The Bell inequality I: Joint measurability*

Abstract. An examination of the derivations of the Bell inequality shows that neither hidden-variable nor locality assumptions are fundamental to them; but every derivation presupposes that two spin projections along different directions for the same particle are jointly measurable, or equivalently, that they have a joint probability distribution. The failure of joint measurability in the quantum case and in the non-quantum ones that are presented is traced to an unsatisfactory choice of system made for describing the physical situation. It is then not possible to draw any philosophical conclusions from the Bell inequality; in particular, Einsteinian realism is not affected by it.

1. Introduction

The debate initiated by Einstein, Podolsky and Rosen [1] and Bohr [2] has in the last few years centred largely on the inequality derived by Bell [3]. His derivation made two explicit assumptions, that a hidden-variable explanation be feasible, and that it be local; that quantum mechanics does not satisfy the inequality is then taken to mean that, in Rohrlich’s words, “local hidden-variable theory is dead” [4]. The presumption of non-locality has variously been used to cast doubt on Einstein’s view of reality [5,6], or to justify the notion of superluminal information transport (and its implications) [7], or even to bolster up parapsychological speculations [8,9]. Since it is this side of the debate which has spilled over into the popular press, a most undesirable mispresentation has been created of what research in physics is about.

I propose to show here that such extrapolations are baseless, because the real assumptions underlying the Bell inequality are neither the hidden-variable nor the locality one, but the assumption that the spin projection of a particle can be measured in more than one direction without any mutual interference; this will below be called the joint-measurability assumption, \( JMA \), either directly or in the equivalent form of the assumption that a joint probability distribution exists for two or more spin projections on the same particle. From the \( JMA \) there follows also the possibility of reordering the sets of experimental data so as to exhibit the fact that they satisfy the Bell inequality.

In section 2 five derivations are presented: Bell’s derivation [3], a quantum-mechanical one, Santos’ quantum-logical one [11], and two probability-theoretic derivations due to Wigner [12] and Holt [13], and Suppes and Zanotti [14], respectively; it will be seen that only the first uses any hidden variables, and only the first two make any locality assumptions. Section 3, using another derivation [15,16,17]

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based on the treatment of experimental data, shows the need for the JMA if the data are to satisfy the inequality. Section 4 examines how failure of the JMA springs from an inadequate selection of the physical system; making the system selection explicit allows a resolution of the conceptual difficulty. None of the conclusions of these three sections are restricted to quantum mechanics; some classical systems that also violate the Bell inequality are therefore presented in section 5. The last section sums up and has some general comments. Some of the weaknesses of the locality concept as used in discussions of the Bell inequality are discussed elsewhere [43].

The upshot of the argument is that in the Einstein-Bohr debate the Bell inequality does not lead unequivocally to a victory for Bohr's side, as is commonly said, and even less so for any of its more recent distortions. To the contrary, reality remains obstinately Einsteinian.

2. The derivation of the Bell inequality

2.1. Bell's hidden-variable derivation

Bell [3] considered the set-up

\[ D_A \leftarrow F_A(\alpha) \leftarrow S \rightarrow F_B(\beta) \rightarrow D_B \]  

(1)

with the source \( S \) emitting pairs of spin \(-\frac{1}{2}\) particles of total spin 0; the particles \( A \) and \( B \) go through filters oriented at angles \( \alpha \) and \( \beta \) in the plane perpendicular to their motion; these could be Stern-Gerlach magnets; the detector \( D_A \) and \( D_B \) register +1 for spin up and -1 for spin down. Actual experiments have mostly used cascade photons from atomic deexcitations (see [18] or [19] for reviews); this differs from the spin case only in details that are inessential for the present purposes.

