Nambu-Goto action and classical rebits in any signature and in higher dimensions

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We perform an extension of the relation between the Nambu-Goto action and classical rebits. Of course, the Cayley hyperdeterminant is the key mathematical tool in such generalization. Using the Wick rotation, we find that in four dimensions such a relation can be established no only with the signature (2+2) but also with any signature. We generalize our result to a curved space-time of (2\(n+1\)+2\(n+1\))-dimensions. Moreover, we also prove that our method can actually extend the Duff’s procedure to any signature in 4\(n\)-dimensions. Here, we shall prove that if one introduces a Wick rotations for various coordinates then one can actually extend the Duff’s procedure to any signature in 4-dimensions. Moreover, we also prove that our method can be extended to curved space-time in (2\(n+2\))-dimensions and (2\(n+1\)+2\(n+1\))-dimensions.

There are a number of physical reasons to be interested on these developments, but perhaps the most important is that eventually our work may be useful on a possible generalization of the remarkable correspondence between black-holes and quantum information theory (see Refs. 7 to 10 and references therein). In fact, from the expression which is, of course, the Polyakov action (see Ref. 12 and references therein) obtained by varying the action (2) with respect to \(g^{ab}\), it is straightforward to show that from (2) one obtains (1) and vice versa. Hence, the actions (1) and (2) are equivalents.

It is convenient to define the induced world-sheet metric

\[
\eta_{\mu\nu} = \partial_\mu x^a \partial_\nu x^a, \tag{4}
\]

Using this definition, the Nambu-Goto action (1) becomes

\[
S = \int d\xi^2 \sqrt{-\epsilon \det{h}}. \tag{5}
\]

It is not difficult to see that in (2+2)-dimensions the expression (4) can be written as

\[
h_{\mu\nu} = \partial_\mu x^{ij} \partial_\nu x^{kl} \varepsilon_{ik} \varepsilon_{jl}, \tag{6}
\]

where \(x^{ij}\) denotes a the \(2 \times 2\) matrix

\[
x^{ij} = \begin{pmatrix}
x^1 + x^3 & x^2 + x^4 \\
-x^2 + x^4 & x^1 - x^3
\end{pmatrix}. \tag{7}
\]
It is important to observe that (7) corresponds to the set $M(2, R)$ of any $2 \times 2$-matrix. In fact, by introducing the fundamental base matrices
\begin{align*}
\delta^{ij} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\varepsilon^{ij} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\eta^{ij} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\lambda^{ij} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{align*}
(8)
on one observes that (7) can be rewritten as the linear combination
\begin{equation}
x^{ij} = x^1 \delta^{ij} + x^2 \varepsilon^{ij} + x^3 \eta^{ij} + x^4 \lambda^{ij}.
\end{equation}
(9)

Let us now introduce the expression
\begin{equation}
h = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} h_{ac} h_{bd}.
\end{equation}
(10)
If one uses (4) one gets
\begin{equation}
h = \det(h_{ab}).
\end{equation}
(11)
However, if one considers (6) one obtains
\begin{equation}
h = D\det(h_{ab}),
\end{equation}
(12)
where $D\det(h_{ab})$ denotes the Cayley hyperdeterminant of $h_{ab}$, namely
\begin{equation}
D\det(h_{ab}) = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} \\
\times \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{mr} \varepsilon_{ns} \partial_a x^{ij} \partial_b x^{kl} \partial_c x^{mn} \partial_d x^{rs}.
\end{equation}
(13)
Of course, (11) and (12) imply that
\begin{equation}
\det(h_{ab}) = D\det(h_{ab}).
\end{equation}
(14)
In turn, (14) means that in $(2 + 2)$-dimensions the Nambu-Goto action (5) can also be written as
\begin{equation}
S = \int d\xi^2 \sqrt{-D\det(h_{ab})}.
\end{equation}
(15)
Note that, since in this case one is considering the $(2 + 2)$-signature one must set $\epsilon = +1$ in (5).

In $(4 + 4)$-dimensions the key formula (6) can be generalized as
\begin{equation}
h_{ab} = \partial_a x^{ijm} \partial_b x^{kl} \varepsilon_{ik} \varepsilon_{jl} \eta_{ms}.
\end{equation}
(16)
While in $(8 + 8)$-dimensions one has
\begin{equation}
h_{ab} = \partial_a x^{ijm} \partial_b x^{kl} \varepsilon_{ik} \varepsilon_{jl} \eta_{ms} \varepsilon_{nr},
\end{equation}
(17)
(see Refs. 5 and 6 for details). So by considering the real variables $x^{i_1 \ldots i_n}$ and properly considering the matrices $\varepsilon_{ij}$ and $\eta_{ij}$ the previous formalism can be generalized to higher dimensions. Of course, in such cases the Cayley hyperdeterminant $D\det(h_{ab})$ must be modified accordingly.

