform force [22]. Moreover, in Ref. [23], in the framework of conformable fractional quantum mechanics, the three-dimensional fractional harmonic oscillator is studied and by using an effective and efficient formalism, Schrödinger equation, probability density, probability flux and continuity equation have been investigated and in Ref. [24]. Fractional calculus has also been studied for the Dirac equation, the resulting wave function, and the energy eigenvalue equation.

There are different methods to solve fractional differential equations analytically. One of the most common, simple, and practical methods used is the Laplace transform [25]. In this paper, the Laplace transform of fractional operators is represented, and some related formula is introduced. Fractional calculus has been considered for modeling viscoelastic systems that cover various fields and subjects [22]. Here, we show that the proposed fractional model has a better result as compared to that of the non-fractional models have shown for probing the different aspects of mechanical physics.

This work is organized as follows. We first review the fractional calculus in Sec. 2. Next, we investigate dynamics of a particle within a viscoelastic medium in Sec. 3. In Sec. 4, by considering a retarding force proportional to the fractional velocity, vertical motion of a body in a resisting medium is studied and in the last section, to provide a better understanding of the motion of a projectile in a resisting viscoelastic medium, we will discuss it under the condition that there exists a retarding force proportional to the fractional velocity.

2. Introduction to Fractional Calculus

The Riemann-Louville fractional integral is defined as

$$I_{0\,|x}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi)\, d\xi \quad x>0, \qquad (1)$$

where $0<\alpha<1$ and $f(x)$ is a continuous function. Also the Caputo fractional derivative is introduced as [22]

$$D_{0\,|t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\nu-1} \frac{d^n}{d\xi^n} f(\xi)\, d\xi, \quad (2)$$

where $n=[\nu]+1$ and $[x]$ implies a Gauss symbol.

The Laplace transform of Caputo fractional derivative can be represented by the following form

$$L\left[D_t^{\alpha} x(t)\right] = \frac{s^m F(s) - s^{m-1} x(0) - s^{m-2} x'(0) - \ldots - x^{(m-1)}(0)}{s^{m-\alpha}}, \qquad (3)$$

and by inserting $\alpha=1$ and $m=2$ in Eq (1), we have

$$L\left[D_t^2 x(t)\right] = s^2 F(s) - s\,x(0) - x'(0). \qquad (4)$$

The Mittag-Leffler functions and the generalized Mittag- Leffler functions for $\alpha', \beta' > 0$ and $z \in C$ are defined as [34]

$$E_{\alpha'}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha'+1)}, \qquad (5)$$

$$E_{\alpha',\beta'}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha'+\beta')}. \qquad (6)$$

For $\alpha', \beta' > 0$, $a \in R$ and $s^{\alpha'} > |a|$ inverse Laplace transform formula has the form

$$L^{-1}\left[\frac{s^{\alpha'-\beta'}}{s^{\alpha'}+a}\right] = t^{\beta'-1} E_{\alpha',\beta'}\left(-a t^{\alpha'}\right). \qquad (7)$$

3. Resisted motion of a particle in a viscoelastic medium

Now let us investigate the dynamics of a particle in a viscoelastic medium. In reality, per cycle of motion the part of the energy is destroyed. In the other words, the measure of damping is determined by the amount of energy lost.

Experimentally, we can consider the horizontal motion in a viscoelastic medium as the simplest example of the resisted motion of a particle. By considering a general order of viscoelastic damping, the frictional force takes the following form

$$F_{\alpha} = -C D_t^{\alpha} x(t), \qquad 0 \le \alpha < 1. \qquad (8)$$

In order to be consistent with the time dimensionality, we consider, the fractional derivative operator as

$$\frac{d}{dt} \to \frac{1}{C_1^{1-\alpha}} \frac{d^{\alpha}}{dt^{\alpha}}, \qquad (9)$$

where C_1 represents the fractional time in the system [37]. Then, in Eq. (8), we change C to $(C/C_1^{1-\alpha})$.

