ON THE HISTORY OF THE
STATISTICAL THEORIES
OF TURBULENCE

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RESUMEN

Presentamos en estas notas una discusión de la introducción de herramientas estadísticas en el estudio de la turbulencia. El punto focal es el período entre las dos guerras mundiales, poniendo especial atención a las llamadas teorías semi-empíricas y en la teoría de G.I. Taylor en 1935. El contexto institucional, que da a estos desarrollos su significado histórico, se menciona brevemente. Las notas continúan con una excursión a los años que siguieron a la Segunda Guerra Mundial, cuando apareció la teoría de Kolmogorov y también las formulaciones equivalentes de Heisenberg, von Weizsäcker y Onsager.

ABSTRACT

In these lectures the introduction of statistical tools in the study of turbulence will be discussed. The focus will be on the interwar period, with attention especially paid to the so-called semiempirical theories developed in the Twenties and to the main shift in techniques and language occurring with the statistical theory of G.I. Taylor in 1935. The institutional context, which gives these developments their full historical meaning, will also be sketched. An excursion will follow into the years immediately after the Second World War, when Kolmogorov's theory, and its equivalent formulations by Heisenberg, von Weizsäcker and Onsager appear on the scence.
1. FLUID MECHANICS AND KINETIC THEORY: EARLY THEORIES OF TURBULENCE

1.1. A Turning Point: 'Statistical' vs 'Empirical' Theories of Turbulence

The four-part paper on "The Statistical Theory of Turbulence" by Geoffrey Taylor, Yarrow Research Professor at Trinity College, Cambridge, published in 1935 in the Proceedings of the Royal Society (1), is unanimously referred to as the most important turning point in the history of the theoretical investigations on turbulent flows. Modern writers just consider it as the starting point for the introduction of statistical thinking in the study of fluid dynamics. To his contemporaries, it marked a striking departure from the main concepts and ideas which dominated the field at that time, and it was regarded as the introduction of a totally new way of looking at the problem of turbulence.

Turbulent flow had been for about half a century the largest single stumbling block to the proper understanding of the properties of the behaviour of fluids for engineers and applied mathematicians. Despite the sizeable amount of work done on the subject, very little had been accomplished, up to the mid-Thirties, on the way of understanding the fundamental physical processes underlying the "hopelessly complicated" (2) properties of the behaviour of actual turbulent flows, and no satisfactory way had been found to deal with the random fluctuations which characterize a turbulent velocity field. In order to render the subject more amenable to mathematical investigation, Taylor limited himself to a simplified version of the problem, introducing the idea of the isotropic turbulence, i.e., one in which "the average value of any function of the velocity components, defined in relation to a given set of axes, is unaltered if the axes of reference are rotated in any manner" (1) or, roughly speaking, one in which the intensity components in all directions are equal.

The importance of Taylor's work lies not so much in his derivation of new results, as in the fact that, using this simplified model of turbulence, he clearly defined the statistical quantities significant to the description of the state of turbulence, i.e., the correlation functions between velocity components at different times or different points in space, and that, by relating his model to the state of isotropic turbulence actually produced in a wind tunnel, he opened the way for meaningful experimen
tional investigations of the relevant statistical properties of a turbulent flow. This started a new trend in theoretical and experimental research as well, and from that point on it was customary to refer to theories of turbulence by drawing a distinction between 'empirical' or 'phenomenological' (i.e. pre-1935) theories, and 'statistical' (i.e. post-1935) theories.

Stated simply this way, this distinction can be misleading. The so-called 'semi-empirical' theories of turbulence of the mid-Twenties and early Thirties were often often referred to (both by their founders and by later authors) as being based on analogies between a turbulent flow and the kinetic theory of gases. This clearly suggests that some kind of statistical ideas were employed in these theories, too. It is interesting to discuss the nature of the statistical concepts which are present in these early theories of turbulence, and the extent of the mentioned analogy with the kinetic theory. Before doing that, it seems appropriate to say something about the scientific community in which these ideas were produced and circulated.

1.2. Applied Mathematics and Mechanics: a New Sector of the Scientific Community

As a problem in classical mechanics, the study of the properties of fluid motion had for a long time attracted the interest and the efforts of a wide range of scientists, which one can variously label as classical physicists, mathematical physicists, applied mathematicians - and, in some cases, theoretically-minded engineers. The common concern for the mathematical investigation of problems in classical mechanics which are of fundamental importance in practical applications slowly comes to identify, toward the beginning of the Twentieth Century, a peculiar figure of scientist, which more and more diversifies himself both from the physicist - no longer interested in any problem in classical mechanics where, in principle at least, there is (supposedly) nothing new to discover - and from the pure mathematician - interested in developing new fields of mathematical investigation, and shunning the mere application of old and consolidated methods of analysis to problems in 'technical physics'. The archetypal model of this professional figure is shaped, in the early years of the century,
in Göttingen under the influence of Felix Klein. Ludwig Prandtl, whose fundamental paper on boundary layer (3) had been almost totally ignored at the Third International Congress of Mathematics in 1904, came to embody the particular blend of pure and applied science fostered by Klein and associated with Göttingen. The model spread through Germany, mostly through the stream of Prandtl's students which came out of Göttingen and, with more or less delay and the permanence of national characteristics, a similar trend was visible all over Europe and—even if later on—on the other side of the Atlantic.

In the years following the end of the First World War, there are clear signs of a widespread feeling for the need of a more marked characterization of the peculiar nature of this kind of scientific activity at the institutional level, and the scientists engaged in the field strive to build up a stronger identity for their profession. After the Naturfor scherversammlung held in Nauheim in 1920, some of the leading exponents of applied mechanics in Germany (Prandtl, von Kármán, von Mises, Trefftz) agreed that it would be desirable to have at the next meeting some sessions specifically devoted to problems in applied mathematics and mechanics, as distinct from the sessions for mathematics and physics (4). In 1921, edited by R. von Mises, professor of applied mathematics at the University of Berlin, a new journal makes its appearance, the Zeitschrift für angewandte Mathematik und Mechanik (ZAMM), which immediately becomes the main reference point for scientists working in the field.

Not sharing the strong nationalistic views of the majority of his colleagues, made stronger by the situation of the defeated Germany in the post-war years, T. von Kármán had more ambitious plans. Kármán, the most brilliant of Prandtl's students, at the time director of the Aerodynamics Institute in Aachen, devoted his main efforts towards the aim of reconstructing the international scientific cooperation broken by the war. In 1922, together with the distinguished mathematician T. Levi-Civita from Rome, Kármán organized a conference on Hydro- and Aerodynamics in Innsbruck (5). There the prospect was discussed of holding at regular intervals international meetings of scientists working in the more general field of applied mathematics and mechanics. Notwithstanding the difficulties and the resistances (sometimes the boycott: Kármán often complained
about the 'Katz-und Mausspiel' between German and French scientists(6),
the program succeeded. The first International Congress of Applied Mechanics (ICAM) was held in Delft in 1924, and an International Committee was formed with the task of planning the future meetings. These were held regularly every four years until the breaking of World War II, in Zürich (1926), Stockholm (1930), Cambridge, U.K. (1934) and Cambridge, Mass. (1938). This international body gave consistence to the emergence and consolidation of a new sector of the scientific community.

1.3. Applied Mechanics and the Problem of Turbulence.

Bordering with the fields of mathematics, physics and engineering, but claiming and identity of its own, this group of scientists dealt with a heavy body of problems in the mechanics of solids and fluids which required for their solution a degree of mathematical investigation far beyond the capabilities of the cut-and-try approach of the practical engineers. From the beginning, turbulence was one such problem, and its weight kept growing in the interwar period. At each ICAM one of the general lectures was devoted to a review of recent advancements in the problem of turbulence, and a steadily growing number of papers and communications were discussed on the subject. The main contributions to the semiempirical theories of turbulence were presented, respectively by L. Prandtl and T. von Kármán at the International Congresses in 1926 and 1930.

At that time, the problem of turbulence was usually considered as consisting of two different parts. The first problem was 'the onset of turbulence', i.e., how and under which conditions in a given boundary the transition occurs from the regular, laminar flow to the irregular, chaotic pattern characteristic of turbulent behavior. The problem was attacked by well-established mathematical methods, studying the stability of the flow superimposing to the laminar solution of the equations small disturbances and determining for which values of the parameters characterizing the flow these small disturbances kept growing instead of dying away. Even if few definite results had been obtained, the line of investigation to be followed seemed to be clearly drawn: more careful experimental values to be checked against the results of heavy manipulation of standard tools in mathematical analysis(7).
clear view of turbulence by satisfactory mathematical penetration,...or in the solution will come only by breaking the boundaries of classical mechanics and turning to statistical considerations. Following the imposing and truly amazing results which have been accomplished by physical statistics in the last years, one will probably take the second point of view, which, if it proved to be true, would be of inestimable, fundamental meaning for the general interpretation of mechanics."

