MEAN FIELD METHODS FOR TOPOLOGICAL EXCITATIONS OF SPIN AND LATTICE GAUGE MODELS

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RESUMEN

Se obtiene una aproximación de campo medio en términos de un parámetro de desorden para modelos Z(N) de espines y de normas, aplicando una transformación de dualidad a una aproximación de Bethe-Peierls. Se reproduce la fenomenología de la condensación de excitaciones topológicas (frustraciones).

ABSTRACT

A mean field approximation is obtained in terms of a disorder parameter for Z(N) spin and gauge models, by applying the duality transformation to a Bethe-Peierls approximation. The phenomenology of the condensation of topological excitations (frustrations) is reproduced.
1. INTRODUCTION

Order-disorder phase transitions in statistical models and field theories are usually interpreted by the spontaneous breaking of a global symmetry. Lattice gauge theories are in this respect different because the gauge invariance cannot be broken spontaneously. The sum over configurations is exactly local invariant; therefore, massless fluctuations, independent on every site of the lattice, force to zero the value of the field and of any other non-gauge invariant operator.

Numerical simulations indicate that Abelian gauge theories exhibit phase transitions between the confining phase and the deconfining one. Therefore, an analytic method is needed in order to determine the mechanism of the transition.

Mean field methods (MF) are usually applied in these problems as phenomenological descriptions to gain intuition and set the basis for more sophisticated tools.

However, the application of MF to gauge models is apparently inconsistent with their exact local symmetry property, since it predicts a non-zero mean value of the field, as in the global symmetric case. Although it has been suggested
how to recover the gauge invariance of the method at the end of the calculation (5), here we shall avoid this problem by introducing a MF description which is explicitly gauge invariant. This MF approach describes the phase transition in Z(N) gauge models at a finite coupling $\beta_c = O((N/2\pi)^2)$ for large N, as the spontaneous breaking of a global Z(N) symmetry. This invariance leads to the conservation of the topological charge associated to the disordered field configurations called "frustrations". The breaking mechanism is due to the occurrence of such configurations in the ground state (condensation of topological excitations (6)).

The "disorder" parameter which detects these configurations is the expectation value of a non local gauge invariant operator, whose expression is identified by the duality transformation in 3d, as we shall see in Section 2.

The method extends to Z(N) gauge model in 4d and applies as well to Z(N) spin model in 2d. These models exhibit two transition points for N>5, related to the condensation of frustrations at $\beta_1 = O((N/2\pi)^2)$ and to the condensation of "vortices" at $\beta_2 = O(1)$, where the vortices (or monopoles) are the topological excitations of the U(1) symmetric model (7), (8).

The results of the MF for frustrations will be analyzed in Section 3: they reproduce the existing phenomenology of the excitations which is based on duality (6), Kosterlitz-Thouless estimations of the free energy (8) and numerical simulations (3).

We obtain correct results for Z(2) gauge and spin models (9).

The Z(N) phase diagram, which has a richer structure, is partially reproduced. In particular, the conjectured scaling law for the Wilson action $\beta_1 = a(1 - \cos(2\pi/N))^{-1}$ is explicitly obtained (10).

A similar approach for vortex excitations is currently under investigation in order to complete the description of the Z(N) phase diagram.

2. THE BETHE-PEIERLS APPROXIMATION FOR FRUSTRATIONS

The MF for frustrations we are going to introduce for Z(N) gauge model in 3d is a kind of Bethe-Peierls approximation (11). The model is defined on a finite cluster and contains a border term, which is determined by a consistency equation for the order parameter. In order to identify the correct form of both the border term and the "disorder" parameter we shall use the duality transformation, which relates in 3d the Z(N) gauge model to the Z(N) spin model.
Let us consider the Bethe-Peierls approximation for the order-disorder transition of $Z(N)$ spin model on the finite cluster $\Lambda$ (see Fig.1).

Fig.1 - The cluster $\Lambda$ has white dots and full lines: $\theta_0$ is the variable on an internal site and $\theta_1$ on a border site. The cluster $\Lambda^*$ has dashed lines; $\theta_{a\beta}$ is the variable associated to a border plaquette (dashed).

