RESEARCH Revista Mexicana de Física **66** (3) 308–314 MAY-JUNE 2020

The Wigner-Dunkl-Newton mechanics with time-reversal symmetry

W. Sang Chung[a] and H. Hassanabadi[b]
[a] *Department of Physics and Research Institute of Natural Science,*
College of Natural Science, Gyeongsang National University, Jinju 660-701, Korea.
e-mail: mimip44@naver.com
[b] *Faculty of Physics, Shahrood University of Technology, Shahrood, Iran.*
e-mail: h.hasanabadi@shahroodut.ac.ir

Received 2 December 2019; accepted 15 January 2020

In this paper, we use the Dunkl derivative concerning to time to construct the Wigner-Dunkl-Newton mechanics with time-reversal symmetry. We define the Wigner-Dunkl-Newton velocity and Wigner-Dunkl-Newton acceleration and construct the Wigner-Dunkl-Newton equation of motion. We also discuss the Hamiltonian formalism in the Wigner-Dunkl-Newton mechanics. We discuss some deformed elementary functions such as the ν-deformed exponential functions, ν-deformed hyperbolic functions and ν-deformed trigonometric functions. Using these, we solve some problems in one dimensional Wigner-Dunkl-Newton mechanics.

Keywords: Dunkl derivative; Wigner-Dunkl-Newton mechanics; time-reversal symmetry.

PACS: 03.65.Ge; 03.65.Fd; 02.30.Gp

DOI: https://doi.org/10.31349/RevMexFis.66.308

1. Introduction

There are several ways to replace the Newtonian mechanics with a deformed version, which is performed by adopting a deformed derivative instead of the Newton derivative in defining the velocity. For example, q-derivative [1-2] gives the q-deformed mechanics [3-4], fractional derivative [5-10] gives the fractional mechanics [11-14].

Let us consider the deformation

$$\frac{d}{dt} \to D_t^A, \tag{1}$$

where D_t^A is the deformed time derivative depending on the deformation parameter A. Then the definition of the velocity is deformed as

$$v_A = D_t^A x. \tag{2}$$

Here the deformed velocity reduces to the ordinary Newton velocity when the special value of A is taken ($q = 1$ in the q-deformed mechanics and $\alpha = 1$ in the fractional mechanics). This deformed mechanics deserves study as a kind of effective theory when we deal with the dynamics of complicated dynamical models.

As another example of the deformed derivative, we can consider Dunkl derivative which has been widely used in many works in various field of physics including quantum system [15-31]. Recently, the authors of Ref. 29 used the Dunkl derivative to discuss the one-dimensional quantum mechanical model, which is called a Wigner-Dunkl quantum mechanics. Here, the momentum operator is expressed in terms of the Dunkl derivative instead of the ordinary derivative;

$$\hat{p} = \frac{1}{i} D_x^\nu, \quad \hat{x} = x, \tag{3}$$

where we set $\hbar = 1$, and the Dunkl derivative is defined as

$$D_x^\nu = \partial_x + \frac{\nu}{x}(1 - \mathcal{R}), \quad \mathcal{R} = (-1)^{x\partial_x}. \tag{4}$$

Here, $\mathcal{R}$ is the spatial parity (or reflection) operator obeying $\mathcal{R} : x \to -x$. Then the Heisenberg relation is deformed as

$$[\hat{x}, \hat{x}] = i(1 + 2\nu\mathcal{R}), \tag{5}$$

which is called a Wigner algebra, and the Wigner parameter ν is assumed to be real. In Ref. [29] the time derivative was not replaced by the Dunkl derivative.