Denoting the individual results registered by \( D_A \) and \( D_B \) by \( a \) and \( b \), respectively,

\[ a = \pm 1, \quad b = \pm 1 \]  

(2)

an experimental correlation coefficient \( r_{ab} \) is obtained from \( n \) repetitions of a measurement pair. A theoretical prediction \( \rho_{ab} \) for this quantity is found as follows. Assume that there are hidden variables \( \lambda \) with a probability \( \mu(\lambda) \); then the locality assumption implies that \( a = a(\alpha, \beta) \) and \( b = b(\beta, \lambda) \). The rotational symmetry of the set-up (1) now gives that

\[ \langle a \rangle = \int a(\alpha, \lambda)d\mu(\lambda) = \langle b \rangle = 0 \]

\[ \langle a^2 \rangle = \int a^2(\alpha, \lambda)d\mu(\lambda) = \langle b^2 \rangle = 1 \]  

(3)
and therefore we have

\[ \rho_{\alpha\beta} = \int a(\alpha, \lambda)b(\beta, \lambda)d\mu(\lambda). \]  

(4)

If the filters in (1) had been oriented along directions \( \alpha' \) and \( \beta' \), two new measurements \( a' \) and \( b' \) (suppressing the arguments and transferring the primes for simplicity) are obtained; three further correlation coefficients \( \rho_{\alpha'\beta}, \rho_{\alpha'\beta'}, \rho_{\alpha'\beta'} \) may then be written, and one has

\[ V_\pm \equiv \rho_{\alpha\beta} - \rho_{\alpha'\beta} \pm (\rho_{\alpha'\beta'} + \rho_{\alpha'\beta'}) = \int [ab - a'b \pm (ab' + a'b')]d\mu. \]  

(5)

Now

\[ ab - a'b \pm (ab' + a'b') = \pm 2 \]  

(6)

since either \( a - a' = 0, a + a' = \pm 2 \) or \( a - a' = \pm 2, a + a' = 0 \). Hence

\[ V = \max (|V_+|, |V_-|) = |\rho_{\alpha\beta} - \rho_{\alpha'\beta}| + |\rho_{\alpha'\beta'} + \rho_{\alpha'\beta'}| \leq 2. \]  

(7)

This is the Bell inequality in the form given by Clauser et al. [20]. Many other inequalities of this kind are known [21,22], but since the discussion below applies with small changes to them, these variants will be ignored.

As is well known [3,18,19], eq. (7) is not compatible with the prediction

\[ \rho_{\alpha\beta} = ((\hat{\sigma}_A \cdot \alpha)(\hat{\sigma}_B \cdot \beta)) = -\cos(\alpha - \beta) \]  

(8)

of quantum mechanics. In (8), \( \alpha \) and \( \beta \) are unit vectors in the directions \( \alpha \) and \( \beta \), while \( \hat{\sigma}'_A \) and \( \hat{\sigma}'_B \) are the spin operators for particles \( A \) and \( B \), the apostrophe indicating that they are normalized to give eigenvalues \( \pm 1 \).

The experimental evidence on the whole favours the quantum-mechanical result (8), though there is more room for doubts than is often conceded [23,24,25,26].

This derivation of Eq. (7) assumes (i) the existence of hidden variables \( \lambda \) that account for the values \( a \) and \( b \), (ii) the locality assumption that \( a \) contains no \( \beta \) dependence, (iii) that for a given \( \lambda \) we know all four values \( a, b, a', b' \), and (iv) that \( \mu \) depends neither on \( \alpha \) nor on \( \beta \). The four assumptions are not independent of each other, of course. The third is equivalent to the JMA; it is needed for the second member of (5).

2.2. A quantum derivation

Consider a system composed of two subsystems \( A \) and \( B \) and described by a density matrix \( \hat{\rho}_{AB} \). Filter procedures, with operators \( \hat{F}(i, \vartheta), i = A, B \), can be carried out on the two subsystems, where \( \vartheta \) is an angular parameter. To agree with Eq. (2) the
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\( \hat{F} \) are taken to have eigenvalues \( \pm 1 \), and to satisfy

\[
\begin{align*}
\text{tr} \, \hat{F}(i, \vartheta) \hat{W}_{AB} &= 0 \\
\text{tr} \, \hat{F}^2(i, \vartheta) \hat{W}_{AB} &= 1
\end{align*}
\]

\( \hat{F}(A, \vartheta) \) and \( \hat{F}(B, \phi) \) commute for all \( \vartheta, \phi \). Then

\[ \rho_{\alpha\beta} = \text{tr} \, \hat{F}(A, \alpha) \hat{F}(B, \beta) \hat{W}_{AB} \]

and thus

\[
V = | \text{tr} \, \hat{F}(A, \alpha) \hat{F}(B, \beta) \hat{W}_{AB} - \text{tr} \, \hat{F}(A, \alpha') \hat{F}(B, \beta) \hat{W}_{AB} | \\
+ | \text{tr} \, \hat{F}(A, \alpha) \hat{F}(B, \beta') \hat{W}_{AB} - \text{tr} \, \hat{F}(A, \alpha') \hat{F}(B, \beta') \hat{W}_{AB} |
\]