The cluster action reads:

$$A_{Z(N)} = \beta \sum_{(x,\mu) \in \Lambda} \cos(\Delta_{\mu} \theta_x) + b \sum_{x \in \delta \Lambda} \cos(\theta_x) \tag{1}$$

where $\beta = T^{-1}$, $\Delta_{\mu} \theta_x = \theta_{x+\mu} - \theta_x$ and $\theta_x = 2\pi n_x / N$, $n_x = 0, 1, \ldots, N-1$ and the last summation runs on the sites of $\delta \Lambda$, the border of $\Lambda$. This border term is responsible for the breaking of the symmetry under the transformation $\theta_x \rightarrow \theta_x + 2\pi n / N$.

The order parameter is determined self-consistently by requiring translation invariance on the cluster $\Lambda$:

$$\langle \cos \theta_0 \rangle_{\beta,b} = \frac{\langle \cos \theta_1 \rangle_{\beta,b}}{b} \quad ; \quad \langle . \rangle_{\beta,b} = \frac{\sum e^A(.)}{\sum e^A} \tag{2}$$

Equation (2) is an implicit equation for $b$ as a function of $\beta$; the solution $b = 0$ is always present and moreover below the critical temperature ($\beta > \beta_c$) a solution $b \neq 0$ appears, which detects the breaking of the symmetry.
For instance, on the simple cluster in Fig.1 the N=2 consistency equation is
\[ \rho = \left(\frac{(t+\rho)}{(1+t\rho)}\right)^5; \rho = \exp(-2b), t = \exp(-2\beta) \]  
(3)

At the critical coupling \( B_c = 0.3466 \) two symmetric solutions start from the origin \( b=0 \) (second order phase transition).

The duality transformation is firstly performed on the partition function of the spin model, eq.(1). It amounts to use a Fourier expansion for each term in exp(\( A \)), as follows(6):
\[ \exp(\beta \cos(\Delta_s \Theta_x)) = \sum_{k \mu} \exp(ik_\mu(x)\Delta_s \Theta_x) a(k_\mu(x), \beta) \]  
(4a)

\[ = \sum_{k \mu} \frac{1}{\sqrt{2\pi}} \exp(ik_\mu \Delta_s \Theta_x - k_\mu^2/2\beta) \]  
(4b)

where in the last term the \( a(k_\mu(x), \beta) \) coefficients have been replaced by their expansion at large N and \( 1<<\beta<(N/2\pi)^2 \) (Villain approximation).

After this substitution the partition function of the spin model reads
\[ Z = \sum_{\Theta} \sum_{k \mu} e^{i \sum k_\mu (\Delta_s \Theta)} \{ \Pi a(k_\mu, \beta) \} e^{-i \sum r \theta} \]
\[ \cdot \{ \Pi' a(r, b) \} \]
\[ = \sum_{\Theta} \sum_{k \mu} \exp(-i \sum \Theta (\Delta_s k_\mu) + i \sum (k_\mu - r) \theta) \{ \Pi a(k_\mu, \beta) \} \]
\[ \cdot \{ \Pi' (a(r, b) a(k_\nu, \beta)) \} \]  
(5)

where \( \Sigma, \Pi \) run over all the internal sites of \( \Lambda \) and \( \Sigma', \Pi' \) run over the border ones, with \( \nu \) orthogonal to \( \delta \Lambda \). In the second step a summation by parts has been used. The steps of the transformation are the standard ones of ref.(6), where also the border term on the finite lattice has been considered. The summation over \( \Theta_x \) variables leads to the conditions
\[ \sum_{\mu} \Delta_s k_\mu(x) = 0 \quad \text{for the internal sites } x, \]
\[ k_\nu(y) = r(y) \quad \text{for the border sites } y \text{ and } \nu \perp \delta \Lambda. \]  
(6)