In this case, the Newtonian equation satisfies the equation of motion as follows

$$m D_t^2 x(t) = -\frac{C}{C_1^{1-\alpha}} D_t^{\alpha} x(t), \qquad (10)$$

with the following initial conditions

$$x'(0) = V_0, \quad x(0) = 0. \qquad (11)$$

On the other hand we know that $F(s) = L[x(t)]$, therefore we have

$$L\left[m D_t^2 x(t)\right] = -L\left[\frac{C}{C_1^{1-\alpha}} D_t^{\alpha} x(t)\right], \qquad (12)$$

then by substituting Eq. (1) and Eq. (2) into Eq. (12), we find the following relation:

$$m\left\{s^2 F(s) - s x(0) - x'(0)\right\} = -\frac{C}{C_1^{1-\alpha}} \left\{s^{\alpha} F(s) - s^{\alpha-1} x(0)\right\}, \qquad (13)$$

which, upon substitution of Eq. (11), becomes

$$m\left\{s^2 F(s) - V_0\right\} = -\frac{C}{C_1^{1-\alpha}} s^{\alpha} F(s), \qquad (14)$$

where

$$F(s) = \frac{m V_0}{m s^2 + \frac{C}{C_1^{1-\alpha}} s^{\alpha}}. \qquad (15)$$

Therefore, we can write

$$x(t) = L^{-1}[F(s)] = V_0 L^{-1}\left[\frac{1}{s^2 + \frac{C}{m C_1^{1-\alpha}} s^{\alpha}}\right], \qquad (16)$$

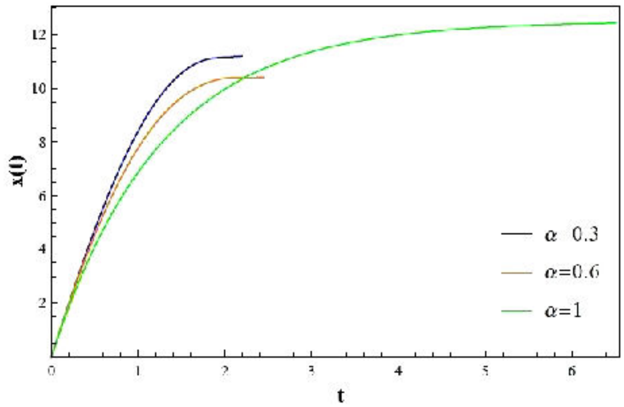

FIGURE 1. Horizontal component of the position $x(t)$ as a function for time t, given by Eq. (18).

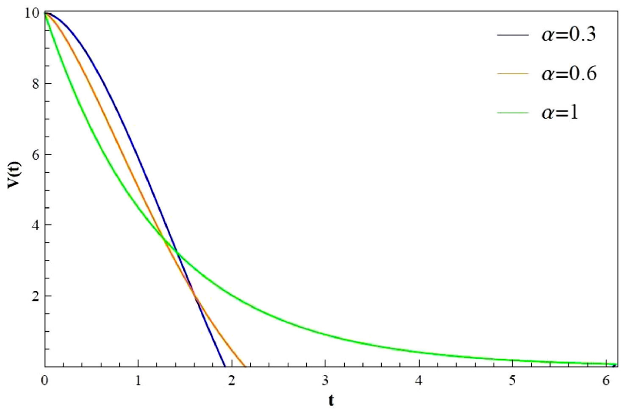

FIGURE 2. Velocity $V(t)$ as a function of time t, given by Eq. (19).

which can be compared with Eq. (5) to obtain the following parameters:

$$\alpha' = 2-\alpha, \qquad \beta' = 2, \qquad a = \frac{C}{mC_1^{1-\alpha}}, \tag{17}$$

after which the solution for $x(t)$ reads

$$x(t) = V_0 t E_{2-\alpha,\,2}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) = V_0 t\Bigg\{\frac{1}{\Gamma(2)} + \frac{-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}}{\Gamma(4-\alpha)} + \frac{\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right)^2}{\Gamma(6-2\alpha)} + \frac{\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right)^3}{\Gamma(8-3\alpha)} + \ldots\Bigg\}. \tag{18}$$

In Fig. 1, $x(t)$ with three different values of α as a function of t with parameters $C = 0.8$, $C_1 = 1.2$, $m = 1$ and $V_0 = 10$ has been plotted. Using the equation above, the velocity can be written as

$$V(t) = \frac{1}{m}t^{-\alpha}V_0\Bigg\{mt^{\alpha}E_{2-\alpha,2}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) - \frac{C}{C_1^{1-\alpha}}t^2\Bigg[E_{2-\alpha,3-\alpha}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) - E_{2-\alpha,4-\alpha}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right)\Bigg]\Bigg\} \tag{19}$$