If we consider the undeformed quantum theory in 1 + 1 dimension, time and Hamiltonian can be regarded as the quantum operators obeying

$$[\hat{t}, \hat{H}] = -i. \tag{6}$$

The relation (6) can be deformed through Dunkl derivative concerning as follows:

$$[\hat{t}, \hat{H}] = -i(1 + 2\nu\mathcal{T}), \quad \nu \in \mathbb{R}, \tag{7}$$

where $\mathcal{T}$ is the temporal parity (or time-reversal) operator obeying $\mathcal{T} : t \to -t$, *i.e.*,

$$\mathcal{T}F(t) = F(-t). \tag{8}$$

Then, the time realization is given by

$$\hat{t} = t, \quad \hat{H} = iD_t^\nu, \tag{9}$$

where the Dunkl time-derivative is defined as

$$D_t^\nu = \frac{d}{dt} + \frac{\nu}{t}(1 - \mathcal{T}). \tag{10}$$

Dunkl time derivative arises in the time-dependent Schrödinger equation for Wigner Dunkl quantum mechanics in the form

$$iD_t^\nu \psi(x,t) = \left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right]\psi(x,t)$$
$$= \left[-\frac{1}{2m}(D_x^\mu)^2 + V\right]\psi(x,t). \tag{11}$$

Another example of Dunkl time derivative is the application of it to the Dunkl electrodynamics [30]. Here the authors introduced the Dunkl field strength tensor of the form

$$F_{ab}^{(\nu)} = D_a^\nu A_b - D_b^\nu A_a, \quad (a = 0,1,2,3), \tag{12}$$

where 0 means time-component and A_a means the Dunkl-deformed electromagnetic 4-potential. Using the Dunkl field strength tensor, they discussed Dunkl-Maxwell theory.

The introduction of the Dunkl derivative concerning time in the quantum theory is related to the Dunkl derivative with respect to time in the classical theory. In this paper, we will introduce the Dunkl derivative with respect to time in the classical Newton mechanics when we define the velocity or acceleration. This gives a new deformed mechanics called a Wigner-Dunkl-Newton (WDN) mechanics. This paper is organized as follows. In Sec. 2, we discuss WDN equation of motion. In Sec. 3, we discuss Hamiltonian formalism in WDN mechanics. In Sec. 4, we discuss the ν-deformed functions. In Sec. 5, we discuss some mechanical examples.

2. WDN equation of motion

In this section, we will introduce the Dunkl time derivative so that it may deform the ordinary Newton mechanics. Now let us introduce the WDN velocity with the help of Dunkl derivative with respect to time as

$$v_\nu(t) = D_t^\nu x(t) = \frac{d}{dt}x(t) + \frac{\nu}{t}(x(t) - x(-t)). \tag{13}$$

The WDN velocity is the same as the ordinary velocity when $x(t)$ is even. But, for odd $x(t)$, we have $v_\nu(t) = (d/dt)x(t) + (2\nu/t)x(t)$. The WDN acceleration is also obtained by acting the Dunkl derivative with respect to time on the WDN velocity,

$$a_\nu(t) = D_t^\nu v(t) = (D_t^\nu)^2 x(t). \tag{14}$$

This can also be written as

$$a_\nu(t) = \frac{d}{dt}v(t) + \frac{\nu}{t}(v(t) - v(-t)) \tag{15}$$

or

$$a_\nu(t) = (D_t^\nu)^2 x(t) = \left(\frac{d}{dt}\right)^2 x(t)$$
$$+ \frac{2\nu}{t}\frac{d}{dt}x(t) - \frac{\nu}{t^2}(x(t) - x(-t)), \tag{16}$$

where we used

$$\mathcal{T} D_t^\nu = -D_t^\nu. \tag{17}$$

The WDN velocity and WDN acceleration depend on the temporal parity of $x(t)$. With WDN velocity and WDN acceleration, the WDN equation of motion reads

$$F = m a_\nu(t) = m D_t^\nu v_\nu(t) = m(D_t^\nu)^2 x(t), \tag{18}$$

where F is a force.