Observing (7) one wonders whether one can consider in (6) other signatures in 4-dimensions besides the $(2 + 2)$-signature. It is not difficult to see that using the Wick rotation in any of the coordinates $x^1$, $x^2$, $x^3$ or $x^4$ one can modify the signature. For instance, one can achieve the $(1 + 3)$-signature if one uses the prescription $x^2 \to i x^2$ in (6). This method lead us inevitably to generalize our method to a complex structure. One simple introduce the complex matrix
\begin{equation}
z^{ij} = z^1 \delta^{ij} + z^2 \varepsilon^{ij} + z^3 \eta^{ij} + z^4 \lambda^{ij},
\end{equation}
(18)
where the variables $z^1$, $z^2$, $z^3$ and $z^4$ are complex numbers. The expression (6) is generalized accordingly as [13]
\begin{equation}
h_{ab} = \partial_a z^{ij} \partial_b z^{kl} \varepsilon_{ik} \varepsilon_{jl}.
\end{equation}
(19)
Thus, in this case, the Cayley hyperdeterminant becomes
\begin{equation}
D\det(h_{ab}) = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} \\
\times \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{mr} \varepsilon_{ns} \partial_a z^{ij} \partial_b z^{kl} \partial_c z^{mn} \partial_d z^{rs}.
\end{equation}
(20)
and consequently the Nambu-Goto action must be written using (20). Of course, the Nambu-Goto action, or the Polyakov action, must be real and therefore one must choose any of the coordinates $z^1$, $z^2$, $z^3$ and $z^4$ in (20) either as pure real or pure imaginary.

Similarly, the generalization to a complex structure can be made by introducing the complex variables $z^{i_1 \ldots i_n}$ and writing
\begin{equation}
D\det(h_{ab}) = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} \\
\times \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{mr} \varepsilon_{ns} \partial_a z^{i_1 \ldots i_n} \partial_b z^{j_1 \ldots j_n} \\
\times \varepsilon_{k_1 \ldots k_n} \partial_c z^{i_1 \ldots i_n} \partial_d z^{j_1 \ldots j_n} \partial_e z^{k_1 \ldots k_n} \partial_f z^{l_1 \ldots l_n},
\end{equation}
(21)
or
\begin{equation}
D\det(h_{ab}) = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} \\
\times \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{mr} \varepsilon_{ns} \partial_a z^{i_1 \ldots i_n} \partial_b z^{j_1 \ldots j_n} \partial_c z^{k_1 \ldots k_n} \partial_d z^{l_1 \ldots l_n},
\end{equation}
(22)
depending whether the signature is $(2^n + 2^n)$ or $(2^{n+1} + 2^{n+1})$, respectively.

One can further generalize our procedure by considering a target curved space-time. For this purpose let us introduce the curved space-time metric
\begin{equation}
g_{\mu \nu} = e^A_{\mu} e^B_{\nu} \eta_{AB}.
\end{equation}
(23)
Here, $e^A_{\mu}$ denotes a vielbein field and $\eta_{AB}$ is a flat metric. The Polyakov action in a curved target space-time becomes
\begin{equation}
S = \int d\xi^2 \sqrt{-g} \det\left( g_{\mu \nu} \partial_a x^{\mu} \partial_b x^{\nu} \right) g_{\mu \nu}.
\end{equation}
(24)
Using (23), one sees that this action can be written as
\begin{equation}
S = \int d\xi^2 \sqrt{-\epsilon} \det\left( g_{\mu \nu} \partial_a x^{\mu} e^{A}_{\mu}(\partial_b x^{\nu}) e^{B}_{\nu} \right) \eta_{AB}.
\end{equation}
(25)
So, by defining the quantity
\[ E^A_\mu = \partial_\mu x^\nu e^A_\nu, \]
The action in (25) reads as
\[ S = \int \sqrt{-\varepsilon} \det g^{ab} E^A_\alpha E^B_\beta \eta_{AB}. \] (27)
Hence, in a target space-time of \((2 + 2)\)-dimensions one can write (27) in the form
\[ S = \int \sqrt{-\varepsilon} \det g^{ab} E^i_\alpha E^j_\beta \varepsilon_{ik} \varepsilon_{jl}, \] (28)
where
\[ E^i_\alpha = \partial_\alpha x^\mu e^A_\mu. \] (29)
Here, we considered the fact that one can always write
\[ \varepsilon^{ij} = e^1_\alpha \delta^{ij} + e^2_\alpha \varepsilon^{ij} + e^3_\alpha \eta^{ij} + e^4_\alpha \lambda^{ij}. \] (30)
In order to clarify the possible application of the generalization of (4) namely
\[ h_{ab} = E^A_\alpha E^B_\beta \eta_{AB} \] (31)
and therefore in \((2 + 2)\)-dimensions this expression becomes
\[ h_{ab} = E^i_\alpha E^j_\beta \varepsilon_{ik} \varepsilon_{jl}, \] (32)
while in \((4 + 4)\)-dimensions and \((8 + 8)\)-dimensions one obtains
\[ h_{ab} = E^i_\alpha E^j_\beta \varepsilon_{ik} \varepsilon_{jl} \eta_{mr}, \] (33)
and
\[ h_{ab} = E^i_\alpha E^j_\beta \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{mr} \varepsilon_{ns}, \] (34)
respectively. The analogue of Cayley hyperdeterminant in this case will be
\[ \det(h_{1,1,2,2}) = \varepsilon_{1111} \varepsilon_{2222} \varepsilon_{3333} \varepsilon_{4444} \] (39)
and therefore the corresponding Nambu-Goto action becomes
\[ S = \int \sqrt{-\varepsilon} \det(h_{1,1,2,2}). \] (40)