In Fig. 2, we have plotted $V(t)$ with three different values of α as a function of t with parameters $C = 0.8$, $C_1 = 1.2$, $m = 1$ and $V_0 = 10$. Then, we have obtained the acceleration as

$$a(t) = \frac{1}{m^2}\frac{C}{C_1^{1-\alpha}}t^{1-2\alpha}V_0\Bigg\{\frac{C}{C_1^{1-\alpha}}t^2E_{2-\alpha,4-2\alpha}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) + (3+\alpha)\Bigg[\frac{C}{C_1^{1-\alpha}}t^2E_{2-\alpha,5-2\alpha} \times\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) - \frac{C}{C_1^{1-\alpha}}t^2E_{2-\alpha,6-2\alpha}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) + mt^{\alpha}\Bigg[E_{2-\alpha,3-\alpha}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) - E_{2-\alpha,4-\alpha}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right)\Bigg]\Bigg]\Bigg\} \tag{20}$$

Special cases:
$1)$ For $\alpha = 1/2$

Recalling Eqs. (4) and (17) and substituting into Eq. (18), the obtained solution becomes

$$x(t) = V_0\,t\,E_{\frac{3}{2},\,2}\left(-\frac{C}{mC_1^{\frac{1}{2}}}t^{\frac{3}{2}}\right) = V_0 t\left\{\frac{1}{\Gamma(2)} + \frac{-\frac{C}{mC_1^{\frac{1}{2}}}t^{\frac{3}{2}}}{\Gamma\left(\frac{7}{2}\right)} + \frac{\left(-\frac{C}{mC_1^{\frac{1}{2}}}t^{\frac{3}{2}}\right)^2}{\Gamma(5)} + \frac{\left(-\frac{C}{mC_1^{\frac{1}{2}}}t^{\frac{3}{2}}\right)^3}{\Gamma\left(\frac{13}{2}\right)} + \ldots\right\}. \tag{21}$$

So, velocity and acceleration can be calculated as follows:

$$V(t) = \frac{1}{m} V_0 \left\{ m E_{\frac{3}{2},2}\left(-\frac{C}{mC_1^{\frac{1}{2}}} t^{\frac{3}{2}}\right) + \frac{C}{C_1^{\frac{1}{2}}} t^{\frac{3}{2}} \left[E_{\frac{3}{2},\frac{5}{2}}\left(-\frac{C}{mC_1^{\frac{1}{2}}} t^{\frac{3}{2}}\right) + E_{\frac{3}{2},\frac{7}{2}}\left(-\frac{C}{mC_1^{\frac{1}{2}}} t^{\frac{3}{2}}\right)\right]\right\} \tag{22}$$

$$\begin{aligned} a(t) = \frac{1}{2m^2} \frac{C}{C_1^{\frac{1}{2}}} \sqrt{t} V_0 \Bigg[& -5m E_{\frac{3}{2},\,\frac{5}{2}}\left(-\frac{C}{mC_1^{\frac{1}{2}}} t^{\frac{3}{2}}\right) + 2\frac{C}{C_1^{\frac{1}{2}}} t^{\frac{3}{2}} E_{\frac{3}{2},\,3}\left(-\frac{C}{mC_1^{\frac{1}{2}}} t^{\frac{3}{2}}\right) \\ & + 5m E_{\frac{3}{2},\frac{7}{2}}\left(-\frac{C}{mC_1^{\frac{1}{2}}} t^{\frac{3}{2}}\right) - 5\frac{C}{C_1^{\frac{1}{2}}} t^{\frac{3}{2}} E_{\frac{3}{2},4}\left(-\frac{C}{mC_1^{\frac{1}{2}}} t^{\frac{3}{2}}\right) + 5\frac{C}{C_1^{\frac{1}{2}}} t^{\frac{3}{2}} E_{\frac{3}{2},5}\left(-\frac{C}{mC^{\frac{1}{2}}} t^{\frac{3}{2}}\right)\Bigg], \end{aligned} \tag{23}$$

in view of other special approaches as follows 2) For $C = 0$, Eq. (21) leads

$$x(t) = V_0\, t \tag{24}$$

4. Vertical motion of a body in a resisting medium

Now let us consider the vertical motion of a body in a resisting medium in which there exists a retarding force proportional to the fractional velocity. In this case, we consider that the body is projected downward with zero initial velocity $v(0) = 0$ in a uniform gravitational field. Then the equation of motion is given by

$$m D_t^2 y(t) = mg - \frac{C}{C_1^{1-\alpha}} D_t^{\alpha} y(t), \qquad 0 < \alpha \leq 1 \tag{25}$$

with the following initial condition

$$y(0) = y_0, \quad y'(0) = 0 \tag{26}$$

Taking the Laplace transform of both side of the Eq. (25), we get

$$\begin{aligned} m\left[s^2 F(s) - s y(0) - y'(0)\right] &= \frac{mg}{s} - \frac{C}{C_1^{1-\alpha}} \\ &\times \left[\frac{sF(s) - y(0)}{s^{1-\alpha}}\right]. \end{aligned} \tag{27}$$