When a moving observer (x', t') moves with the uniform WDN velocity u_ν relative to a fixed observer (x,t), the acceleration and velocity for a moving observer, a'_ν and v'_ν are related to those for a fixed observer, a_ν and v_ν as follows:

$$a'_\nu = a_\nu \tag{19}$$

$$v'_\nu = v_\nu - u_\nu \tag{20}$$

$$x' = x - \frac{u_\nu}{1+2\nu}t, \quad t' = t, \tag{21}$$

The last equation gives the Wigner-Dunkl-Galilei (WDG) transformation.

For the time -reversal (temporal parity), any function can be decomposed into the function with even temporal parity and the function with odd temporal parity, *i.e.*, any function F is given by

$$F = F_e + F_o, \tag{22}$$

where

$$\mathcal{T} F_e = F_e, \quad \mathcal{T} F_o = -F_o. \tag{23}$$

Thus, the WDN equation of motion for odd and even part reads

$$F_e = m\left(\ddot{x}_e + \frac{2\nu}{t}\dot{x}_e\right) \tag{24}$$

$$F_o = m\left(\ddot{x}_o + \frac{2\nu}{t}\dot{x}_o - \frac{2\nu}{t^2}x_o\right). \tag{25}$$

3. Hamiltonian formalism in WDN mechanics

In the WDN mechanics, the work is not well defined because we have no information for the inverse of the Dunkl derivative (Dunkl integral). Nevertheless, we can obtain the conserved Hamiltonian by introducing the deformed Poisson bracket.

From the time-dependent Schrödinger equation for Wigner Dunkl quantum mechanics, we know the Hamiltonian for classical variables x, p is given by

$$H = \frac{p^2}{2m} + V(x) = E = const. \tag{26}$$

From the deformed commutator relation (5), we can define the deformed Poisson bracket (DPB) as

$$\{f(x,p), g(x,p)\}_{DPB} = D_x^\nu f \partial_p g - D_x^\nu g \partial_p f. \tag{27}$$

Indeed, the Eq. (27) gives the relation

$$\{x, p\}_{DPB} = 1 + 2\nu. \tag{28}$$

From the time-dependent Schrödinger equation for Wigner Dunkl quantum mechanics, the time evolution of some classical quantity $A(x,p)$ is defined as

$$D_t^\nu A = \{A, H\}_{DPB}, \tag{29}$$

which gives the Dunkl-Wigner-Hamilton equation

$$D_t^\nu p = -D_x^\nu V \tag{30}$$

$$D_t^\nu x = (1+2\mu)\frac{p}{m}. \tag{31}$$

Thus, WDN equation of motion reads

$$m(D_t^\nu)^2 x = -(1+2\mu) D_x^\nu V. \tag{32}$$

The evolution of the Hamiltonian is

$$D_t^\nu H = \{H, H\}_{DPB} = 0, \tag{33}$$

which implies that the Hamiltonian is constant, *i.e.*, a conserved quantity. From now on we define the force corresponding to the conserved Hamiltonian as

$$F = -(1+2\nu) D_x^\nu V, \tag{34}$$

where we call V a WDN potential energy in the WDN mechanics. Like the ordinary Newton mechanics, if there does not exist potential energy obeying the Eq. (34), we have the WDN equation of motion,

$$F = m(D_t^\nu)^2 x. \tag{35}$$

4. The ν-deformed functions

In this section, we discuss the ν-exponential function, ν-deformed hyperbolic functions, and ν-deformed trigonometric functions. First, consider the following ν-deformed differential equation

$$D_t^\nu y(t) = a y(t), \quad y(0) = 1. \tag{36}$$

We will denote the solution of the above equation by

$$y(t) = e_\nu(at), \tag{37}$$

which we call ν-exponential function.