3. Conclusions and comments

We have generalized the Duff’s procedure concerning the combination of the Nambu-Goto action and the Cayley hyperdeterminant in target space-time of \((2 + 2)\)-dimensions. Such a generalization first corresponds to a curved worlds with \((2^{2n} + 2^{2n})\)-signature or \((2^{2n+1} + 2^{2n+1})\)-signature. Using complex structure we may be able to extend the procedure to any signature. Further, we generalize the method to \(p\)-branes.

It turns out that these generalization may be useful in a number of physical scenario beyond string theory and \(p\)-branes. In fact, since the quantity \(\varepsilon^{ij} \cdots \alpha\) can be identified with a \(n\)-complex rebot one may be interested in the route leading to oriented matroid theory \([14]\) (see also Ref. 15 and 16). In this direction, using the phireotope concept (see Ref. 17 and references therein), which is a complex generalization of the concept of chirotope in oriented matroid theory, a link between super-\(p\)-branes and qubits (in this context) has already been established \([17]\). Thus, it may be interesting for further developments to explore the connection between the results of the present work and supersymmetry via the Grassmann-Plücker relations (see Refs. 8 and 9 and references therein). It is worth mentioning that such relations are natural mathematical notions in information theory linked to \(n\)-qubit entanglement. Indeed, in such a case, the Hilbert space can be broken in the form \(C^{2n} = C^L \otimes C^l\) with \(L = 2n - 1\) and \(l = 2\). This allows a geometric interpretation in terms of the complex Grassmannian variety \(Gr(L, l)\) of \(2\)-planes in \(C^{2n}\) via the Plücker embedding. In this context, the Plücker coordinates of Grassmannians \(Gr(L, l)\) are natural invariants of the theory (see Ref. 9 for details). However, it has been mentioned in Ref. 18, and proved in Refs. 19 and 20, that for normalized qubits the complex 1-qubit, 2-qubit and the 3-qubit are deeply related to division algebras via the Hopf maps, \(S^3 = S_1 \rightarrow S^2\), \(S^7 = S^4 \rightarrow S^4\), and \(S^{15} = S^8 \rightarrow S^8\), respectively. In order to clarify the possible application of these observations in the context of our formalism let us consider the general complex state \(|\psi\rangle \in C^{2n}\),

\[ |\psi\rangle = \sum_{i_1 i_2 \cdots i_n = 0}^1 C^{i_1 i_2 \cdots i_n} |i_1 i_2 \cdots i_n\rangle, \] (41)
learns that the SL(2 + 2) time Lorentz in sheet diffeomorphism. The second two factors are space-SL and Spin-dimensions, namely Spin(2, 2) ⊃ SL(2, R) ⊃ SU(2) ⊃ SO(1, 3). It is interesting to make the following observations. First, one finds that a 3-rebit and 4-rebit have 8 and 16 real degrees of freedom, respectively. Thus, one learns that the 4-rebit can be associated with the 16 degrees of freedom of a 3-qubit. It turns out that this is the kind of embedding discussed in Ref. 9. Second, one may expect the quantum development of the Nambu-Goto action in n-dimensions to consider the 16-dimensions of target space-time as the maximum dimension required by division algebras via the Hopf map $S^{15} \to S^8$. Finally, the question arises whether in our generalized formalism one may also find hidden symmetries of the Nambu-Goto action in the sense of Ref. 1. In (2+2)-dimensions the hyperdeterminant turns out to be invariant under
\[
[SL(2, R) \times SL(2, R) \times SL(2, R)] \times S_3. \tag{42}
\]
Here, the first SL(2, R) is a global subgroup of the world-sheet Lorentz-invariance. The second two factors are space-time SL, namely Spin(2, 2) ⊃ SL(2, R), by complexifying the $x^a$ one may take different real forms, Spin(2, 2) ⊃ SL(2, R) $\times$ SL(2, R), Spin(1, 3) ⊃ SL(2, C), Spin(4) $\cong$ SU(2) $\times$ SU(2) to obtain various signatures. However, only in (2 + 2)-dimensions one has the three factors SL(2, R) in the same footing and hence additional S3. In the case of (4 + 4)-dimensions one may consider the chain of maximal embeddings and branches,
\[
so(4, 4) \supset s(2, R) \oplus so(2, 3) \supset so(1, 1) \oplus sl(2, R) \oplus sl(2, 2). \tag{43}
\]
However, these subgroups are not full symmetry groups and therefore it is difficult to reveal hidden discrete symmetries of the Nambu-Goto action in this case. In other cases the analysis seems even more difficult, but this motivate us to explore in more detail these developments.

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