Solving the Eq. (27) with respect to $f(s)$, we have

$$\begin{aligned} F(s) = & \frac{g}{\left[s^3 + \frac{C}{mC_1^{1-\alpha}} s^{\alpha+1}\right]} + \frac{y_0}{\left[s + \frac{C}{mC_1^{1-\alpha}} s^{\alpha-1}\right]} \\ & + \frac{\frac{C}{mC_1^{1-\alpha}} y_0}{\left[s^{3-\alpha} + \frac{C}{mC_1^{1-\alpha}} s\right]}, \end{aligned} \tag{28}$$

which can be rewritten as

$$\left\{ \begin{array}{l} \frac{g}{\left[s^3 + \frac{C}{mC_1^{1-\alpha}} s^{\alpha+1}\right]} \\ \\ \beta' = 3, \ \ \alpha' = 2-\alpha \end{array} \right., \quad \left\{ \begin{array}{l} \frac{y_0}{\left[s + \frac{C}{mC_1^{1-\alpha}} s^{\alpha-1}\right]} \\ \\ \beta' = 1, \ \ \alpha' = 2-\alpha \end{array} \right.,$$

$$\left\{ \begin{array}{l} \frac{\frac{C y_0}{mC_1^{1-\alpha}}}{\left[s^{3-\alpha} + \frac{C}{mC_1^{1-\alpha}} s\right]} \\ \\ \beta' = 3-\alpha, \ \ \alpha' = 2-\alpha \end{array} \right. . \tag{29}$$

Using the inverse Laplace transform $y(t) = L^{-1}[F(s)]$, we have

$$\begin{aligned} y(t) = & y_0 E_{2-\alpha,1}\left(-\frac{C}{mC_1^{1-\alpha}} t^{2-\alpha}\right) \\ & + g t^2 E_{2-\alpha,3}\left(-\frac{C}{mC_1^{1-\alpha}} t^{2-\alpha}\right) + \frac{C}{mC_1^{1-\alpha}} y_0 t^{2-\alpha} \\ & \times E_{2-\alpha,3-\alpha}\left(-\frac{C}{mC_1^{1-\alpha}} t^{2-\alpha}\right). \end{aligned} \tag{30}$$

For the special case when $\alpha = 1$, we obtain

$$\begin{aligned} y(t) = & y_0 E_{1,1}\left(-\frac{C}{m} t\right) + g t^2 E_{1,3}\left(-\frac{C}{m} t\right) \\ & + \frac{C}{m} y_0 t^{2-\alpha} E_{1,2}\left(-\frac{C}{m} t\right), \end{aligned} \tag{31}$$

which can be expanded in series as

$$y(t) = y_0 + \frac{1}{2!} g t^2 + \frac{1}{3!}\left(-\frac{C}{m}\right) g t^3 + \frac{1}{4!}\left(\frac{C}{m}\right)^2 g t^4 + \ldots, \tag{32}$$

so that in the limit of $C \to 0$,

$$y(t) = y_0 + \frac{1}{2!} g t^2. \tag{33}$$

On the other hand, for $\alpha = 1/2$ we will have

$$\begin{aligned} y(t) = & y_0 E_{\frac{3}{2},1}\left(-\frac{C}{mC_1^{\frac{1}{2}}} t^{\frac{3}{2}}\right) + g t^2 E_{\frac{3}{2},3}\left(-\frac{C}{mC_1^{\frac{1}{2}}} t^{\frac{3}{2}}\right) \\ & + \frac{C}{mC_1^{\frac{1}{2}}} y_0 t^{\frac{3}{2}} E_{\frac{3}{2},\frac{5}{2}}\left(-\frac{C}{mC_1^{\frac{1}{2}}} t^{\frac{3}{2}}\right). \end{aligned} \tag{34}$$

For simplicity above equation can be written as

$$\begin{aligned} y(t) = & y_0 \frac{1}{\Gamma(1)} \\ & + g t^2 \left\{ \frac{1}{\Gamma(3)} + \frac{-\frac{C}{mC_1^{\frac{1}{2}}} t^{\frac{3}{2}}}{\Gamma\left(\frac{9}{2}\right)} + \frac{\frac{C^2}{m^2 C_1} t^3}{\Gamma(6)} + \ldots \right\}. \end{aligned} \tag{35}$$

where, after using $\Gamma(3) = 2!$, the result is read as

$$y\left(t\right) = y_0 + \frac{1}{2}gt^2 - \frac{\frac{C}{mC_1^{\frac{1}{2}}}gt^{\frac{7}{2}}}{\Gamma\left(\frac{9}{2}\right)} + ... \qquad (36)$$