They are solved as usual, by writing \( k_\mu \) in terms of the variables \( \Theta_\mu^* \), defined on the links of the dual cluster \( \Lambda^* \), (see Fig.1).
Finally we get
\[ Z = Z_0 \sum_{\phi_p^*} \exp(A^*) \]
where
\[ A^* = -\frac{1}{2} \sum_{(u,v)} \epsilon_{uv} (\Delta_u \phi_v - \Delta_v \phi_u)^2 - \frac{1}{2} \sum_{(u,v)} \epsilon_{uv} (\Delta_u \phi_v - \Delta_v \phi_u)^2 \]
where \( Z_0 \) is a normalization factor and the action is quadratic when the Fourier coefficients have the Villain form (eq.(4.b)). In such an approximation the dual couplings are \( \beta = (1/\beta)(N/2\pi)^2 \) and \( \tilde{\beta} = (1/\beta)(N/2\pi)^2 \).

The border term in the gauge action (eq.(8)) is responsible for the symmetry breaking associated to the condensation of frustrations. In order to see how it works, let us derive the duality relations for the order parameters in eq.(2):

\[ <\cos \theta_0>_{, \beta, b} = \exp((\beta + \frac{1}{2}) (\cos(\theta_{a\beta} - 2\pi/N) - \cos(\theta_{a\beta})))_{\, b^*, b^*} \]  
\[ <\cos \theta_1>_{, \beta, b} = \exp(b^* (\cos(\theta_{a\beta} - 2\pi/N) - \cos(\theta_{a\beta})))_{\, b^*, b^*} \]  

where \( \theta_{a\beta}^* = \Delta_a \theta_{a\beta}^* - \Delta_\beta \theta_a^* \) is a border plaquette of the cluster \( \Lambda^* \) in Fig.1. The duality relations are exact for \( N=2 \) with \( \text{th}(\beta^*) = \exp(-2\beta) \) and \( \text{th}(b^*) = \exp(-2b) \); for large \( N \) they hold in the Villain quadratic approximtion. The r.h.s. of eqs. (9a,b) are respectively the expectation values of frustrated configurations on the cube \( C \in \Lambda^* \) and on the nearest neighbour external cube which shares a plaquette \( \theta_{a\beta}^* \).

The frustration is a disordered configuration which is detected in the statistical average by twisting of \( 2\pi/N \) the angle of the plaquettes orthogonal to a dual line \( L \) starting at \( C \) and ending at the border (see Fig.2; on the smallest cluster of Fig.1 this line has length one).

\[ \text{Fig.2} - \text{The Z(N) frustration on the cube C of the infinite lattice, having unit charge.} \]
In the large $\beta^*$ phase the energy of an isolated frustration is proportional to the length of the line; therefore, this excitation is forbidden by a volume effect in the infinite lattice limit. \(^{(8)}\)

We introduce the consistency equation for the gauge theory by equating the r.h.s. of eqs. (9a,b): according to the discussion of eq. (2) it has the unique solution $\beta = \infty$ in the ordered phase $\beta^* > \beta_c^* = 0 ((N/2\pi)^2)$; this solution reproduces the previous infinite volume effect by "freezing" the cluster boundary. The additional solution $\beta^* < \infty$, which appears below the transition point ($\beta^* < \beta_c^*$), allows a non zero value of the disorder parameter, corresponding to the condensation of frustrations.

Here we report some physical remarks and address to ref. \(^{(9)}\) for a more thorough discussion.

First of all the previous description of the phase transition occurring in the 3d $Z(N)$ gauge model at $\beta_c = 0 ((N/2\pi)^2)$ is based on an explicitly gauge invariant disorder parameter (see eqs. (9a,b)): this description is "complementary" to the usual order-disorder one, in the sense that the broken and the unbroken phases are interchanged.

The symmetry broken by the condensation mechanism can be identified as a "dual" realization of the global symmetry of the spin theory\(^{(9)}\). This symmetry leads to the conservation of the topological charge, i.e. the number of frustrated cubes modulo $N$ for $Z(N)$, and breaks at the condensation point, when the vacuum contains an arbitrary number of frustrations.