Considering parity, we can set the solution of the Eq. (36) as

$$y(t) = y_e(t) + y_o(t), \tag{38}$$

where $y_e(t)$ is the even function obeying $\mathcal{T} y_e(t) = y_e(t)$, while $y_o(t)$ is the odd function obeying $\mathcal{T} y_o(t) = -y_o(t)$. Inserting the Eq. (38) into the Eq. (36) and splitting the Eq. (36) into the even part and odd part we get

$$\frac{dy_e(t)}{dt} = a y_o(t)$$
$$\frac{dy_o(t)}{dt} + \frac{2\nu}{t} y_o(t) = a y_e(t). \tag{39}$$

From the parity of $y_e(t)$ and $y_o(t)$, we can set

$$y_e(t) = \sum_{n=0}^{\infty} a_n t^{2n}$$
$$y_o(t) = \sum_{n=0}^{\infty} b_n t^{2n+1}. \tag{40}$$

Inserting the Eq. (40) into the Eq. (39), we get the following recurrence relations

$$2(n+1) a_{n+1} = a b_n \tag{41}$$

$$(2n+1+2\nu) b_n = a a_n \tag{42}$$

Inserting the Eq. (42) into the Eq. (41), we have

$$a_{n+1} = \frac{a^2}{2(n+1)(2n+1+2\nu)} a_n, \tag{43}$$

which gives

$$a_n = \frac{1}{n! \left(\nu + \frac{1}{2}\right)_n} \left(\frac{a}{2}\right)^{2n} \tag{44}$$

$$b_n = \frac{1}{n! \left(\nu + \frac{1}{2}\right)_{n+1}} \left(\frac{a}{2}\right)^{2n+1}. \tag{45}$$

Thus, we have the following solution of the Eq. (36):

$$y(t) = e_\nu(at) = \cosh_\nu(at) + \sinh_\nu(at), \tag{46}$$

where ν-deformed hyperbolic functions are defined as

$$\cosh_\nu(at) = \sum_{n=0}^{\infty} \frac{1}{n! \left(\nu + \frac{1}{2}\right)_n} \left(\frac{at}{2}\right)^{2n}$$
$$= {}_0F_1\left(; \nu + \frac{1}{2}; \frac{a^2 t^2}{4}\right) \tag{47}$$

$$\sinh_\nu(at) = \sum_{n=0}^{\infty} \frac{1}{n! \left(\nu + \frac{1}{2}\right)_{n+1}} \left(\frac{at}{2}\right)^{2n+1}$$
$$= \frac{at}{2\nu+1} {}_0F_1\left(; \nu + \frac{3}{2}; \frac{a^2 t^2}{4}\right), \tag{48}$$

and

$${}_0F_1(; a; t) = \sum_{n=0}^{\infty} \frac{1}{n!(a)_n} t^n, \tag{49}$$

and

$$(a)_0 = 1, \quad (a)_n = a(a+1)(a+2)\cdots(a+n-1). \tag{50}$$

Here, the parity relation for the ν-deformed hyperbolic functions are as follows:

$$\mathcal{T}\cosh_\nu(at) = \cosh_\nu(at) \tag{51}$$

$$\mathcal{T}\sinh_\nu(at) = -\sinh_\nu(at). \tag{52}$$

It can be easily checked that the ν-deformed hyperbolic functions reduce to $\cosh(at)$ and $\sinh(at)$ in the limit $\nu \to 0$. Acting the ν-derivative on the ν-exponential function and the ν-deformed hyperbolic functions, we have

$$D_t^\nu e_\nu(at) = a e_\nu(at) \tag{53}$$

$$D_t^\nu \cosh_\nu(at) = a\sinh_\nu(at) \tag{54}$$

$$D_t^\nu \sinh_\nu(at) = a\cosh_\nu(at). \tag{55}$$

If we replace $t \to it$ in the Eq. (46), we have the ν-deformed Euler relation

$$e_\nu(iat) = \cos_\nu(at) + i\sin_\nu(at), \tag{56}$$

where

$$\cos_\nu(at) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\left(\nu+\frac{1}{2}\right)_n}\left(\frac{at}{2}\right)^{2n} = {}_0F_1\left(;\nu+\frac{1}{2};-\frac{a^2t^2}{4}\right) \tag{57}$$

$$\sin_\nu(at) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\left(\nu+\frac{1}{2}\right)_{n+1}}\left(\frac{at}{2}\right)^{2n+1} = \frac{at}{2\nu+1}\,{}_0F_1\left(;\nu+\frac{3}{2};-\frac{a^2t^2}{4}\right). \tag{58}$$