5. Motion of a projectile in a resisting medium

In this section we are interested in considering motion of a projectile in a resisting viscoelastic medium in which there exists a retarding force proportional to the fractional velocity. In this case we have the following equations

$$mD_t^2 x\left(t\right) = -\frac{C}{C_1^{1-\alpha}}D_t^{\alpha}x\left(t\right),$$

$$mD_t^2 y\left(t\right) = -mg - \frac{C}{C_1^{1-\alpha}}D_t^{\alpha}y\left(t\right), \qquad 0 < \alpha < 1 \quad (37)$$

with the initial conditions

$$x\left(0\right) = 0, \qquad y\left(0\right) = 0,$$

$$x'\left(0\right) = V_0\cos\theta, \qquad y'\left(0\right) = V_0\sin\theta. \qquad (38)$$

Taking the Laplace transform of both on both sides of Eq. (39), we can find

$$F\left(s\right) = \frac{V_0\cos\theta}{ms^2 + \frac{C}{C_1^{1-\alpha}}s^{\alpha}}, \qquad (39)$$

$$G\left(s\right) = -\frac{mg}{ms^3 + \frac{C}{C_1^{1-\alpha}}s^{\alpha+1}} + \frac{V_0\sin\theta}{ms^2 + \frac{C}{C_1^{1-\alpha}}s^{\alpha}}. \qquad (40)$$

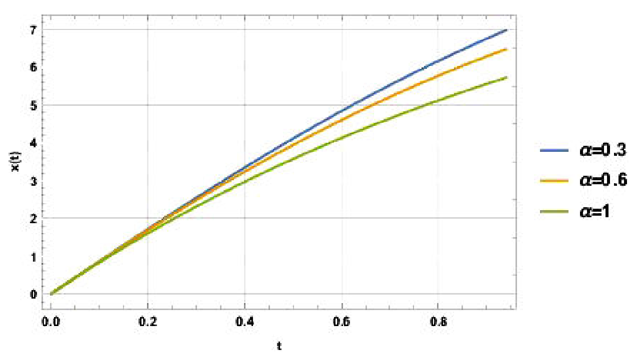

FIGURE 3. Horizontal component of the position $x(t)$ as a function of time t, given by Eq. (41).

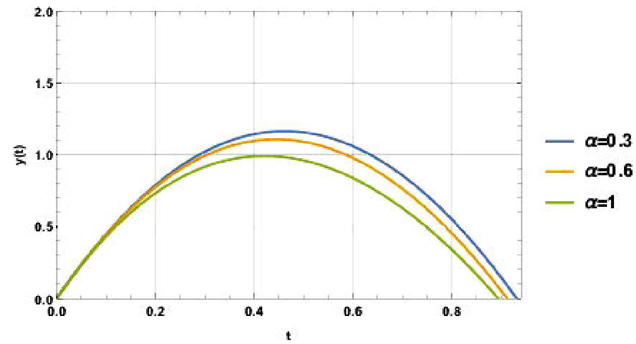

FIGURE 4. Vertical component of the position $y(t)$ as a function of time t, given by Eq. (42).

where $F\left(s\right)$ and $G\left(s\right)$ are Laplace transforms of $x\left(t\right)$ and $y\left(t\right)$, respectively. Using the inverse Laplace transform and properties of Mittag-Leffler function, we have

$$x\left(t\right) = V_0\cos\theta\, t\, E_{2-\alpha,2}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right), \qquad (41)$$

$$y\left(t\right) = -gt^2E_{2-\alpha,3}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) + V_0\sin\theta\, t\, E_{2-\alpha,2}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right). \qquad (42)$$

In Fig. 3, we have plotted $x(t)$ with three different values of α as a function of t with parameters $C = 0.8$, $C_1 = 1.2$, $m = 1$, $\theta = \pi/6$ and $V_0 = 10$. Also, in Fig. 4, we have plotted $y(t)$ with three different values of α as a function of t with parameters $C = 0.8$, $C_1 = 1.2$, $m = 1$, $g = 10$, $\theta = \pi/6$ and $V_0 = 10$.