It is tempting to carry out the trace over the \( B \) variables, to find \( \hat{W}_A = \text{tr}_B \hat{W}_{AB} \)

\[
V \leq | \text{tr}_A [ \hat{F}(A, \alpha) - \hat{F}(A, \alpha')] \hat{W}_A | + | \text{tr}[ \hat{F}(A, \alpha) + \hat{F}(A, \alpha')] \hat{W}_A | \leq 2 \tag{9}
\]

where use has been made of Eq. (6). But this is not valid since, in general,

\[ \text{tr} \, \hat{F}(B, \vartheta) \hat{W}_{AB} \neq \text{tr} \, \hat{F}(B, \phi) \hat{W}_{AB}. \]

Now when the operators \( \hat{F}(B, \vartheta) \) and \( \hat{F}(A, \vartheta + \pi) \) act on states \( \hat{W}_{AB} \) with total spin 0, they behave exactly alike (not, of course, acting on \( \hat{F}(i, \phi) \hat{W}_{AB} \)). Hence by writing \( V \) in terms of operators on \( A \) only, the trace over the \( B \) coordinates is immediate, and we find

\[
V = | \text{tr}_A \, \hat{F}(A, \alpha) \hat{F}(A, \beta + \pi) \hat{W}_A - \text{tr}_A \, \hat{F}(A, \alpha') \hat{F}(A, \beta + \pi) \hat{W}_A | \\
+ | \text{tr}_A \, \hat{F}(A, \alpha) \hat{F}(A, \beta' + \pi) \hat{W}_A + \text{tr}_A \, \hat{F}(A, \alpha') \hat{F}(A, \beta' + \pi) \hat{W}_A |
\]

Therefore, provided \( \hat{F}(A, \vartheta) \) and \( \hat{F}(A, \phi) \) commute for \( \vartheta \neq \pi \), Eq. (9) follows and the Bell inequality (7) holds; but if they do not commute, the inequality cannot be derived.

This formalism is local, for though \( \text{tr}_B \, \hat{F}(B, \vartheta) \hat{W}_{AB} \) depends on \( \vartheta \), \( \text{tr}_A \text{tr}_B \hat{F}(B, \vartheta) \hat{W}_{AB} \) does not. This derivation assumes (i) locality, (ii) the JMA. The latter takes the forms of the commutation requirement for two operators on the same particle, since only then does the inequality follow. The quantum nature of the proof excludes hidden variables.
2.3. A quantum-logical derivation

This derivation is due to Santos [11]. He defines a separation between two propositions \( x \) and \( y \) on the quantum lattice as

\[
s(x, y) = p((x \cap y') \cup (x' \cap y))
\]

(10)

where \( x' \) is the orthocomplement of \( x \) and \( p(x) \) is the probability measure that defines the quantum state of the system. Now the quantum separation (10) satisfies the triangle inequality

\[
s(x, y) + s(y, z) \geq s(x, z)
\]

provided \( x, y, z \) all belong to one Boolean sublattice. Since this cannot be the case for spin measurements, two of which would have to be done on the same particle, Santos derives the inequality

\[
s(x_1, y_1) + s(x_2, y_1) + s(x_2, y_2) \geq s(x_1, y_2)
\]

(11)

valid when all four propositions belong to one Boolean sublattice. Since this is not generally the case, Santos adds a “locality condition”: We have no reason to identify the proposition \( x_1 \) when measured with \( y_1 \), written \( (x_1/y_1) \), with the proposition \( x_1 \) when measured with \( y_2 \); since these propositions concern incompatible spin components, a complete measurement has in fact 8 propositions rather than 4. He then defines as local any theory in which, for a fixed \( i \), the propositions \( (x_i/y_k) \) are identified for all \( k \) whenever the region in which the \( y_k \) are measured is spatially separated from that of \( x_i \). In such a theory, he concludes, (11) will hold, and from it the Bell inequality follows quite straightforwardly.