It could be pointed out that here von Mises is both suggesting a technical step to be taken and expressing the marked sympathies toward indeterminism and rejection of classical, causal mechanics which he clearly showed in other occasions just at the same period, as P. Forman indicates in his study of Weimar culture and the repudiation of causality by German physicists\(^\text{(10)}\). Less sensitive than the physicists' community to philosophical considerations, and essentially untouched by the ongoing conceptual revolution of quantum mechanics, the applied mathematicians took some time to accept and develop von Mises's technical suggestion. The first paper to attract wide recognition and to inspire a line of research, in which statistical concepts taken from the kinetic theory of gases were introduced, was L. Prandtl's report at the Zürich ICAM in 1926, "Über die ausgebildete Turbulenz"\(^\text{(11)}\). Preceded by a shorter communication in the ZAMM in 1925\(^\text{(12)}\), it claimed to be "still far from conclusive", but to "concern, rather, the first steps in a new path which I hope will be followed by many others".

1.4. Fluid Mechanics and Kinetic Theory: L. Prandtl

Right at the beginning of his paper, Prandtl clearly defines the limits of the task he sets to himself:

The researches on the problem of turbulence which have been carried on in Göttingen for about five years have unfortunately left the hope of thorough understanding of turbulent flow very small. The photographic and kinetographic pictures have shown us only how hopelessly complicated this flow is....

Such pictures have hitherto been utilized only for statistical information on the magnitude of the velocity variations. Otherwise, we have not been able to learn much from them. That which I would call the "great problem of developed tur
bulence", involving a thorough understanding and a quantitative calculation of the processes through which new phenomena are constantly being developed and also involving the determination of that mixing force which is produced in each individual instance by the contest of new development and damping, is not likely therefore to be soon solved.

It is always possible, however, if we forego a deeper understanding of the phenomena of turbulence, to follow theoretically various phenomena in a logical way controlled by experiments, especially regarding the mean motion in a given turbulent flow. Even the determination of the mean velocity, as a function of the place, is a technically very important task. The first step in this direction may be so characterized that the apparent viscosity forces, produced by the mixing, will be represented in such a form that they can be introduced into the hydrodynamic differential equations for the mean motion of turbulent flows\(^{13}\).

The apparent viscosity forces here mentioned by Prandtl are those arising in a turbulent flow as a consequence of the fluctuating, irregular velocities. Let us consider, as Prandtl does, a two-dimensional flow, such as the flow between two parallel plates; one is interested in determining the distribution of the mean velocity in the x direction as a function of the distance y from the wall. The Navier-Stokes equation for the x-component of the velocity \(u\) is

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

where \(v\) is the y-component of the velocity, \(P\) the pressure, \(\rho\) the mass density and \(\mu\) the kinematic viscosity coefficient.

This equation may be easily solved (assuming steady conditions) in the case of laminar flow where \(v\) is zero and no fluctuations around the mean values are present; the result is a parabolic profile for the velocity. In the case of turbulent flow, following Reynolds, one can write \(u = U + u\), \(v = v\) where \(U\) is the mean value of the x-component of the velocity and \(u\) and \(v\) the fluctuating parts of the x and y components of the velocity respectively. Taking into account the continuity equation and time-averaging according to the rules

\[
\bar{U} = U , \quad \bar{u} = \bar{v} = 0 ,
\]
where the bars over the quantities denote the time-averages, one obtains, from the above equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial x} - \rho u v \right).$$

The fluctuating velocity components do not disappear from the equation after the averaging process due to the presence of a non linear term, and they appear as additional 'turbulent viscous forces', in the form of the factor $\rho u v$ showing the existence of a nonzero correlation between the fluctuating velocity components.

The existence of turbulent shearing stresses, much larger than the molecular viscous force $\mu (\partial u/\partial y)$, had long been known from empirical evidence. Following the French engineer J. Boussinesq they were often represented in the form $\varepsilon (\partial u/\partial y)$ where $\varepsilon$, known variously in the literature as 'Austauschgrössen' or 'eddy viscosity', is a quantity dependent on the particular turbulent flow and varying from point to point. Prandtl's problem was to find a way to relate the turbulent stress $\rho u v$ to the mean velocity in order to allow the closure of the hydrodynamic equations and the determination of the velocity profile for a turbulent flow.

To do so, he pictured the motion of the turbulent flow as a mixing process of colliding fluid particles, introducing an analogy between transport phenomena in kinetic theory and turbulent exchange.

At this point it is essential to introduce the characteristic length for the turbulent condition, which here plays a similar role to that of the mean free path in the kinetic gas theory. It can be considered as the diameter of the simultaneously moved fluid masses, but also as the distance traversed by such a fluid mass before it loses its individuality by mixing with neighbouring masses... Following the second definition, we will designate this length as the 'mixing path' (or 'mixing length') and represent it by $l$. If we assume that a moving fluid mass, situated in a current with velocity decrease crosswise to the current, possesses a velocity equal to the mean
velocity at the point where it originated, and that it is shifted by the mixing length \( l \) transversely to the current, its velocity will then differ from the mean velocity at the new point, and this velocity difference, in its first approximation, equals \( l(\partial u/\partial y) \), when the mean direction of flow relative to the x-axis is chosen. The mean oscillation \( u \) can therefore be put proportional to \( l(\partial u/\partial y) \). The transverse motion \( v \) can be considered as being produced in the manner that two fluid masses with different \( u \), which find themselves in one another's presence, either come together or move farther apart. The velocities \( v \) thus produced can therefore be put proportional to \( u \). The apparent shearing stress \( (\tau = \rho uv) \) therefore becomes

\[
\rho = \rho l^2 \frac{(\partial u)^2}{\partial y}
\]

when the proportionality factors of \( u \) and \( v \) and also the correlation factor, which would come into the formation of the mean product, are collected into the still unknown length \( l \).

A comparison with the Boussinesq formula \( \tau = \epsilon(\partial u/\partial y) \) shows that one can express the 'eddy viscosity' \( \epsilon \) in the form

\[
\epsilon = \rho l^2 \frac{\partial u}{\partial y} = \rho lv.
\]

In the kinetic theory, molecular viscosity is given by the product \( \rho \lambda c \), where \( \lambda \) is the mean free path and \( c \) is the average velocity of the heat motion; the mixing length and fluctuating velocity thus play an analogous role in turbulent flow theory as the mean free path and the heat motion do in the kinetic theory of gases.

Prandtl was well aware that his formula, though qualitatively satisfactory (it gives resistances proportional to the square of the velocity, as it was empirically known to be the case for turbulent friction), suffered from serious drawbacks. The whole line of reasoning, and the very idea of a 'mixing length', rest on a questionable picture of the mixing process in turbulent flow; a 'fluid particle' is a rather ill-defined concept. Moreover, very little can be said from the theoretical point of view about the mixing length \( l \) (one can just say that it must go to zero as one approaches the limiting walls, where transversal motion is
prevented by the boundary conditions). It has to be determined for every particular flow configuration from empirical data. The statistical properties of the turbulent velocity field, which are embodied into the mixing length, still elude theoretical investigation. And, since \( l \) is a theoretically unknown quantity, one is still not able to solve the hydrodynamic equation and derive the velocity profile. Prandtl's proposal, as he himself admitted, was just a first suggestion. The next step, leading to what is still regarded as the main result of the semiempirical theories, was taken by von Kármán in 1930\(^{(16)}\).

1.5. Von Kármán's Similarity Hypothesis

Von Kármán's aim was "to make the laws of turbulent flow in channels more accessible to calculation with a minimum of arbitrary assumptions". A first assumption consists in neglecting the viscous term in the hydrodynamic equation; this is certainly a wrong approximation in the immediate proximity of the walls, where viscous force dominates over the turbulent stress, but experimental evidence suggests it to be a reasonable assumption all over the remaining section of the channel. A second assumption, of essentially statistical character, is made about the properties of the fluctuating velocities. It is assumed that "the state of flow in the neighbourhood of two points corresponding to different values of \( y \) differ from each other only by the scale of velocities and length", or, in other words, one assumes "similarity of the state of fluctuation independent of the position of the considered point in respect to the mean motion".