One can also express the ν-deformed trigonometric functions as

$$\cos_\nu(at) = 2^{\nu-1/2}\Gamma\left(\nu+\frac{1}{2}\right)(at)^{1/2-\nu}J_{\nu-1/2}(at) \tag{59}$$

$$\sin_\nu(at) = \frac{1}{2\nu+1}2^{\nu+1/2}\Gamma\left(\nu+\frac{3}{2}\right) \times (at)^{1/2-\nu}J_{\nu+1/2}(at). \tag{60}$$

Figure 1 shows the plot of $y=\cos_\nu(t)$ for $\nu=0$ (Gray), $\nu=0.2$ (Brown), and for $\nu=-0.2$ (Pink). Figure 2 shows the plot of $y=\sin_\nu(t)$ for $\nu=0$ (Gray), $\nu=0.2$ (Brown), and for $\nu=-0.2$ (Pink).

Acting the ν-derivative on the ν-deformed trigonometric functions, we get the following relations

$$D_t^\nu \cos_\nu(at) = -a\sin_\nu(at),$$

$$D_t^\nu \sin_\nu(at) = a\cos_\nu(at). \tag{61}$$

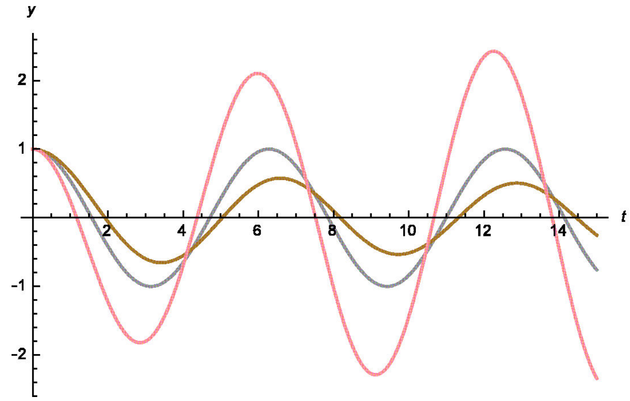

FIGURE 1. Plot of $y=\cos_\nu(x)$ for $\nu=0$ (Gray), $\nu=0.2$ (Brown), and $\nu=-0.2$ (Pink).

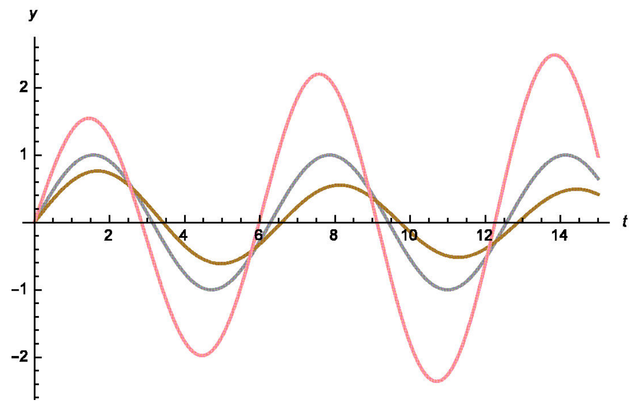

FIGURE 2. Plot of $y=\sin_\nu(x)$ for $\nu=0$ (Gray), $\nu=0.2$ (Brown), and $\nu=-0.2$ (Pink).

5. Some examples

Let us discuss some examples for the WDN mechanics in one dimension.