The quantum-logical derivation goes through, therefore, in two cases. Either the four propositions belong to a Boolean sublattice and are therefore jointly measurable; this is the JMA. Or the theory is “local”; but if \( (x_1/y_1) \) and \( (x_1/y_2) \) can be identified, that is to say, if their truth values are identical, then \( x_1 \) can be measured with \( y_1 \) and also with \( y_2 \), and so \( y_1 \) and \( y_2 \) can be measured jointly. The second case reduces to the first. To make this clear, consider the 8n propositions \( (x_i/y_k)_m \), \( m = 1..n, i = 1,2 \); the truth value of \( (x_i/y_k)_m \) is not in general that of \( (x_i/y_k)_{m'} \). It is, however, possible (under not very restrictive conditions) to reorder one of the proposition sets so that they are equal in most cases. What is not possible is to do this simultaneously for all 8n propositions, unless the JMA holds. This will be shown in section 3.

2.4. The Wigner-Holt probability-theoretic derivation

We give this derivation, due to Wigner [12], in a revised form of Holt’s [13]. Although Wigner begins his discussion in terms of hidden variables, both his notation and the nature of his argument make them superfluous; we do not therefore introduce
them. A probability space in which the four possible results $a, a', b, b'$ form the four axes is considered. To the $2^4$ points in this space probabilities $p_{aa'bb'}$ are assigned, which satisfy

$$\sum_{aa'bb'} p_{aa'bb'} = 1$$

If we assume that they also satisfy Eq. (3), the correlation coefficients are

$$\rho_{\alpha\beta} = \sum_{aa'bb'} ab p_{aa'bb'}$$

and so on. One has immediately that

$$\rho_{\alpha\beta} - \rho_{\alpha'\beta} \pm (\rho_{\alpha\beta'} + \rho_{\alpha'\beta'}) = \sum_{aa'bb'} [ab - a'b \pm (ab' + a'b')]p_{aa'bb'}$$

Eq. (6) then yields the Bell inequality in the form of Eq. (7).

Locality, in the sense that the value found for $a$ in one measurement does not depend on whether $b$ or $b'$ is determined with it, cannot be expressed here since only probabilities are given. Not even weak locality, in the sense that the probability of $Q$ is independent of what is measured with it, can be stated since the probabilities depend on all four variables. Only related conditions, such as

$$Pr(a|b = 1) = Pr(a|b = -1)$$

or

$$Pr(a|b = 1) = Pr(a|b' = 1)$$

may be stipulated, in the form

$$\sum_{a'b'} p_{aa'1b'} = \sum_{a'b'} p_{aa'-1b'}$$

or

$$\sum_{a'b'} p_{aa'1b'} = \sum_{aa'b} p_{aa'b1}$$

But the Bell inequality is derived whether such conditions hold or not. Thus no locality condition is involved in this derivation. On the other hand, the existence of a joint probability distribution is expressly required; we show in section 4.1 below that this is equivalent to the JMA.
2.5. A pure probability-theoretic derivation

Suppes and Zanotti [14] have given what seems to be the most general derivation of the Bell inequality, based only on probability theory. They establish the following theorem:

Let \( x, y, z \) be three random variables taking \( \pm 1 \) as values and having 0 expectations; their correlation coefficients \( \rho(x, y), \rho(x, z), \rho(y, z) \) are given. Then a necessary and sufficient condition for a joint probability distribution of the three variables to exist is that the inequalities

\[
-1 \leq \rho(x, y) + \rho(x, z) + \rho(y, z) \leq 1 + 2 \min \{ \rho(x, y), \rho(x, z), \rho(y, z) \}
\]

be satisfied. The proof is based on the fact that of the eight discrete values forming the joint distribution, seven are determined by the known means and correlations and by their sum being 1. Eq. (13) is then the condition that the eighth probability lie between 0 and 1.

This theorem is now applied to the triplets \((a, a', b), (a, a', b'), (a', b, b')\) to give, e.g.

\[
-1 \leq \rho_{a\alpha'} + \rho_{a\beta'} + \rho_{a'\beta'}
\]

\[
-1 \leq -\rho_{a\alpha'} - \rho_{a'\beta} + \rho_{a\beta}
\]

which sum to

\[
-2 \leq \rho_{a\beta} - \rho_{a'\beta} + \rho_{a\beta'} + \rho_{a'\beta'}
\]

Combining this with similar relations yields the Bell inequality (7).