The average velocity around a point at \( y = 0 \) can be developed in powers of \( y \) as

\[
U = U'_o y + U''_o \frac{y^2}{2} + \ldots
\]

where \( U'_o \) and \( U''_o \) are the first and second partial derivatives of \( U \) with respect to \( y \) evaluated at \( y = 0 \). Accordingly, the stream function \( \psi = \psi(x,y) \) defined by the relations
can be expressed as

\[ \psi(x, y) = \frac{1}{2} U'_o y^2 + \frac{1}{6} U''_o y^3 + \ldots + \psi(x, y) \]  

where \( \psi(x, y) \) is the stream function of the fluctuating motion. If now one sets

\[ x = 1 \xi \quad y = 1 \eta \quad \psi(x, y) = Af(\xi, \eta) \]  

the similarity condition requires that only \( \xi \) and \( \eta \) be dependent on the point, i.e. on \( U'_o, U''_o \) etc., while the function \( f(\xi, \eta) \) be independent. The Navier-Stokes equation written by means of the stream function, the vorticity equation, is

\[ \frac{\partial \psi}{\partial y} \frac{\partial \Delta \psi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \Delta \psi}{\partial y} = \mu \Delta \psi \]

Neglecting the viscous term and keeping only the first terms in expression (1) for \( \psi \), we may write

\[ \left[ U'_o y + \frac{\partial \psi}{\partial y} \right] \frac{\partial \Delta \psi}{\partial x} - \frac{\partial \psi}{\partial x} U''_o - \frac{\partial \psi}{\partial x} \frac{\partial \Delta \psi}{\partial y} = 0 \]

Using the relations given by (2) this last expression may be written as

\[ U'_o \ln \frac{A}{1} \frac{\partial \Delta f}{\partial \xi} \frac{A}{1 \xi} \frac{\partial \Delta f}{\partial \xi} + A^2 \left[ \frac{\partial f}{\partial \eta} \frac{\partial \Delta f}{\partial \xi} - \frac{\partial f}{\partial \xi} \frac{\partial \Delta f}{\partial \eta} \right] = 0 \]

This last equation is independent of \( A, 1, U'_o, U''_o \), as requested by the similarity hypothesis, if the following relations hold true:

\[ U'_o 1 \alpha \frac{A}{1} \quad U''_o \alpha \frac{A}{1^3} \]
or
\[ 1 \alpha \frac{U'}{U''_o}, \quad A \alpha \frac{U'^3}{U''^2_0} = 1^2 U'_o. \]

The shearing stress is thus given by
\[ \tau = -\rho u'\nu' = \rho \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} = \rho \frac{A^2}{l^2} \left( \frac{\partial f}{\partial \xi} \frac{\partial f}{\partial \eta} \right). \]

The term in parenthesis in the far right hand side of this last expression is now a constant characteristic of the particular turbulent flow. Prandtl's result comes up again
\[ \tau \propto \rho l^2 U'_o^2 \]

which comes as no surprise, since, as von Kármán remarks

Prandtl set the component of fluctuation v proportional to the length l and the absolute value \(|\partial U/\partial y|\) of the velocity gradient, from which we obtain \(\tau = \rho l^2 (\partial U/\partial y)^2\) in agreement with (Kármán's result). Considered from the standpoint of the fluctuation theory, this assumption signifies that the correlation between \(u\) and \(v\) can be taken as independent of the position. Evidently this assumption corresponds to our assumption of similarity and, therefore, it is not to be wondered that we confirm this expression.

Our similarity consideration, however, gives essentially more than this in that we obtain an equation determining the mixing length, which to the first approximation appears as the ratio of the two first differential quotients of the mean velocity. It remains to be seen what the consequence of this relation will be in respect to the velocity distribution (17).

In the case of flow in a channel of width \(2h\), if \(T_o\) represents the skin friction at the surface, the value of \(\tau\) at a distance \(y\) from the middle of the channel is given by
\[ \tau = T_o \frac{y}{h}. \]
By putting together this expression with those previously obtained for 1 and \( \tau \) one has
\[
\tau_o \frac{y}{h} = k^2 \rho \frac{u'}{u''^2}
\]
or
\[
\frac{u''}{u^2} = k \frac{\sqrt{h}}{\sqrt{\tau_o / \rho}} \cdot \frac{1}{\sqrt{h} - \sqrt{y}}
\]
where \( k \) is a constant. This expression may be integrated ((\( \partial U / \partial y \)) going to infinity at \( y = h \), i.e., at the wall)
\[
U' = \frac{1}{2k} \frac{\sqrt{\tau_o / \rho}}{\sqrt{h}} \cdot \frac{1}{\sqrt{h} - \sqrt{y}}
\]
Integrating again, one obtains
\[
U = U_{\text{max}} + \frac{1}{k} \frac{\sqrt{F_o}}{\rho} \left( \log \left(1 - \frac{\sqrt{y}}{h}\right) + \frac{\sqrt{y}}{h}\right)
\]
for the velocity as a function of the distance from the wall (given by \( h - y \)).

The quantity \( k \) still remains in the formula as an unknown constant to be experimentally determined. Nonetheless, one now had a theoretically satisfactory result to substitute for the various power laws which had been empirically suggested to account for existing experimental data.

In the final summary of his paper, von Kármán pointed the limits of his results, showing at the same time his personal satisfaction:

Although the determination of the characteristic length of fluctuation \( l \) with the help of the similarity assumption has justified itself to some extent, the assumption should be better founded. The proof of the existence of a solution of the flow equations limited to a relatively narrow width of fluctuation. This calculation should also give the value of
the constant k. It is also questionable, whether or not the assumption is applicable to other more complicated cases. I wish to emphasize that this is not at all certain...

It is possible, however, that through the introduction of the equations of flow, the way to a rational theory of turbulence will be opened, leading to a rational treatment of many problems associated with turbulent flow...(18)

Kármán was clearly proud to have accomplished more than his old teacher Prandtl had been able to do, by rederiving his results in a more satisfactory way, without using dubious hypotheses on the mixing process, and pushing the analysis to the final goal of obtaining a theoretical expression for the velocity profile. Surprisingly, he maintained that his theory was basically a refined version of a kinetic theory of turbulence. Thus "Kinetische Theorie der Turbulenz" was the original title he had envisioned for his paper, and only in the final version did he change it to "Mechanische Ähnlichkeit und Turbulenz". Writing at the end of 1929 to J.M. Burgers, the leading Dutch fluid dynamicist in Delft, he said: "I have tried something like a kinetic theory of vortexes, and I use the word 'kinetic' as opposed to 'statistic' just as the kinetic theory of gases following Maxwell is opposed to the statistical viewpoint following Gibbs and the modern statistics(19)." Burgers, a former student of Paul Ehrenfest, was at the time attempting to introduce the ideas of statistical ensembles in the study of a collection of vortexes(20) and Kármán was stressing the difference between Burgers' approach and his own. As far as the kinetic theory is concerned, however, it seems fair to say that the actual relationship between the kinetic theory of gases and Prandtl's and Kármán's 'kinetic' theories of turbulence is nothing more than a very crude analogy. These theories, anyway, opened the road to a certain degree of theoretical insight into the study of turbulent shear stress and resistance, which was their ultimate goal; and much work was done in the following years trying to adjust them to the most various flow configurations and to obtain experimental data allowing to determine the numerical constants which still appeared as unknown quantities in the equations.
1.6. Atmospheric Turbulence, Diffusion and Random Walks, i.e., G.I. Taylor.

In 1932 G.I. Taylor presented his version of a 'mixing length' theory, which differed from Prandtl's in the fact that in the mixing process Taylor considered vorticity instead of momentum as the conserved quantity (20). Taylor had become interested in the problem of resistance and fluid friction through his acquaintance with the Cambridge aerodynamicist B.M. Jones and the experimental work which was conducted at the Aeronautical Laboratory at Cambridge and at the National Physical Laboratory in Teddington. With the emergence of Prandtl's and Kármán's theories he was led to develop a theory of his own, taking over again an idea which he had introduced many years earlier, when his interest in turbulent phenomena had first been aroused in a completely different context. Taylor's preoccupation with turbulent flow dates in fact from the period immediately preceding the first World War. A. Schuster, professor of physics at Manchester, had given money to Cambridge University in order to establish a temporary readership in dynamical meteorology; Taylor was appointed in 1911 and started doing research on turbulent transfer processes in the lower atmosphere. Following the tragedy of the Titanic in 1912, the British Government sent Royal Majesty's ship 'Scotia' for a scientific expedition to look for icebergs in the Northern Atlantic and report about their position. Taylor was on the Scotia as the meteorologist, and spent six months along the Banks of Newfoundland flying instrument-carrying kites from the deck to collect data about vertical temperature and wind velocity distribution. In the resulting paper, published in 1915 (22), and devoted to the turbulent transfer of heat and momentum in the atmosphere, he had introduced the same idea of a mixing length which appeared ten years later in Prandtl's papers. As it dealt with atmospheric phenomena, Taylor's work was not paid much attention by the applied mathematicians busy with their problems of friction in channels and pipes. Taylor, meanwhile, was still attracted by atmospheric problems, and turned his attention to the properties of diffusion of particles in air produced by fluctuating turbulent velocities. Stimulated by his knowledge of K. Pearson's work in statistics, he envisioned turbulent diffusion as a kind of random process, referring it to statisti-
cal problems widely studied by Pearson, such as the drunkard's walk or the random migration of insects. The result was a very elegant paper on "Diffusion by continuous movements", published in 1921\(^{(23)}\).