5.1. Particle at rest

Let us consider that a particle is at rest, and its position is $x(0)$. In ordinary Newton mechanics, we have $v(t)=0$ which gives $x(t)=x(0)$. In WDN mechanics, this case is replaced with

$$v_\nu(t) = 0, \tag{62}$$

or

$$\frac{dx(t)}{dt} + \frac{\nu}{t}(1-\mathcal{T})x(t) = 0. \tag{63}$$

Let us set

$$x(t) = x_e(t) + x_o(t). \tag{64}$$

Inserting the Eq. (64) into the Eq. (63) and splitting into the even part and the odd part we get

$$\frac{dx_e(t)}{dt} = 0, \tag{65}$$

$$\frac{dx_o(t)}{dt} + \frac{2\nu}{t}x_o(t) = 0. \tag{66}$$

Solving these with the initial position $x(0)$ we get

$$x_e(t) = x(0), \qquad x_o(t) = 0, \tag{67}$$

which gives the same result as the ordinary Newton mechanics,

$$x(t) = x(0). \tag{68}$$

5.2. Uniform WDN velocity

Let us consider that a particle moves with the uniform WDN velocity, $u_\nu = const$, and its initial position is $x(0)$. In WDN mechanics, we have

$$v_\nu(t) = u_\nu, \tag{69}$$

or

$$\frac{dx(t)}{dt} + \frac{\nu}{t}(1-\mathcal{T})x(t) = u_\nu. \tag{70}$$

Solving this equation, we have

$$\frac{dx_e(t)}{dt} = 0, \tag{71}$$

$$\frac{dx_o(t)}{dt} + \frac{2\nu}{t}x_o(t) = u_\nu. \tag{72}$$

From the Eq. (71), we get

$$x_e(t) = x_e(0). \tag{73}$$

In the Eq. (72), we set

$$x_o(t) = \sum_{n=0}^{\infty} c_n t^{2n+1}. \tag{74}$$

Inserting the Eq. (74) into the Eq. (72) we get

$$c_0 = \frac{u_\nu}{1+2\nu}, \quad c_1 = c_2 = \cdots = 0. \tag{75}$$

Thus, we have

$$x(t) = x(0) + \frac{u_\nu}{1+2\nu}t. \tag{76}$$

Because u_ν is even, we have $(1-\mathcal{T})u_\nu = 0$. Thus, the WDN acceleration becomes zero.

5.3. Uniform WDN acceleration

Let us consider that a particle moves with the uniform WDN acceleration, $a_\nu = const$, and its initial position is $x(0)$, and its initial velocity $v(0)$. In WDN mechanics, we have

$$a_\nu(t) = a_\nu \tag{77}$$

or

$$\frac{dv_\nu(t)}{dt} + \frac{\nu}{t}(1-\mathcal{T})v_\nu(t) = a_\nu \tag{78}$$

Solving this equation, we have

$$v_\nu(t) = v(0) + \frac{a_\nu}{1+2\nu}t \tag{79}$$

or

$$\frac{dx(t)}{dt} + \frac{\nu}{t}(1-\mathcal{T})x(t) = v(0) + \frac{a_\nu}{1+2\nu}t, \tag{80}$$

which gives

$$\frac{dx_e(t)}{dt} = \frac{a_\nu}{1+2\nu}t \tag{81}$$

$$\frac{dx_o(t)}{dt} + \frac{2\nu}{t}x_o(t) = v(0). \tag{82}$$

Thus, we have

$$x(t) = x(0) + \frac{v(0)}{1+2\nu}t + \frac{a_\nu}{2(1+2\nu)}t^2. \tag{83}$$

5.4. Resisted motion with linear damping

Let us consider that a particle moves in the viscous medium with the resistance proportional to the WDN velocity, and its initial position is $x(0)$, and its initial WDN velocity $v(0)$. In WDN mechanics, we have

$$ma_\nu(t) = -m\gamma v_\nu(t) \tag{84}$$

or

$$D_t^\nu v_\nu(t) = \frac{dv_\nu(t)}{dt} + \frac{\nu}{t}(1-\mathcal{T})v_\nu(t) = -\gamma v_\nu(t). \tag{85}$$