This Bethe-Peierls approximation can be extended to treat the 4d gauge model and the 2d spin model. Duality transforms these models into themselves; the form of the action is invariant for $N=2$ and for large $N$, again in the Villain form; the dual couplings are the same as in the 3d case. The topology of the frustrations in these models is known by duality arguments: in the 2d spin model the frustration is on a plaquette, i.e. a twist $\pi/N$ is added to the difference of spin angles on links orthogonal to a dual line starting at the plaquette and going to the border. In the 4d gauge theory there is a dual line of frustrated 3d cubes, corresponding to a family of twisted plaquettes on a surface.

This approach has some formal advantages on standard MF methods in these theories. Also in 4d we have explicit gauge invariance of the method: furthermore, selfduality suggests how to recover gauge invariance on the standard MF method\(^{(4)}\), just by expanding the size of the cluster and considering a Bethe-Peierls approximation for the Wilson "line" operator,
the dual of the disorder parameter. In 2d the Mermin-Wagner theorem is partially satisfied: the magnetization vanishes in the intermediate Coulomb phase, while the frustration—which should vanish as well—is different from zero.

3. RESULTS AND DISCUSSION

The MF for frustrations has been applied to $Z(2)$ gauge models in 3d and 4d and to the spin model in 2d, and extended to the related $Z(N)$ symmetric models. For the $N=2$ case our results agree with the existing phenomenology and the numerical estimates. The transition is second order in 2d and 3d, and it is first order in 4d; Figs. 3 and 4 show the possibility of the method to reproduce the theory both in the strong and the weak coupling phases, with increasing accuracy by approximating the theory on larger clusters.

![Fig. 3 - Plot of the internal energy as a function of $t=\theta^*$: the MF for frustrations (full line) on a 3X3X2 cluster, compared with the weak coupling series expansion (dashed line) and Montecarlo points.](image-url)
Fig. 4 - The same as in Fig. 3, but in 4d: the frustration MF (full line) on a $2^d$ cluster is compared with the strong and the weak coupling expansions (dashed lines).

For the $N \geq 5$ cases the method describes the transition occurring at $\beta = O(N^2)$; a maximum variation is observed in the internal energy in correspondence to the other transition point which is present in 2d and 4d.

The scaling law of the transition temperature in any dimension, for the cosine action,

$$\beta_1 = \alpha (1 - \cos(2\pi/N))^{-1}$$

is reproduced by the MF on the simplest cluster and the values of $\alpha$ are reported in Table 1.

<table>
<thead>
<tr>
<th>Model</th>
<th>MF</th>
<th>Montecarlo</th>
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<tbody>
<tr>
<td>2d spin</td>
<td>1.1</td>
<td>0.67</td>
</tr>
<tr>
<td>3d gauge</td>
<td>1.61</td>
<td>1.5</td>
</tr>
<tr>
<td>4d gauge</td>
<td>0.5</td>
<td>0.78</td>
</tr>
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</table>

Table 1 - Estimates of the coefficient $\alpha$ in eq. (10) by the MF and the Montecarlo simulations: the MF values are the same as in ref. (8), where they are obtained by free energy estimates.
According to our method, the transition is second order in 2d and 3d and first order in 4d. In 2d there are indications that the \( Z(N) \) models for \( N>4 \) have two Kosterlitz-Thouless type transitions of the same kind as \( U(1) \). This singular behavior is not reproduced by the MF, because it is due to the infrared singularity of the fluctuations.

In the 4d \( U(1) \) gauge theory there have been recent indications\(^{(14)} \) for a first order phase transition at \( \beta_2=0(1) \). For the \( Z(N) \) case one would expect two first order phase transitions. However, finite size effects are not yet under control\(^{(15)} \) and the order of the transition is not clearly predicted.

These first attempts show that disordered configurations (corresponding to non local operators) can be treated by semiclassical MF approximations at the same level as ordered configurations (local operators), at least using the specific prescription provided by the duality transformation. The formulation of similar methods in other disordered statistical systems, like frustrated models and spin glasses, could provide non trivial qualitative physics.

REFERENCES