Before discussing the continuous version of the problem of random migration in one dimension, Taylor proceeds to a 'slight extension' of the discontinuous case. Imagine a point moving along a straight line with uniform velocity \(v\) which, after a time interval \(\tau\), either reverses its motion or continues moving forward, and let the process be repeated \(n\) times. The mean square value of the distance traveled by the point after a time \(n\tau = T\), \(X_n^2\), will be

\[
X_n^2 = [ (x_1 + x_2 + \ldots + x_n)^2 ]
\]

where the bracket indicates that the mean value is taken. Hence

\[
X_n^2 = nd^2 + 2 [ x_1 x_2 + x_1 x_3 + \ldots + x_1 x_n ]
\]

with

\[
d = v\tau.
\]

It there is no correlation between any of the \(x_i\), \([x_r, x_s] = 0\) for all \(r \neq s\) and

\[
x_n = v\sqrt{T\tau},
\]

a familiar result showing that the standard deviation is in this case proportional to the square root of the total time. Taylor now proceeds to consider the case in which there is correlation between successive displacements:

Actually in a turbulent fluid or in any continuous motion there is necessarily a correlation between the movement in any one short interval of time and the next. This correlation will evidently increase as the interval of time diminishes, till, when the time is short compared with the time during which a finite change in velocity takes place, the coefficient of correlation tends to the limiting value unity\(^{(24)}\).
Let us assume that \( x_r \) and \( x_{r+1} \) are correlated by a correlation coefficient \( c \), so that the correlation between \( x_r \) and \( x_{r+s} \) is given by the coefficient \( c^s \). Then one has

\[
X_n^2 = nd^2 + 2d^2 \left[ nc + (n-1)c^2 + (n-2)c^3 + ... + c^n \right].
\]

Summing the series, and putting \( n = T/\tau \) \( d = v \tau \), one gets

\[
X_n^2 = v^2 \left\{ \frac{(1+c)}{(1-c)} \tau T - \frac{2c^2(1-c)^n \tau^2}{(1-c)^2} \right\}.
\]

If we now let \( \tau \) become indefinitely small, \( c \) will tend to its limiting value of 1. Calling \( A \) the limit to which the quantity \( \tau/(1-c) \) tends as \( \tau \to 0 \), \( c \to 1 \), one sees that \( X_n^2 \) has the limiting value

\[
X_n^2 = v^2 \left\{ 2AT - 2A^2 \left[ 1 - \exp \left( -\frac{\tau}{A} \right) \right] \right\}.
\]

For small values of \( T \) this reduces to \( x_n = vT \), which is what one would expect since the correlation coefficient between successive displacements has not appreciably departed from the value of 1. When \( T \) is large, on the contrary, one has \( X_n = v \sqrt{2AT} \). The memory of the correlation has been lost.

Taylor now goes on the investigate the statistical properties of the curve of any function varying irregularly with time, say \( p(t) \), characterized by the constancy of the standard deviation \( \sqrt{[p^2]} \) over successive periods of time. Obviously the value of \( [p^2] \) does not identify the curve; one has to specify the values of \( [(dp/dt)^2] \), \( [(d^2p/dt^2)^2] \), etc. as well. Taylor shows that the successive derivatives of the curve \( p(t) \) are correlated between each other in the following way

\[
\begin{bmatrix}
  p \\
  \frac{d}{dt} p \\
  \frac{d^2}{dt^2} p \\
  \frac{d^3}{dt^3} p \\
  \frac{d^4}{dt^4} p \\
  \frac{d^n}{dt^n} p \\
  \frac{d^{n+1}}{dt^{n+1}} p
\end{bmatrix} = (-1)^n \begin{bmatrix}
  d^n p \\
  d^n \left( \frac{d}{dt} p \right) \\
  d^n \left( \frac{d^2}{dt^2} p \right) \\
  d^n \left( \frac{d^3}{dt^3} p \right) \\
  d^n \left( \frac{d^4}{dt^4} p \right) \\
  d^n \left( \frac{d^n}{dt^n} p \right) \\
  d^n \left( \frac{d^{n+1}}{dt^{n+1}} p \right)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  p \\
  \frac{d}{dt} p \\
  \frac{d^2}{dt^2} p \\
  \frac{d^3}{dt^3} p \\
  \frac{d^4}{dt^4} p \\
  \frac{d^n}{dt^n} p \\
  \frac{d^{n+1}}{dt^{n+1}} p
\end{bmatrix} = 0
\]
The correlations to which \( p \) and its differential coefficients must be subject in order that their standard deviations may be constant, have now been established. We can, therefore, now use these standard deviations to define some statistical properties of the curve.

In analysing any actual curve, it may be very difficult and tedious to obtain these standard deviations. There is, however, another method of defining the statistical properties of the curve which is equivalent to that given above, but which is likely to be much more manageable in practice. This method will now be considered\(^{(24)}\).

This other method consists in considering the correlation coefficient between values of the curve at two different times. This is defined by

\[
\begin{align*}
[ P_t P_{t+\xi} ] &= R_\xi \sqrt{[P_t^2]} \sqrt{[P_{t+\xi}^2]} \\
\end{align*}
\]

which reduces, since the standard deviation of \( p \) is constant, to

\[
R_\xi = \frac{[ P_t P_{t+\xi} ]}{[ P_t^2 ]}
\]

Now expand \( P_{t+\xi} \) in powers of \( \xi \)

\[
P_{t+\xi} = P_t + \xi \frac{dp}{dt} + \frac{\xi^2}{2!} \frac{d^2p}{dt^2} + \ldots
\]

Hence,

\[
[ P_t P_{t+\xi} ] = [P_t^2] + \xi [ \frac{dp}{dt} ] + \frac{\xi^2}{2!} [ \frac{d^2p}{dt^2} ] + \ldots
\]

and

\[
R_\xi = 1 - \frac{\xi^2}{2! [P_t^2]} \left[ \left( \frac{dp}{dt} \right)^2 \right] + \frac{\xi^4}{4! [P_t^2]} \left[ \left( \frac{d^2p}{dt^2} \right)^2 \right] + \ldots
\]

After these preliminaries one can discuss the problem of diffusion
by continuous movements:

The theorems which have just been proved will now be used to find out what are the essential properties of the motion of a turbulent fluid which makes it capable of diffusing through the fluid properties such as temperature, smoke content, colouring matter or other properties which adhere to each particle of the fluid during its motion.

Consider a condition in which the turbulence in a fluid is uniformly distributed so that the average conditions of every point in the fluid are the same. Let \( u \) be the velocity parallel to a fixed direction, which we will call the axis of \( x \), of the particle on which our attention is fixed. It will now be shown that the statistical properties which were defined above (now in relation to \( u \) instead of \( p \)) are sufficient to determine the law of diffusion, i.e., the law which governs the average distribution of particles initially concentrated at one point, at any subsequent time (23).

Suppose that one knows \( u^2 \) and \( R_\xi \) (this is now the correlation coefficient between the value of \( u \) for a particle at any time and the value of \( u \) for the same particle a time later \( \xi \)). Now

\[
\int_0^t [u_t u_\xi] d\xi = [u_t^2] \int_0^t R_\xi d\xi
\]

and since \( [u^2] \) does not vary with \( t \) and \( R_\xi \) is an even function of \( \xi \),

\[
\int_0^t [u_t u_\xi] d\xi = [u^2] \int_0^t R_\xi d\xi.
\]

One also has

\[
\int_0^t u_t u_\xi d\xi = \left[ u_t \int_0^t u_\xi d\xi \right] = [u_t X],
\]

where \( X \) is the distance traversed by a particle in time \( T \). Hence

\[
[u^2] \int_0^t R_\xi d\xi = [u_t X] = \frac{1}{2} \frac{d}{dt} [x^2]
\]

and

\[
[x^2] = 2[u^2] \int_0^T R_\xi d\xi dt.
\]
Thus the problem of diffusion, in a simplified type of turbulent motion, is reduced to the consideration of a single quantity, the correlation coefficient.