Differentiating $x\left(t\right)$ and $y\left(t\right)$ with respect to the time, the velocity can be calculated as

$$x'\left(t\right) = \frac{1}{m}t^{-\alpha}V_0\cos\theta\left\{m\,t^{\alpha}\,E_{2-\alpha,2}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) - \frac{C}{C_1^{1-\alpha}}\,t^2\left[E_{2-\alpha,3-\alpha}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) - E_{2-\alpha,4-\alpha}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right)\right]\right\}, \qquad (43)$$

and

$$y'\left(t\right) = \frac{1}{m}t^{-\alpha}\left[-2gmt^{1+\alpha}E_{2-\alpha,3}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) - 2\frac{C}{C_1^{1-\alpha}}gt^3E_{2-\alpha,5-\alpha}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) + mt^{\alpha}V_0\,\sin\theta E_{2-\alpha,2}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) - \frac{C}{C_1^{1-\alpha}}t^2V_0\sin\theta E_{2-\alpha,3-\alpha}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right) + \frac{C}{C_1^{1-\alpha}}t^2\left(gt + V_0\,\sin\theta\right)E_{2-\alpha,4-\alpha}\left(-\frac{C}{mC_1^{1-\alpha}}t^{2-\alpha}\right)\right]. \qquad (44)$$

If we denote the range and the time required for the entire trajectory by R' and T' respectively, the following representation is obtained

$$y\left(t=T'\right)=0. \tag{45}$$

Now consider the case that $\alpha=1-\varepsilon$ and ε is sufficiently small. In this case we have

$$\begin{aligned} E_{2-\alpha,\ell} \sim & \frac{1}{\left(-\frac{C}{mC_1^{\varepsilon}}t\right)^{\ell-1}}\left[e^{-\frac{C}{mC_1^{\varepsilon}}t}-\sum_{n=0}^{\ell-2}\left(-\frac{C}{mC_1^{\varepsilon}}t\right)^{n}\right] \\ & +\varepsilon\sum_{n=0}^{\infty}\frac{n}{\Gamma\left(n+\ell\right)}F\left(n+\ell-1\right)\left(-\frac{C}{mC_1^{\varepsilon}}t\right)^{n}+\varepsilon\ln t\sum_{n=0}^{\infty}\frac{n}{\Gamma\left(n+\ell\right)}\left(-\frac{C}{mC_1^{\varepsilon}}t\right)^{n}. \end{aligned} \tag{46}$$

By using Eqs. (45) and (46), up to a first order in ε, we have

$$T'=\frac{2V_0\sin\theta}{g}\left(1-\frac{CV_0\sin\theta}{3mgC_1^{\varepsilon}}\right)+\varepsilon\frac{2C{V_0}^2\sin^2\theta}{3mg^2C_1^{\varepsilon}}\left(-\gamma+\frac{13}{6}+\ln\frac{2V_0\sin\theta}{g}\right), \tag{47}$$

which, when α goes to 1, can be simplified into

$$T'\rightarrow T=\frac{2V_0\sin\theta}{g}\left(1-\frac{CV_0\sin\theta}{3mgC_1^{\varepsilon}}\right). \tag{48}$$

The range is obtained from the relation $R'=x(T')$ as

$$R'=\frac{V_0^2\sin 2\theta}{g}\left(1-\frac{4CV_0\sin\theta}{3mgC_1^{\varepsilon}}\right)+\varepsilon\frac{2C{V_0}^3\sin^2\theta\cos\theta}{9mg^2C_1^{\varepsilon}}, \tag{49}$$

which can be reduced, when α goes to 1, to we can also have

$$R'\rightarrow R=\frac{V_0^2\sin 2\theta}{g}\left(1-\frac{4CV_0\sin\theta}{3mgC_1^{\varepsilon}}\right). \tag{50}$$

Therefore, the change due to the fractional resistance is given by

$$\Delta R=R'-R=\varepsilon\frac{2C{V_0}^3\sin^2\theta\cos\theta}{9mg^2C_1^{\varepsilon}}>0. \tag{51}$$

Thus, the range becomes larger for the fractional resistance when compared with the linear resistance case.

6. Conclusion

In this article, we have considered fractional calculus as a new tool in studying interesting aspects of classical mechanics. First, we have briefly discussed the basic concepts of fractional calculus and we have presented an interpretation of fractional derivative and solution of fractional equations analytically. Then, by considering the modeling of viscoelastic systems within the fractional calculus framework, we have investigated applications of this approach in three different problems in classical mechanics including the study of resisted motion of a particle in a viscoelastic medium, the vertical motion of a body in a resisting medium and the motion of a projectile in a resisting medium. The obtained results satisfy the ordinary results of classical mechanics in. It has also been proved that the ordinary solutions are obtained provided the fractional effects are removed. Thus, the results demonstrate that the proposed fractional model presents an enhanced description as compared to that of the non-fractional models have shown when probing the different aspects of mechanical physics.