In this derivation no hidden variables occur, and a locality assumption would be meaningless. The only assumption required is the JMA, which here takes the form of the assumption that the four three-dimensional distributions exist (equivalent to the existence of the four-dimensional distribution, since they are discrete-valued). We note that explicit use is made of the correlations \( \rho_{a\alpha'} \) and \( \rho_{\beta'\beta} \) which are not well defined in quantum theory.

2.6. Summary

Only the first of these derivations effectively uses hidden variables, while in the second they are impossible. Only the first two make any locality assumption; Santos' locality condition will be seen below to have a different origin. But in all of them either the JMA is assumed to hold or a joint probability distribution is taken to exist. Since quantum mechanics, which does not satisfy the JMA, violates the Bell inequality, it is the JMA that appears as the key assumption in deriving the inequality.
The irrelevance of the hidden-variable hypothesis has previously been noted by Eberhard [29] and Stapp [30]. The nature of the locality assumptions made in these derivations is further discussed elsewhere [43].

3. The reordering problem

Another derivation of the Bell inequality is due to Eberhard [15] and Stapp [16]; it was given a simpler form by Peres [17]. In the derivations of section 2, the values of $a, a', b, b'$ are seen as "possessed" by the particles so that they can figure in the same equations, even though they are not jointly measurable. To sidestep the resulting conceptual problems (to which we return below), here the experimental observations $a_i$ and $b_i$, $i = 1 \ldots n$, are considered. They yield an experimental correlation coefficient

$$r_{\alpha\beta} = \frac{1}{n} \sum_{i=1}^{n} a_i b_i,$$

(14)

in terms of which we have

$$\rho_{\alpha\beta} = r_{\alpha\beta}$$

(Here and below, $\doteq$ is to be read "equal in the limit of $n$ large enough", i.e. when any difference is of the order of the expected statistical fluctuations.)

If measurement $b'_i$ along the direction $\beta'$ had been made, a correlation coefficient

$$\rho_{\alpha\beta} = \frac{1}{n} \sum_{i=1}^{n} a_i b'_i,$$

(15)

would be obtained, and similarly for the other two. Here it is argued that locality implies that for a given $i$, the $a_i$ in Eq. (14) is equal to that in (15), so that the four correlation coefficients may be combined as

$$\rho_{\alpha\beta} - \rho_{\alpha'\beta'} \pm (\rho_{\alpha\beta'} + \rho_{\alpha'\beta'}) = \frac{1}{n} \sum_{i=1}^{n} [a_i b'_i - a_i b'_i \pm (a_i b'_i + a_i b'_i)] \leq 2$$

where Eq.(6) has been applied to each summand in the second member so as to yield the Bell inequality (7).

The "locality" argument used here, essentially the same as that employed by Santos (see section 2.3), stipulates a condition such as

$$a_i(\alpha, \beta) = a_i(\alpha),$$
or perhaps better

$$a_i(\alpha, \beta) = a_i(\alpha, \beta').$$  \hspace{1cm} (16)$$

But a functional dependence of this kind can be attributed to a theoretical quantity, not to an experimental result, for which we can at most say that it was found together with this or that other result. In Santos' notation the condition becomes

$$(a_i/b_i) = (a'_i/b'_i)$$

If the JMA holds, then in fact $a_i, b_i$ and $b'_i$ can be measured together and (16) holds. If the JMA does not hold, then $(a_i/b_i)$ and $(a'_i/b'_i)$ belong to independent experimental runs, and (16) cannot be true as it stands. It may be made true by reordering the data from one of the runs, provided that

$$\frac{1}{n} \sum_{i=1}^{n} a_i(\alpha, \beta) = \frac{1}{n} \sum_{j=1}^{n} a_j(\alpha, \beta')$$  \hspace{1cm} (17)$$

(It can be shown [43] that (17) is meaningful where (16) is not.) This condition is given by Stapp [30], but he did not see that it is not sufficient. This becomes clear in an explicit description of the reordering process for all four sets of $n$ data pairs:

1. In each series from one run, place the pairs with positive $a$ at the top, and within each part of a given sign, place the pairs with positive $b$ at the top. Pairs must not, of course, be disassociated. This is always possible. Then

2. Provided Eq. (17) and its equivalent for $\alpha'$ hold, the $a$ values in columns 1 and 2 are paired, as are those in columns 3 and 4.