If follows immediately that, when the time $T$ is so small that $R_\zeta$ does not differ appreciably from unity, the integral on the right hand side of the last expression equals $T^2/2$, so that

$$[x^2] = [u^2]T^2$$

or

$$\sqrt{[x^2]} = T\sqrt{[u^2]}.$$  

On the other hand, it seems reasonable to assume that after a certain time interval $T_1$ the correlation coefficient will fall to zero. In this case

$$\lim_{t \to \infty} \int_0^t R_\zeta \, d\xi$$

will be finite and equal to $I$, say. Then, for $T > T_1$

$$\frac{d}{dt} [x^2] = 2 [u^2] I$$

and

$$\sqrt{[x^2]} = \sqrt{2[u^2]} IT$$

so that the standard deviation is again proportional to the square root of the time. As Taylor remarks, "this, therefore, is a property which a continuous eddying motion may be expected to have which is exactly analogous to the properties of discontinuous random migration in one dimension".
1.7 A turning point for a turned problem

It is evident that all the roots of Taylor's approach to the statistical theory of turbulence of 1935 are to be found in the paper we have discussed. The 'uniformly distributed' turbulence of 1921 leads directly to the more rigorous definition of isotropic turbulence, and the idea that the fundamental characteristics of turbulence are expressed by the correlation functions is clearly laid out in this work. One might well wonder why fourteen years were necessary for these ideas to be taken again and force their way through a wider audience. As far as Taylor himself is concerned, an explanation could be found in his personal style of work and habit of research. He defined himself a classical physicist, and often stated that he was willing to pursue a line of theoretical investigation only if at the same time the chance was given of checking it against the results of simple experiments he could perform by himself. Due to the rather abstract character of his study of 1921, he explicitly handed over the problem to the mathematicians, publishing the paper in the Proceedings of the London Mathematical Society. He took the matter over again in 1935 when, with the development in the early Thirties of the technique of the hotwire anemometer, it was possible to measure with sufficient precision the fundamental quantities he was interested in: correlations.

Why Taylor's paper on diffusion failed to have any effect upon the community of applied mathematicians working on the problem of turbulence is not difficult to see if one compares it with the typical products of Prandtl and von Kármán. It is quite clear that Taylor's problem is actually a completely different one. His 'simplified and uniform' turbulence has very little to do with the inhomogeneous turbulent flow in a channel which is the central problem for the leading hydrodynamicists of the day. Only much later, in 1935, will these ideas spread and produce a dramatic change in the approach to the problem of turbulence. This will in turn be possible because the problem itself will have by that time modified its nature. The discussion of this shift is the subject of the next chapter.
The year 1935 is usually considered as a crucial moment in the development of the theories of turbulence, due to the publication, in the Proceedings of the Royal Society, of a four-part paper by G. I. Taylor, of Trinity College, Cambridge, entitled "Statistical Theory of Turbulence" (27). This paper had a great impact on the community of applied mathematicians working in the field of fluid dynamics and the scientists involved with experimental research on the properties of turbulent flows. It started a new trend of copious production of theoretical and experimental results as well. From that moment on, it was customary to refer to theories of turbulence by drawing a clear distinction between 'empirical' or 'phenomenological'.

In 1935 Taylor approached the problem in much the same way as he had done in 1921. At that time, his work had passed almost unnoticed. Fourteen years later the same ideas aroused widespread interest. We will discuss the reasons for such a sharp change of attitudes towards Taylor's theoretical proposal.

Taylor begins by pointing at the conceptual difficulties encountered in the various 'mixing length' theories existing at that time. It is convenient to quote directly from the introduction of his paper:

At an early stage in the development of the theory of turbulence the idea arose that turbulent motion consists of eddies of more or less definite range of sizes. This conception combined with the already existing ideas of the Kinetic Theory of Gases led Prandtl and me independently to introduce the length 1 which is often called a "Mischungsweg" and is analogous to the "mean free path" of the Kinetic Theory. The length 1 could only be defined in relation to the definite but quite erroneous conception that lumps of air behave like molecules of a gas, preserving their identity till some definite point in their path, when they mix with their surroundings and attain the same velocity and other properties as the mean value of the corresponding property in the neighbourhood. Such a conception must evidently be regarded as a very rough representation of the true state of affairs. If we consider a number of particles of small volumes of fluid starting from some definite level and carrying, say, heat in a direction transverse to the mean stream lines, their average distance from the level at which they started will go on increasing indefinitely so that we can only consider a "Mischungsweg" in relation to some arbitrary time of flight during which we must consider that the particles preserve their individual properties distinct from those
of their surroundings. Clearly this is an arbitrary conception and if pursued logically probably leads to a definitely wrong result...
The difficulty... led me, some years ago, to introduce the idea that the scale of turbulence and its statistical properties in general can be given an exact interpretation by considering the correlation between the velocities at various points of the field at one instant of time or between the velocity of a particle at one instant of time and that of the same particle at some definite time, $\xi$, later.$^2$.

The work Taylor is referring to is, clearly, his "Diffusion by Continuous Movements" of 1921. There he had shown that in a field of "uniformly distributed" turbulence two quantities were sufficient to characterize the diffusive properties of the turbulence, namely its intensity $\overline{[u^2]}$, defined as the mean square root of the velocity fluctuations (supposed constant throughout space), and the correlation coefficient between values of the turbulent velocity of the same fluid particle at two different instants of time,

$$R_\xi = \frac{\left[ u_t u_{t+\xi} \right]}{u^2}.$$

After summarizing the main results of the 1921 paper (see chapter 1), Taylor introduced length $l_1$ defined by

$$l_1 = \sqrt{[u^2]} \int_0^T R_\xi \ dx$$

which could be interpreted as giving a measure of the scale of turbulence, similar to the mixing length of the older theories, with the important difference that the new length was now tied to a fundamental statistical feature of the turbulent field, bearing no relation to ambiguously defined mixing processes. It could be defined even if there was no mixture at all.

Shifting from a Lagrangian to an Eulerian viewpoint, another typical length could be introduced, considering now the correlation coefficient between velocities at the same time at two different points in the fluid
A and B spaced a distance $y$ apart $R_y$, defined by

$$R_y = \frac{\left[ u_A - u_B \right]}{\left[ u^2 \right]}$$

where $u_A$ and $u_B$ denote the velocities at points A and B respectively. It is to be expected that this correlation coefficient, which is equal to unity for $y = 0$, will fall to zero as $y$ increases. If $Y$ is a distance such that $R_y = 0$ for any $y > Y$ a length $l_2$ may be defined by

$$l_2 = \int_0^Y R_y \, dy$$

analogous in the Eulerian system to the length $l_1$ in the Lagrangian system. Taylor suggests this length as an "average size of the eddies".

Taylor then looked into the general problem of utilizing the fundamental statistical properties of turbulence just defined in order to characterize the significant physical features of the system. In a turbulent flow, due to the presence of viscosity, the relevant quantity is the rate of dissipation of energy:

The rate of dissipation of energy in a fluid at any instant depends only on the viscosity, $\mu$, and on the instantaneous distribution of velocity. If, therefore, the representation of the essential statistical properties of the velocity field can be expressed by the $R_y$ curve and similar correlation curves it must be possible to deduce from them the rate of dissipation of energy. This would in general involve a complicated analysis, but the problem can be much simplified if the field of turbulent flow is assumed to be isotropic. In isotropic turbulence the average value of any function of the velocity components, defined in relation to a given set of axes, is unaltered if the axes of reference are rotated in any manner (29).

The general expression for the mean rate of dissipation of energy $[\dot{W}]$, obtained from the equations of motion, is given by
\[
[W] = \mu \left\{ 2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 \right] + 2 \left[ \left( \frac{\partial v}{\partial y} \right)^2 \right] + 2 \left[ \left( \frac{\partial w}{\partial z} \right)^2 \right] + \left[ \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] + \left[ \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 \right] + \left[ \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 \right] \right\}.
\]

In the case of isotropic turbulence Taylors investigated the mean values of any quadratic function of the nine quantities \((\partial u_\alpha / \partial x_\beta)\), which contains forty five terms. With ingenious calculations he showed that when the turbulence is isotropic all these terms are simply related to each other, so that it is sufficient to know just one of them, say \((\partial u/\partial y)^2\). Hence, the expression for the mean rate of dissipation reduces to

\[
[W] = 7.5\mu \left[ \left( \frac{\partial u}{\partial y} \right)^2 \right].
\]

The quantity \((\partial u/\partial y)^2\) is related to the way in which the value of \(R_y\) falls off from its initial value of 1 as \(y\) increases from zero. As we saw before, Taylor had proved in 1921 that the correlation coefficient could be expressed as a power series

\[
R_y = 1 - \frac{y^2}{2[u^2]} \left[ \left( \frac{\partial u}{\partial y} \right)^2 \right] + \frac{y^4}{4![u^2]} \left[ \left( \frac{\partial^2 u}{\partial y^2} \right)^2 \right] + \ldots.
\]

Therefore

\[
\left[ \left( \frac{\partial u}{\partial y} \right)^2 \right] = 2[u^2] \lim_{y \to 0} \frac{1 - R_y}{y^2}
\]

and one may introduce a new length \(\lambda\) defined by

\[
\frac{1}{\lambda^2} = \lim_{y \to 0} \frac{1 - R_y}{y^2}
\]
so that

$$[w] = 15 \mu \frac{[u^2]}{\lambda^2}.$$  

Since the length $\lambda$ is related to the correlation length $R_y$, the mean rate of dissipation of energy is expressed as a function of the intensity and scale of turbulence.