Solving this equation, we have

$$v_\nu(t) = v(0)e_\nu(-\gamma t). \tag{86}$$

5.5. Harmonic oscillator

In WDN mechanics, the WDN equation for the harmonic oscillator is given by

$$ma_\nu(t) = -kx \tag{87}$$

or

$$m(D_t^\nu)^2 x = -kx. \tag{88}$$

Solving the above equation with initial condition $x(0) = A, v_\nu(0)$, we have

$$x(t) = A\cos_\nu\left(\sqrt{\frac{k}{m}}t\right). \tag{89}$$

Thus, the motion becomes non-periodic unless $\nu = 0$.

6. Conclusion

From the introduction of the Dunkl derivative concerning time in the quantum theory [15], we proposed a new deformed mechanics called WDN mechanics, where the WDN velocity and WDN acceleration are defined by the Dunkl derivative for time. We discussed Hamiltonian formalism in WDN mechanics. For the Dunkl time derivative, we found some deformed elementary functions such as the ν-deformed exponential functions, ν-deformed hyperbolic functions, and ν-deformed trigonometric functions. Using these functions, we solved some problems in one-dimensional WDN mechanics. Some problems remain unsolved for WDN mechanics. For example, the work is not well defined for WDN mechanics. For this reason, we obtained the conserved Hamiltonian from the deformed Poisson bracket. We think that these problems and their related topics will become clear soon.

Acknowledgement

The authors thank the referee for a thorough reading of our manuscript and the constructive suggestion.

1. F. H. Jackson, A Basic-sine and cosine with symbolical solutions of certain differential equations, *Proc. Edinburgh Math. Soc.* **22** (1903) 28, https://doi.org/10.1017/S0013091500001930.
2. F. H. Jackson, On q-Functions and a certain Difference Operator, *Trans. Roy. Soc. Edinburgh* **46** (1909) 253, https://doi.org/10.1017/S0080456800002751.
3. A. Lavagno, A. M. Scarfone, and P. Narayana Swamy, Classical and quantum q-deformed physical systems, *Eur. Phys. J. C* **47** (2006) 253, https://doi.org/10.1140/epjc/s2006-02557-y.
4. P. Caban, A. Dobrosielski, A. Krajewska, and Z. Walczak, On q-deformed Hamiltonian mechanics, *Phys. Lett. B* **327** (1994) 287, https://doi.org/10.1016/0370-2693(94)90730-7.
5. S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives* (Gordon and Breach Science Publishers, Philadelphia, 1993).
6. K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (John Wiley and Sons, New York, 1993).
7. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* (Elsevier Science, Amsterdam, 2006).
8. I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999).
9. K. Oldham and J. Spanier, *The Fractional Calculus* (Academic Press, New York, 1974).
10. R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.* **264** (2014) 65, https://doi.org/10.1016/j.cam.2014.01.002.
11. M. Klimek, Lagrangian fractional mechanics -a noncommutative approach, *Czechoslov. J. Phys.* **55** (2005) 1447, https://doi.org/10.1007/s10582-006-0024-7.
12. F. Riewe, Mechanics with fractional derivatives, *Phys. Rev. E* **55** (1997) 3581, https://doi.org/10.1103/PhysRevE.55.3581.
13. W. S. Chung, Fractional Newton mechanics with conformable fractional derivative, *J. Comput. Appl. Math.* **290** (2015) 150, https://doi.org/10.1016/j.cam.2015.04.049.
14. W. S. Chung and H. Hassanabadi, Dynamics of a Particle in a Viscoelastic Medium with Conformable Derivative, *Int. J. Theor. Phys.* **56** (2017) 851, https://doi.org/10.1007/s10773-016-3228-z.
15. C. F. Dunkl, Differential-difference operators associated to reflection groups, *Trans. Am. Math. Soc.* **311** (1989) 167, https://doi.org/10.1090/S0002-9947-1989-0951883-8.
16. C. F. Dunkl, *Orthogonal Polynomials of Several Variables* (Cambridge University Press, Cambridge, 2001), https://doi.org/10.1017/CBO9781107786134.
17. L. M. Yang, A Note on the Quantum Rule of the Harmonic Oscillator, *Phys. Rev.* **84** (1951) 788, https://doi.org/10.1103/PhysRev.84.788.
18. I. Cherednik, A unification of Knizhnik-Zamolodchikov and Dunkl operators via affine Hecke algebras, *Invent. Math.* **106** (1991) 411, https://doi.org/10.1007/BF01243918.
19. E. M. Opdam, Harmonic analysis for certain representations of graded Hecke algebras, *Acta Math.* **175** (1995) 75, https://doi.org/10.1007/BF02392487.
20. K. Hikami, Dunkl Operator Formalism for Quantum Many-Body Problems Associated with Classical Root Systems, *J. Phys. Soc. Jpn. b* (1996) 394, https://doi.org/10.1143/JPSJ.65.394.
21. S. Kakei, Common algebraic structure for the Calogero-Sutherland models, *J. Phys. A* **29** (1996) L619, https://doi.org/10.1088/0305-4470/29/24/002.
22. L. Lapointe and L. Vinet, Exact operator solution of the Calogero-Sutherland model, *Commun. Math. Phys.* **178** (1996) 425, https://doi.org/10.1007/BF02099456.
23. V. X. Genest, L. Vinet, and A. Zhedanov, The singular and the 2:1 anisotropic Dunkl oscillators in the plane, *J. Phys. A* **46** (2013) 325201, https://doi.org/10.1088/1751-8113/46/32/325201.
24. V. X. Genest, M. E. H. Ismail, L. Vinet, and A. Zhedanov, The Dunkl Oscillator in the Plane II: Representations of the Symmetry Algebra, *Commun. Math. Phys.* **329** (2014) 999, https://doi.org/10.1007/s00220-014-1915-2.