3. Let $i_+, j_+, k_+, l_+$ be the numbers of pairs in the four columns, respectively, where $a$ and $b$ are positive, and $i_-, j_-, k_-, l_-$ the corresponding numbers of negative-negative pairs. The two pairings of $b$ values are then achieved if the number of cases in which the $b$ differs in sign between columns 1 and 2 is the same as that between columns 3 and 4; that is to say

$$|i_+ - j_+| = |k_+ - l_+|$$

$$|i_- - j_-| = |k_- - l_-|$$  \hspace{1cm} (18)$$

for the positive-$a$ part and the negative-$a$ part, respectively.

Conditions (18) imply the Bell inequality; we show this for the simpler case where on account of rotational invariance each column has altogether $p$ positive $a$ values and $q$ positive $b$ values. Then the product sum in the first column is

$$(ab) = 2(i_+ + i_- - p) - n$$
and so

\[ \rho_{\alpha\beta} = \frac{2n(i_+ + i_- - p) - n^2 - (2p - n)(2q - n)}{[n^2 - (2p - n)^2]^{\frac{1}{2}}[n^2 - (2q - n)^2]^{\frac{1}{2}}} \]

The sum of the two Equations (18) then takes the form

\[ |\rho_{\alpha\beta} - \rho_{\alpha'\beta'}| = |\rho_{\alpha'\beta} - \rho_{\alpha'\beta'}| \]  

(19)

For only two correlation coefficients we always have

\[ |\rho_{\alpha'\beta} - \rho_{\alpha'\beta'}| + |\rho_{\alpha'\beta} + \rho_{\alpha'\beta'}| \leq 2 \]  

(20)

Combining (19) and (20) yields the Bell inequality (7).

That there are non-reorderable sets of data, and that in fact their number is much larger than that of the reorderable ones, is evident because the total number of different sets is, for even \( n \),

\[ N_{\text{tot}} = \frac{1}{n!} \left( \frac{n}{2n} \right)^8 \]

while that of the reorderable sets is only

\[ N_{\text{reord}} = \frac{1}{n!} \left( \frac{n}{2n} \right)^4 \]

Two examples of non-reorderable sets are

\[
\begin{array}{ccccccc}
\alpha\beta & \alpha'\beta' & \alpha'\beta & \alpha'\beta' \\
++ & ++ & -- & --
\end{array}
\quad
\begin{array}{ccccccc}
\alpha\beta & \alpha'\beta' & \alpha'\beta & \alpha'\beta' \\
++ & ++ & ++ & --
\end{array}
\quad
\begin{array}{ccccccc}
\alpha\beta & \alpha'\beta & \alpha'\beta' & \alpha'\beta' \\
++ & ++ & -- & --
\end{array}
\quad
\begin{array}{ccccccc}
\alpha\beta & \alpha'\beta & \alpha'\beta' & \alpha'\beta' \\
++ & ++ & ++ & --
\end{array}
\]

The first satisfies the Bell inequality, the second does not.

As regards the Eberhard-Stapp-Peres derivation of the beginning of this section, it is now clear that it needs the property of reorderability in order to go through; the data must thus have the structure they would have if they came from a jointly measurable experiment.
4. The joint-measurability assumption

4.1. Joint measurability and joint probability distribution

The joint-measurability assumption refers to the possibility of measuring two (or more) physical quantities without mutual interference; this last expression is to be understood in the sense that neither measurement affects the value obtained by the other, not necessarily in the sense that they are carried out simultaneously. In the case of spin projections, the JMA is violated, but it is still possible to pass a particle through a second Stern-Gerlach magnet after its spin projection has been measured a first time (assuming that detection does not absorb it). It is of course possible to calculate the quantum-mechanical correlation coefficients for this case, and they satisfy, as expected, the Bell inequality [31].