Taylor then turned to the problem of the decay of turbulence. The kinetic energy per unit volume of a turbulent field is given by

$$\frac{1}{2} \rho [u^2 + v^2 + w^2]$$

which reduces to

$$\frac{3}{2} \rho [u^2]$$

for isotropic turbulence. The rate of decay of the kinetic energy per unit volume is therefore given by

$$- \frac{3}{2} \rho \frac{d}{dt} [u^2].$$

Consider now a field of turbulence produced in a wind-tunnel downstream of a square-mesh grid in a flow with uniform translational speed $u$. The rate of decay in the region downstream from the grid, where it may be shown that the turbulence is actually isotropic, can be written as

$$- \frac{3}{2} \rho u \frac{d}{dx} [u^2]$$

and equated with the mean rate of dissipation $[w]$. That is,

$$- \frac{3}{2} \rho u \frac{d}{dx} [u^2] = 15 \mu \frac{[u^2]}{\lambda^2}.$$
Furthermore Taylor suggested that in the field of turbulence produced by the grid, the scale of turbulence would be dictated by the length of the squares of the square-mesh grid $M$, and that a relation should therefore exist between $\lambda$ and $M$. He found that

$$\frac{\lambda}{M} = A \sqrt{\frac{u'}{\rho \mu}}$$

with $u' = \sqrt{\left[u^2\right]}$ and $A$ a universal constant for all grids of a definite type. Using this relation one finds the following simple law for the decay of turbulence

$$\frac{U}{u'} = \frac{5x}{A^2 M} + \text{constant.}$$

where $x$ denotes the distance from the grid.

This is essentially the content of Part I of Taylor's paper. Parts II, III and IV contain no new ideas, but are devoted to comparison of theoretical predictions with experimental wind-tunnel data, discussion of measurements and suggestions for further experimental work. We will come back later to this aspect of Taylor's paper, and turn now to a short discussion of the roots of his ideas on isotropic turbulence and his main source of inspiration.

We know that what led Taylor to his paper on diffusion in 1921 was his interest on heat diffusion problems in atmospheric turbulence and his acquaintance with K. Pearson's work on statistics, especially on random walks. From 1921 to 1935 Taylor did not publish anything along the same lines, but there is clear evidence that this problem was continuously present in his mind. In the introduction to the paper on diffusion he said:

In the course of the work no discussion of the convergency of the series used is attempted. The work must therefore be regarded as incomplete. The author feels that such questions might be examined with advantage by a pure mathematician, and it is in the hope of interesting one of them that he wishes to offer this paper to the London Mathematical Society.$$^{(30)}$$
The pure mathematician which became interested to the problem was Norbert Wiener. I was first attracted to investigate Wiener's role in the story by a remark which can be found in his autobiography; speaking of his early work, when he occupied himself with Lebesgue integration and measure theory, Wiener wrote:

I was an avid reader of the journals, and in particular of the Proceedings of the London Mathematical Society. There I saw a paper by G.I. Taylor concerning the theory of turbulence. This is a field of essential importance for aerodynamics and aviation, and Sir Geoffrey has for many years been a mainstay of British work in these subjects. The paper was allied to my own interests, inasmuch as the paths of air particles in turbulence are curves and the physical results of Taylor's papers involve averaging or integration over families of such curves. With Taylor's paper behind me, I came to think more and more of the physical possibilities of a theory for averages over curves. The problem of turbulence was too complicated for immediate attack, but there was a related problem which I found to be just right for the theoretical considerations of the field which I had chosen for myself. This was the problem of the Brownian motion, and it was to provide the subject of my first major mathematical work.

From what little is left of the correspondence between the two men in Wiener's papers, it seems that things went actually the other way round: it was Taylor who saw Wiener's paper on Brownian motion (published in the Proceedings of the London Mathematical Society in March of 1924), and wrote him sending a reprint of his paper in April 1924:

I take the liberty of sending you the enclosed paper on the chance that it might be possible to extend your methods given in a recent paper "The average value of a functional," Proc. Lond. Math. Soc. Vol. 22 p. 454, to the problem which I there considered. My paper was written entirely from a physical point of view and at the time I wrote it I was not aware of Einstein's work. It appears to be about a similar problem to yours and Einstein's but the results seem to be of quite a different type. The difference perhaps lies in the condition 3 p. 455, namely that the probability that the particle considered wanders a given distance in a given time is independent of the "direction in which it wanders".

Wiener, as Einstein, was considering the special case in which there is no correlation between successive displacements. Wiener's answer is not pre-
served in his correspondence, but from a second letter from Taylor dated June 1924 we can infer that he showed interest in his work, and sent him more reprints of his papers on irregular motion.

Taylor kept following Wiener's more abstract work; in a letter dated January 1932 he writes to Wiener that he is studying his paper on Acta Mathematica (this is one of Wiener's fundamental papers, the one on "Generalized Harmonic Analysis"), and inquires about the possibilities of using Wiener's method in "solving the problem of two-dimensional turbulence when attacked in a similar manner to that used for one-dimensional turbulence in my paper 'Diffusion by continuous movements'". 1932 is the year in which Taylor published his vorticity transfer theory, which is of mixing length type; it is clear that he was also thinking at the same time of extending his old method using correlations. The influence of Wiener's work on harmonic analysis was even more apparent in a subsequent development of Taylor's theory, of which we will speak later.

There is, therefore, no quantum jump from 1921 to 1935, but rather a slow maturation of ideas and refinement of tools from Taylor's part. There is, on the contrary, a quite marked quantum jump from the early semiempirical theories of turbulence and Taylor's statistical approach in 1935. If one looks at the main characteristics of Prandtl's and von Kármán's theories in the late Twenties, and, most important, at the kind of problem they were attempting to solve, it is clear that with Taylor's theory we enter a different world; when one refers to the problem of turbulence, one is speaking, in both cases, of two very different things. "The great problem of developed turbulence", as Prandtl defined it, was in the Twenties the problem of deriving some theoretical expression for the values of the tangential stresses (the Reynolds stresses) responsible for the transport phenomena in a fluid flowing along a given boundary, and for the resistance experienced by a body immersed in the fluid. The terms "problem of developed turbulence" and "problem of resistance" were often utilized as synonymous. One attempted to find a way of introducing into the mean flow the effects of the tangential stresses due to the presence of correlations between different components of the turbulent velocity at the same point in the fluid, and to understand how this affected
the shape of the velocity profile in particular flow configurations with marked characteristics of inhomogeneity and anisotropy. No wonder that Taylor's paper on diffusion failed to attract much attention, referring as it was to the "simplified, uniformly distributed turbulence". The first reference to Taylor's work on diffusion appears in von Kármán's general lecture on turbulence at the Fourth International Congress of Applied Mechanics in 1934(35), mentioned as being "a radically new approach" that accounts for the slowness of its acceptance. The shift from the problem of developed turbulence as it is conceived in the late Twenties and early Thirties, to the problem of the statistical properties of turbulence as emphasized in the late Thirties implies not only a change in the technical instruments involved but a drastic shift in the nature of the problem itself. One does not look any more to the characteristics of the mean flow but focuses on the random patterns of the fluctuations. The asymmetric, inhomogeneous state of the flow along a fixed boundary is replaced, as the object of the analysis, by the isotropic state of turbulent flow where no boundaries are present. The key words change. From 'shear stresses', 'wall distance', 'law of resistance', 'turbulent mixing', one goes to 'correlation functions', 'random walks', 'isotropic turbulence', 'Fourier-component analysis'. It is a shift in technical language which is highly revealing of the changing character of the problem under investigation as much as of the kind of mathematics and ideas involved. This can be immediately realized by pointing at the mere fact that in isotropic turbulence there are no tangential stresses. The "great problem of turbulence", as defined by Prandtl, simply disappears.

From this point of view, the question one is led to ask is not so much "why did it take so long for such ideas to be widely accepted and developed", but rather "why, then, were they accepted and developed in the mid-Thirties, if they were so radically new?".

Let me first substantiate the statement that Taylor's ideas were accepted and developed, by giving a short account of the content of two important papers published in 1938, one by T. von Kármán and L. Howarth, the other by Taylor himself.