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1. M. Dalir and M. Bashour, Applications of fractional calculus, *Appl. Math. Sci.* **4** (2010) 1021.

2. M. M. Meerschaert and C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations, *Appl. Numer. Math.* **56** (2006) 80. https://doi.org/10.1016/j.apnum.2005.02.008

3. H. Jafari, C.M. Khalique, and M. Nazari, An algorithm for the numerical solution of nonlinear fractional-order Van der Pol oscillator equation, *Math. Comput. Model*. **55** (2012) 1782. https://doi.org/10.1016/j.mcm.2011.11.029.

4. H. Jiang *et al.*, Analytical solutions for the multi-term time-space Caputo-Riesz fractional advection-diffusion equations on a finite domain, *J. Math. Anal. Appl.* **389** (2012) 1117. https://doi.org/10.1016/j.jmaa.2011.12.055.

5. R. Almeida and D. Torres, Calculus of variations with fractional derivatives and fractional integrals, *Appl. Math. Lett.* **22** (2009) 1816. https://doi.org/10.1016/j.aml.2009.07.002.

6. M. Gülsu, Y. Öztürk, and A. Anapali, Numerical approach for solving fractional relaxation-oscillation equation, *Appl. Math. Model.* **37** (2013) 5927. https://doi.org/10.1016/j.apm.2012.12.015.

7. D. Baleanu, Fractional variational principles in action, *Phys. Scr.* **136** (2009) 014006. https://doi.org/10.1088/0031-8949/2009/T136/014006.

8. A. Iomin, Fractional-time quantum dynamics, *Phys. Rev. E* **80** (2009) 022103. https://doi.org/10.1103/PhysRevE.80.022103.

9. W. Bu, Y. Tang and J. Yang, Galerkin finite element method for two dimensional Riesz space fractional diffusion equations, *J. Comput. Phys.* **276** (2014) 26. https://doi.org/10.1016/j.jcp.2014.07.023.

10. W. Bu *et al.*, Finite difference/finite element method for twodimensional space and time fractional Bloch-Torrey equations, *J. Comput. Phys.* **293** (2015) 264. https://doi.org/10.1016/j.jcp.2014.06.031.

11. M. Efe, Battery power loss compensated fractional order sliding mode control of a quadrotor UAV, *Asian J. Control* **14** (2012) 413. https://doi.org/10.1002/asjc.340.

12. Y. Li, Y. Chen and H. Ahn, Fractional-order iterative learning control for fractional-order linear systems, *Asian. J. Control* **13** (2011) 54. https://doi.org/10.1002/asjc.253.

13. B. Jin *et al.*, The Galerkin finite element method for a multiterm time-fractional diffusion equation, *J. Comput. Phys.* **281** (2015) 825. https://doi.org/10.1016/j.jcp.2014.10.051.

14. F. Mainardi and G. Spada, Creep, relaxation and viscosity properties for basic fractional models in rheology, *Eur. Phys. J. Spec. Top.* **193** (2011) 133. https://doi.org/10.1140/epjst/e2011-01387-1.

15. R. Lewandowski and B. Chorażyczewski, Identification of the parameters of the Kelvin-Voigt and the Maxwell fractional models, used to modeling of viscoelastic dampers, *Comput. Struct.* **88** (2010) 1. https://doi.org/10.1016/j.compstruc.2009.09.001.

16. R. Lewandowski and Z. Pawlak, Dynamic analysis of frames with viscoelastic dampers modelled by rheological models with fractional derivatives, *J. Sound Vib.* **330** (2011) 923. https://doi.org/10.1016/j.jsv.2010.09.017.

17. A. Kilbas, H. Strivatava and J.Trujillo, *Theory and Application of Fractional Differential Equations* 1st ed. (Elsevier Science, Amsterdam, 2006).

18. I. Pdolubny, *Fractional Differential Equations* 1st ed. (Academic Press, New York, 1998).

19. R. Hilfer, *Application of fractional Calculus in Physics* (World Scientific, Singapore, 2011), https://doi.org/10.1142/3779.

20. R. Herrmann, *Fractional calculus*, 1st ed. (World Scientific, Singapore, 2011), https://doi.org/10.1142/8072.

21. K. Diethelm, *The Analysis of Fractional Differential Equations* (Springer-Verlag, Berlin, 2010), https://doi.org/10.1007/978-3-642-14574-2.