The JMA characterizes the experimental set-up but may also be expressed theoretically. This occurs in the Bell derivation of section 2.1, where the use of Eq. (6) requires that e.g. \( ab - a'b = (a - a')b \), so that for every \( \lambda \) in the range of integration \( a = a(\alpha, \lambda) \) and \( a' = a(\alpha', \lambda) \) must be simultaneously known; this is not possible except coincidentally unless they refer to the same particle and are thus jointly measurable. Bell's original derivation, instead of Eq. (6), used integrals such as \( \int aa'bb'd\mu(\lambda) \), to which this point also applies. The quantum derivation also requires the JMA, so that the operators \( \hat{F}(A, \alpha) \) and \( \hat{F}(A, \alpha') \) commute.

In the two probability-theoretic derivations of section 2 the Bell inequality follows when a joint probability distribution for the four measured quantities is assumed. If we assume that the theoretical model on which this distribution is based provides an adequate account of the data from the experiment, then this assumption is equivalent to the JMA: if the JMA holds, a sequence of joint measurements can be carried out of the four variables, and from them experimental estimates can be made of one fourth-order correlation coefficient, four third-order ones, and six second-order ones; four of the last enter into the Bell inequality. To account for all these, a joint probability distribution must exist to provide theoretical values for them. Inversely, given a joint probability distribution, the JMA is needed so that the corresponding experimental correlations can be measured.

The two coefficients that do not enter into the usual form of the Bell inequality are \( \rho_{aa'} \) and \( \rho_{gg'} \). If the JMA holds, these should be experimentally measurable and theoretically accounted for. Neither in the spin-projection case nor in the cascade-photon one can they be measured or derived from quantum theory. On the other hand, the models of sections 2 and 3 all furnish unambiguous predictions for these coefficients:

\[ 1 \text{If we take } a \text{ and } a' \text{ to be found from separate runs in an experimental series, then it must be assumed that it is possible to reproduce the } \lambda \text{ exactly, or at least to determine sufficiently well to decide which } a \text{ goes with which } a'; \text{ as de Baere [31] has noted, this is an impossible task. He deduces, quite correctly, that the violation of Bell's inequality does not have anything to do with locality. The present point of view seems preferable, however, in that it involves no presupposition about hidden variables.} \]
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Section 2.1  \[ \rho_{a\alpha'} = \int a(\alpha, \lambda)a(\alpha', \lambda)d\mu(\lambda) \]

Section 2.2  \[ \text{tr } \hat{F}(A, \alpha)\hat{F}(A, \alpha')\hat{W}_{AB} \]

Section 2.3  \[ s(a_1, a'_2) \]

Section 2.4  \[ \sum_{aa'bb'} aa'\rho_{aa'bb'} \]

Section 3  \[ \frac{1}{n} \sum a_i a'_i \]

and the Suppes-Zanotti derivation of section 2.5 uses these correlations explicitly. This makes it plain that the inability of these models to predict correctly the situation in quantum theory lies in their acceptance of the JMA.\(^2\)

The equivalence of the existence of a joint probability distribution and the Bell inequality has been proved earlier [14,34,35]. This equivalence does not conflict with our conclusions concerning reorderability, the condition for which, Eqs. (18), is not equivalent either to the Bell inequality or to the joint probability distribution, i.e. the JMA.

4.2. The JMA and physical reality

The JMA could be thought to raise the same doubts about physical reality that were dissipated when locality was shown not to be involved in the deduction of the Bell inequality. For if two quantities cannot be measured together, it is not clear that they can be said actually to exist; nevertheless, spin projections satisfy the EPR criterion [1] or physical reality. But this is to ignore that properties do not exist independently and are not “possessed” in the same sense as objects are possessed by their owner, who can sell them but cannot sell his age, say. Hence properties come into being and disappear with the physical systems with which they are associated.

Thus there arises the possibility of using too limited a model to describe their relation in the system. If a model is used which exceeds the relevant system, no problems develop; but if a subsystem is modelled, difficulties can appear. In going from an adequate model to a submodel, a property may be affected in one of three ways: (i) it may remain unchanged, as for instance the temperature of a (not too small) fraction of a macroscopic system in thermal equilibrium; (ii) it may change in value, as for instance the mass of the same fraction; or (iii) it may cease to exist, as for instance the binding energy of a hydrogen atom, which is not a property of either component particle.