Kármán was greatly impressed by Taylor's work and soon saw the possibility for improvement. In the academic year 1936-37 he worked closely
with the young British mathematician Leslie Howarth, a student of S. Goldstein in Cambridge, which came to Caltech as a King's College Research Fellow (according to W. Sears, Kármán told him "I will get Leslie to teach me tensor theory"). In their joint paper, they developed a general theory of isotropic turbulence, introducing the correlation tensor, which describes the correlation coefficient between two arbitrary velocity components at two arbitrary points, and showing how in isotropic turbulence it is completely determined by a single scalar function. They rederived Taylor's results in a more elegant way, and then turned to the discussion of the dynamics of isotropic turbulence, obtaining an equation for the time development of the correlation function which contains, due to the non-linearity of the equations, non-vanishing terms describing triple correlations. Using an approach similar to the one Kármán had developed in his 1930 theory, that is, introducing a similarity hypothesis, they succeeded in reducing the number of unknowns and deriving a satisfactory differential equation.

Taylor's contribution introduced the idea of the spectrum of turbulence; he resolved the fluctuating velocity variation in a turbulent field at a fixed point into its harmonic components, introducing a spectral function which gives the contribution to the velocity field from the continuous range of frequencies. He then showed how the spectrum curve is nothing else than the Fourier transform of the correlation curve. From this moment the use of the spectral representation of a turbulent velocity field became commonly used. Once again, behind Taylor's work on spectral analysis we find the influence of Wiener; Taylor used (and made explicit mention of the source) some theorems which had been proved by Wiener in his book on "The Fourier Integral" published in 1933 as a direct filiation of the work on generalized harmonic analysis.

These theoretical investigations found immediate ground for application in the wide range of new problems posed by wind-tunnel testing during the Thirties. With the growing importance, after World War I, of wind tunnel research as a tool for improving aircraft design, some puzzling situations began to emerge from the huge quantity of empiric material. It was soon found that experiments performed in geometrically identical arrangements gave widely different results in different wind tun-
nels. In 1923 the National Physical Laboratory began the circulation of two airship models for comparative tests in a large number of the wind tunnels of the world. The results in the U.S. wind tunnels showed variations of 50 percent from a mean value. It was realized that the reason for these discrepancies was to be found in the degree of turbulence present in the main flow, which affected in a sensible way the measurements. It was found, for example, that the intensity of turbulence in the main flow was responsible for the premature transition of the boundary layer along a wing section from laminar to turbulent; as a result the drag coefficient for a given profile could vary by a factor of two.

One had therefore interest in having wind tunnels with as low a turbulence factor as possible, in order to better approximate the conditions in free air, and in finding a rational way of classifying wind tunnels according to their turbulence factor, to make possible comparisons between results obtained in different tunnels. A first step, suggested by Prandtl, was to classify wind tunnels in order of magnitude of turbulence in terms of the curves of drag coefficient of a sphere. One defined as the 'turbulence factor' for a given wind tunnel the ratio between the Reynolds number at which the drag coefficient of a sphere assumed a given value in free air, and the measured Reynolds number at which it assumed the same value in the wind tunnel stream (due to the premature transition to turbulent flow in the boundary layer, this happens at a Reynolds number lower than the one calculated for free air). More refined measurements showed that things were more complex; the diameter of the sphere acted as an isolated, independent parameter. One had to take into account not only the intensity, but also the scale of the turbulence in the flow.

The apparatus which allowed to obtain reliable direct measurements of the intensity of turbulence, the hot-wire anemometer, was developed and greatly improved in the early Thirties by the Göttingen group under Prandtl, at the National Physical Laboratory in Great Britain, and in the U.S. at the National Bureaus of Standards by H. Dryden and his coworkers. Working in cooperation and with the financial assistance of the National Advisory Committee for Aeronautics, this group improved in the early Thirties the hot-wire anemometer to the point that they were able to follow fluctuations in the velocity to the upper frequency of about 5000 cycles.
One of the main results of the measurements performed with this instrument in order to study the characteristics of wind-tunnel turbulence was to show that the turbulence some distance downstream from a regular array of grids is highly isotropic. The mean value of the fluctuation of the velocity component was independent of the particular direction studied.

Just at this moment, and when it had become clear that the scale of the turbulence played a role as important as its intensity, Taylor's theory of isotropic turbulence appeared. It came at the right moment to perfectly fit a practical need. Correlations, which are the basic ingredients of Taylor's definition of scale, can easily be measured by adequately coupling two hot-wire anemometers. Taylor's theoretical framework made it possible to compare results, design new experiments, and make sense of existing data. Parts II, III and IV of Taylor's paper give just the detailed proof that the numerical results obtained in wind tunnels in the previous years at Göttingen, Caltech, at the NPL, and by the Dryden group, can all be explained by the theory exposed in Part I. In particular, the law of decay of turbulence behind grids was confirmed in a spectacular way. The NBS-NACA group immediately performed a series of accurate measurements utilizing Taylor's theory as a guide\(^{(39)}\). In Dryden's words

> these measurements give information about the decay of turbulence and change in scale, which has an interest from the point of view of the theory of turbulence and a practical bearing on the design of wind tunnels to secure low turbulence\(^{(40)}\).

It is significant that the main advertising campaign for what was sometimes called 'Taylor-Kármán's statistical theory' was conducted neither by Taylor, which was quite a shy person, nor by Kármán, which certainly was not a shy person, but by Hugh Dryden, in a long series of technical papers, general lectures and review articles\(^{(41)}\).

We have commented on a wide range of factors which come together in explaining the development and the reception of the new statistical theory: the pure mathematician developing the appropriate tools to deal with the intricacies of chaotic motion; someone able to translate these ideas...
into a simpler mathematical language, applying them to a specific problem and pointing to its practical value; a scientific community with theoretically minded people receptive to new ideas and willing to pursue new lines of research; a group of experimentalists ready to appreciate theoretical contributions which can shed light on their specific problems; a growing concern with the specific problem, which is in turn dictated by wider external factors; the existence of experimental facilities and the development of an appropriate technology; an institutional setting allowing these factors to merge and synthetize. All these things grow and slowly converge in the early Thirties. Taylor's theory of 1935 is Taylor's work; it marks a turning point in the history of the theories of turbulence.

3. KOLMOGOROV, HEISENBERG, VON WEIZSÄCKER, ONSAGER: A CASE OF SIMULTANEOUS DISCOVERY

3.1 Introduction

In the early Forties, the Russian mathematician A.N.Kolmogorov, the German physicists W. Heisenberg and C. von Weizsäcker, and the Swedish chemical physicist L. Onsager proposed independently of each other, what is basically the same theoretical model to deal with the problem of developed turbulence. By means of different formulations and techniques they were led to the same fundamental results. These contributions are altogether referred to, in the literature that followed, as "Kolmogorov's theory". It is worth to discuss them and bring forth their common characteristics, since they represent an important turning point in the complex itinerary of the theoretical researches on turbulence.

3.2 Kolmogorov

First to be published are three short papers by Kolmogorov in the Proceedings of the Soviet Academy of Sciences (42), dealing with the statistics of small-scale structures in isotropic and homogeneous turbulence. The opening sentence of the first paper gives a clear idea of the kind of approach: "In considering the turbulence it is natural to assume the com-
ponents of the velocity... as random variables in the sense of the theory of probabilities". It is just the case that such an approach could actually look "natural" only to a probabilist like Kolmogorov. The sharp judgment expressed by von Kármán in 1937 towards the attempts to use probability in attacking the problem of turbulence ("...Gebelein attempted to solve the problem by looking at if from the high tower of the general theory of probability; I believe that this tower is too high to allow one to see the simple facts") gives enough indication as to how "natural" such a language appeared to the applied mathematician dealing with the subject.

Apart from the formalized language, Kolmogorov's work exhibits a rather simple structure that can be easily summarized. The basic physical mechanism of the problem are incorporated as three hypotheses:

a) turbulence is homogeneous and isotropic at small scale;
b) at small scale the statistical properties depend only on the kinematic viscosity \( \nu \) and the dissipated energy \( \varepsilon \); and
c) in the inertial range ("small scales, but not too small") the statistical properties depend only on the dissipated energy.

From these three hypotheses it is possible to obtain the relevant statistical quantities by the simple use of dimensional analysis. From hypothesis b) one has that the only meaningful length at small scale \( \eta \); known as Kolmogorov's length, is given by

\[
\eta = \left( \frac{\nu^3}{\varepsilon} \right)^{1/4}.
\]

This length represents the wavelength at which viscous effects become dominant. Also, the only meaningful velocity at small scale \( \nu \), known as Kolmogorov's velocity is given by

\[
\nu = (\nu \varepsilon)^{1/4}.
\]
and represents the typical velocity associated with vortexes of scale $\eta$.