25. V. X. Genest, L. Vinet, and A. Zhedanov, The Dunkl oscillator in three dimensions, *J. Phys. Conf. Ser.* **512** (2013) 012010, `https://doi.org/10.1088/1742-6596/512/1/012010`.
26. V. X. Genest, A. Lapointe, and L. Vinet, The Dunkl-Coulomb problem in the plane, *Phys. Lett. A* **379** (2015) 923, `https://doi.org/10.1016/j.physleta.2015.01.023`.
27. M. Salazar-Ramírez, D. Ojeda-Guillén, R. D. Mota, and V. D. Granados, SU(1, 1) solution for the Dunkl oscillator in two dimensions and its coherent states, *Eur. Phys. J. Plus* **132** (2017) 39, `https://doi.org/10.1140/epjp/i2017-11314-3`.
28. M. Salazar-Ramírez, D. Ojeda-Guillén, R. D. Mota, and V. D. Granados, SU(1, 1) solution for the Dunkl-Coulomb problem in two dimensions and its coherent states, *Mod. Phys. Lett. A* **33** (2018) 1850112, `https://doi.org/10.1142/S0217732318501122`.
29. W. S. Chung and H. Hassanabadi, One-dimensional quantum mechanics with Dunkl derivative, *Mod. Phys. Lett. A* **34** (2019) 1950190, `https://doi.org/10.1142/S0217732319501906`.
30. E. J. Jang, S. Park, and W. S. Chung, Electrostatics with reflection symmetry and exact solution of the Dunkl-Laplace equation in cylindrical coordinates, *J. Korean Phys. Soc*. **68** (2016) 379, `https://doi.org/10.3938/jkps.68.379`.
31. R. D. Mota, D. Ojeda-Guillén, M. Salazar-Ramírez, V. D. Granados, Exact solution of the relativistic Dunkl oscillator in (2+1) dimensions, *Ann. Phys*. **411** (2019) 167964, `https://doi.org/10.1016/j.aop.2019.167964`.