22. W. S. Chung and H. Hassanabadi, Dynamics of a Particle in a Viscoelastic Medium with Conformable Derivative, *Int. J. Theor. Phys.* **56** (2017) 851. https://doi.org/10.1007/s10773-016-3228-z.

23. F. S. Mozaffari, H. Hassanabadi, H. Sobhani and W. S. Chung, On the Conformable Fractional Quantum Mechanics, *J. Korean Phys. Soc.* **72** (2018) 980. https://doi.org/10.3938/jkps.72.980.

24. F. S. Mozaffari, H. Hassanabadi, H. Sobhani and W. S. Chung, Investigation of the Dirac Equation by Using the Conformable Fractional Derivative, *J. Korean Phys. Soc.* **72** (2018) 987, https://doi.org/10.3938/jkps.72.987.

25. I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications* **198** (Academic press 1998).

26. E. C. Grigoletto and E. C. de Oliveira, Fractional Versions of the Fundamental Theorem of Calculus, *Appl. Math.* **4** (2013) 23. https://doi.org/10.4236/am.2013.47A006.

27. D. Baleanu and T. Avkar, Lagrangians with linear velocities within Riemann-Liouville fractional derivatives, *Nuovo Cimento B* **119** (2004) 73. https://doi.org/10.1393/ncb/i2003-10062-y.

28. K. Diethelm, Efficient Solution of Multi-Term Fractional Differential Equations Using P(EC)^{m}E Methods, *Computing* **71** (2003) 305. https://doi.org/10.1007/s00607-003-0033-3.

29. K. Diethelm, N.J. Ford, and A.D. Freed, Detailed Error Analysis for a Fractional Adams Method, *Numer. Algorithms* **36** (2004) 31. https://doi.org/10.1023/B:NUMA.0000027736.85078.be.

30. S. Mulish and D. Baleanu, Hamiltonian formulation of systems with linear velocities within Riemann-Liouville fractional derivatives, *J. Math. Anal. Appl.* **304** (2005) 599. https://doi.org/10.1016/j.jmaa.2004.09.043.

31. F. Barpi and S. Valente, Creep and fracture in concrete: a fractional order rate approach, *Eng. Fract. Mech.* **70** (2002) 611. https://doi.org/10.1016/S0013-7944(02)00041-3.

32. S. Mulish and D. Baleanu, Formulation of Hamiltonian Equations for Fractional Variational Problems, Czechoslov. *J. Phys*. **55** (2005) 633. https://doi.org/10.1007/s10582-005-0067-1.

33. D. Baleanu and S. Mulish, Lagrangian Formulation of Classical Fields within Riemann-Liouville Fractional Derivatives, *Phys. Scr.* **72** (2005) 119. https://doi.org/10.1238/Physica.Regular.072a00119.

34. D. Craiem *et al.,* Fractional Calculus Applied to Model Arterial Viscoelasticity, *Lat. Am. Appl. Res.* **38** (2008) 141.

35. R. Churchill, *Operational Mathematics* 3rd ed. (McGraw-Hill, New York, 1972).

36. S. Kazem, Exact Solution of Some Linear Fractional Differential Equations by Laplace Transform, *Int. J. Nonlinear Sci.* **16** (2013) 3.

37. J. F. Gómez-Aguilar, J. J. Rosales-García, J. J. Bernal-Alvarado, T. Córdova-Fraga and R. Guzmán-Cabrera, Fractional mechanical oscillators, *Rev. Mex. Fis.*, **58** (2012) 348.

38. J. J. R. García, M. G. Calderon, J. M. Ortiz and D. Baleanu, Motion of a particle in a resisting medium using fractional calculus approach, *Proc. Romanian Acad. A*, **14** (2013) 42.

39. A. O. Contreras, J. J. R. García, L. M. Jiménez and J. M. Cruz-Duarte, Analysis of projectile motion in view of conformable derivative, *Open Phys.* **16** (2018) 581. https://doi.org/10.1515/phys-2018-0076.

40. B. Ahmad, H. Batarfi, J. J. Nieto, Ó. Otero-Zarraquiños and W. Shammakh, Projectile motion via Riemann-Liouville calculus *Adv. Diff. Eqs.*, **2015** (2015) 63. https://doi.org/10.1186/s13662-015-0400-3.

41. A. Ebaid, Analysis of projectile motion in view of fractional calculus, *Appl. Math. Model.* **35** (2011) 1231. https://doi.org/10.1016/j.apm.2010.08.010.