For $r<<L$, where $L$ is the length defining large scales, from hypothesis b) and dimensional analysis,

$$<(u(o) - u(r))^2 > = v^2 \beta(\frac{r}{\eta})$$

where $\beta$ is a universal function. For the inertial range, $\eta << r << L$, the function $\beta$ can be obtained with the help of hypothesis c). One finds that

$$<(u(o) - u(r))^2 > \propto (\varepsilon r)^{2/3}$$

Using the same type of arguments, one finds that the energy spectrum $E(k)$ is given by

$$E(k) \propto \varepsilon^{2/3} k^{-5/3}$$

for $1/L << k << 1/\eta$. It is interesting to remark that this last result, known in the literature as the "5/3 law" or "Kolmogorov's law", does not appear explicitly in these papers, though it follows as a rather immediate consequence.

The very way in which Kolmogorov's work is structured is typically mathematico-deductive: one starts with a few hypotheses and proceeds to the formal deduction of conclusions (the calculation of correlation functions). It is remarkable how the justification of the hypotheses by phenomenological considerations is confined to a footnote, showing a completely different attitude from that of an engineer. On the other hand, it is easy to realize that, besides the language of probability theory, Kolmogorov does not rely on particularly original phenomenological ideas, nor does he introduce new techniques (the whole deduction of his results is based essentially on the use of dimensional analysis, daily bread for engineers and applied mathematicians). Take, for instance, the
description of the energetic balance in turbulence:

From the energetical point of view it is natural to imagine the process of turbulent mixing in the following way: the pulsations of the first order absorb the energy of the motion and pass it over successively to pulsations of higher orders. The energy of the finest pulsations is dispersed in the energy of heat due to viscosity.

This description is essentially identical to the one suggested about twenty years earlier by the British meteorologist L.F. Richardson in a sort of poem:

Big whorls have little whorls, that feed on their velocity, and little whorls have lesser whorls, and so on to viscosity (in the molecular sense) \(44\).

3.3 Heisenberg and von Weizsäcker

In the last months of 1945, while being held as "war prisoners" together with other German scientists near Cambridge, Great Britain, Heisenberg and von Weizsäcker developed in close collaboration a theory of small-scale turbulence. However, their results were published in two successive papers only in 1948 \(45\), even though Kolmogorov’s, unknown to them in 1945, had been spread in Western scientific circles thanks mainly to the work of G.K. Batchelor \(46\). Although different, the two theories lead to the same physical results. The papers of Heisenberg and von Weizsäcker present the same theory in different styles: Heisenberg’s treatment is more analytical and formal, while von Weizsäcker’s approach is more phenomenological introducing explicit connections with astrophysical problems.

Let us take a quick look at their theory. The first equation of Reynolds’s hierarchy in \(k\)-space for homogeneous and isotropic turbulence is

\[
\frac{\partial E(k)}{\partial t} = -2\nu k^2 E(k) + T(k)
\]

where \(T(k)\) is a term coming from the nonlinear part of the Navier-Stokes equation. This term contains contributions due to the triple correlation functions of the velocity field harmonics. The problem is how to write
T(k) is terms of the energy spectrum so as to 'close' the equation; a problem analogous to the one faced by Prandtl in 1925 in his attempt to find an expression for the Reynolds stresses tied to the properties of the mean motion and solved by the introduction of the "mixing length". In analogy with Prandtl's procedure, Heisenberg and von Weizsäcker proposed that

$$\int_0^k T(k') dk' = -2 \nu_T(k) \int_0^k k'^2 E(k') dk'$$

where $\nu_T(k)$ is a sort of "turbulent viscosity". The analogy with Prandtl suggests that $\nu_T(k)$ must have contributions from wave numbers larger than $k$. From this and dimensional analysis, it follows that

$$\nu_T(k) = c \int_k^{\infty} \left( \frac{E(k')}{k'}^3 \right)^{1/3} dk'$$

One obtains, combining these results,

$$-\partial_t \int_0^k E(k') dk' \approx \varepsilon = 2 \left[ \nu + c \int_k^{\infty} \left( \frac{E(k')}{k'}^3 \right)^{1/2} dk' \right] \int_0^k k'^2 E(k') dk'$$

for $k >> 1/L$. The solution of the equation in the inertial range is

$$E(k) \approx k^{-5/3}$$

It may be noted that Heisenberg and von Weizsäcker's theory contain no relevant technical novelties: the basic ingredients are the idea of turbulent viscosity, mutated from Prandtl, and, once again, dimensional analysis.

3.4 Onsager

Onsager's contribution appears in a manuscript sent to C. Lin, working at the time at Caltech's Guggenheim Aeronautical Laboratory as
von Karman's assistant, dated June 1945, and in a lecture given that same year, an abstract of which is published in the Physical Review (48). A later paper in 1949 (49) is less interesting because in the meantime he had become acquainted with the work of Kolmogorov, Heisenberg and von Weizsäcker, and his presentation shows signs of the influence of their theories. However, the 1945 abstract in the Physical Review contains enough to understand the kind of proposal, and the deep link with Kolmogorov's ideas:

...it has not been pointed out up to now that the subdivision of energy must be a stepwise process...
For such a cascade mechanism that part of the energy density associated with large wave numbers should depend only on the total rate of dissipation \( \epsilon \). Dimensional considerations then require that the energy for a component of wave number \( k \) be equal, apart from a universal factor, to \( \epsilon^{2/3} k^{-11/3} \).

A rather amusing incident gives a fair idea of the nonusual character of Onsager's contribution for the fluid dynamics experts of that time. In addition of the manuscript sent to Lin, Onsager also sent a sort of summary of his ideas to von Kármán, at that time in Paris as scientific advisor for the U. S. Air Force. von Kármán commented to Lin: "I received a letter and a kind of manuscript from a certain Mr. Lars Onsager. I find his letter somewhat "screwy" so I would be glad to have your opinion whether the paper is worth while reading. Perhaps you could indicate to me in a few lines what the idea is, if any" (50).

3.5 A New Look on Turbulence

This incident, by itself, only shows that at the time von Kármán was busy with completely different matters, and that he therefore paid no attention to Onsager's manuscript; however, the very fact that the latter was unknown to him gives an idea of the distance separating their respective scientific environments. Moreover, it seems reasonable to take von Kármán's reaction to Onsager's proposal as typical of the kind of attitude adopted by scientists engaged at that moment in research on turbulence toward these new theories.
What cannot but disconcert the applied mathematicians of the time are not so much the main techniques and ideas used in these works (as we said, they are basically ingredients, and combinations of them, familiar long since to experts in the field), as the way to put the problem and the languages it is dealt with, typical of different scientific traditions. It is true that the concept of isotropic turbulence, the use of correlation functions to define the statistical properties, and the Fourier transform for the spectral representation, had been introduced by G. Taylor and quickly accepted by researchers in the field. However, Taylor had taken care to show the immediate relevance of his approach on applicative and technological grounds. On the contrary, a common characteristic of the theories we are discussing is that they were presented without any trace of applicative interests; they deal with the problem because of its intrinsic interest of a fundamental character. It may be defined as a typical approach of theoretical physics.

This "simultaneous discovery" is of interest because it comes to light when, on the science of turbulent research, new disciplinary sectors appear. It coincides with the moment when some sectors of 'pure' research show a return of interest towards a line of investigation that had been abandoned for a long time, to be destined to the attention of different professional competences.

3.6 Different Backgrounds of the Same Theory

The itineraries, through which our four scientists reached the formulations of their theories, are not only independent, but also very different from each other, as different as their respective scientific and institutional backgrounds.

Kolmogorov comes from the great tradition of Russian mathematics; the presentation of his papers clearly shows his origin as a leader in probability theory (his text where the foundations of the axiomatic approach are laid down goes back to 1933\(^{(51)}\)). To gain a better understanding of his contribution to our story, a detailed investigation of the characteristics acquired by the relation between pure and applied mathematics in the Soviet Union during the period between the two World Wars
would be highly needed.

Heinsenberg's contribution reveals the specific competence he had acquired working under Sommerfeld's direction in his doctoral thesis, that dealt with problems of stability and transition to turbulence in fluid mechanics. Von Weizsäcker, a "second generation" theoretical physicist after the quantum revolution was led to the study of turbulence by the nature of the astrophysical problem he was dealing with in the early Forties, namely the elaboration of a theory of the origin of the solar system based on the hypothesis of a condensation of planets starting from a nebula. The description of the evolution of such an object is a problem of the dynamics of a viscous fluid at high Reynolds numbers. The results obtained through his collaboration with Heisenberg led von Weizsäcker to modify his original model.

Finally, Onsager was a chemical physicist. His interest in the theory of turbulence came from non-equilibrium thermodynamics and nonlinear phenomena.
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25. ibid., p. 9.
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