Too small a model can be extended by embedding it in a larger one. The following situations can then arise:

1. The extension may be unique, as far as the property of interest is concerned. In this case the “small” model provides all necessary information, and the prop-

\(^2\)The relevance of these correlations to an adequate interpretation of the Bell inequality was first stressed by Lochak [33]. The point was somewhat cavalierly dismissed by d’Espagnat [6], simply because in many derivations they do not appear explicitly.
property may even remain constant, as in case (i) above. The extension may give conceptual clarity.

2. When the extension is not unique, parameters that specify the choice are needed; the choice is usually dictated by experimental reasons, not theoretical ones.

2.1. The choices are mutually exclusive, e.g. through involving incompatible experimental set-ups.

2.2. If the choices do not exclude each other, one must distinguish. Let the property of interest be a function \( q = q(\phi, \xi) \) of the choice parameter(s) \( \phi \) and any needed small-model parameters \( \xi \). Then

2.2.1. If \( q \) is a 1:1 function of \( \phi \), then for another value \( \phi' \) but the same \( \xi \) we have

\[ q' = f(q, \phi', \xi) \]

a well-defined function, and the correlation \( \rho(q, q') \) must exist.

2.2.2. The function \( q \) has no inverse, \( f \) does not exist, nor does the correlation between \( q \) and \( q' \).

Case 2.2.1. is the case in which inequalities of the Bell type are expected to exist. In case 1 they cannot be defined, because the relevant parameters do not exist, e.g. the angle parameters in the spin-projection case. In case 2.1. the JMA does not hold, and the inequalities cannot be derived. Case 2.2.2. is marginal: though the different quantities are jointly measurable, they are so each in a different set-up, and the Bell inequality may not be satisfied. The measurement of total spin (actually of the multiplicity \( 2s+1 \)) corresponds to case 1, since the orientation of the inhomogeneous magnetic field is irrelevant. Spin projections correspond to case 2.1: the JMA does not hold for them. An example of case 2.2.2. will be discussed in the next section.

This then means that a spin projection is not a property of the particle by itself, as it is just after emission from the source, but of the joint system (particle + Stern-Gerlach magnet); this is made more comprehensible by the quantum picture, of a wave function with cylindrical symmetry before the interaction with the inhomogeneous field, of planar symmetry in the density matrix after it. But in the absence of a deeper understanding of the physical nature of spin it does not appear possible to furnish a more precise description.

The above account in no way conflicts with the philosophical picture of an independently existing physical reality. The process of creation of a property is a real and physical process; it does not (normally) occur in the process of detection, and even less when measured results are apprehended by the physicist. All that has been done is to complete the Einsteinian picture where it was not sufficiently detailed.

The classification given above is not absolute. A non-linear transformation of the physical quantity that expresses the property of interest may change it from category to category. This is trivially obvious when \( q \) can take a continuous range of values. Dichotomic variables (such as the projections of a total spin \( \frac{1}{2} \)) are exceptional only in that two of the four possible transformations are linear, while the other two eliminate the dependence on \( \phi \) and so translate it to case 1.
5. Some classical examples

The concept of system extension, and the resulting possibility of the JMA not holding, is not restricted to quantum systems. Cases 1 and 2.2.1 hardly need illustrating; accordingly we present examples for the other two.

5.1. Classical violations of the Bell inequality

That classical systems which violate the Bell inequality should exist seems first to have been suggested by Suppes and Zanottii [14]; the first detailed description is given by Scalera [37] and consists of two ribbons issuing helicoidally, with a common angular velocity, from a source; whenever their inclinations are along preassigned directions, they are marked, and two detectors measure the energy on each ribbon between two successive markers. The correlation between these energies is then, under suitable conditions, shown to violate the Bell inequality.

Such a model could not easily be realized in the laboratory; this is remedied by Notarrigo [38], whose model uses two rows of equal masses linked by equal springs and moving only in the planes transverse to the rows. In each row, one end initiates the motion under random conditions which are the same for the two rows; the other masses start at rest. The other ends of the rows are fixed and reflect the excitation wave. In each row one mass is constrained to move along a certain direction, the “easy” direction; this is different for the two rows. Beyond the filter so established the motion is along the easy direction only; for one of the masses there, a reference point is chosen and an event is noted if, during a time window chosen to give a predetermined event probability, the mass passes the reference point moving outward. The quantity of interest is then the correlation coefficient between events on the two rows. It is obtained from a computer simulation, and found to approximate the quantum dependence, Eq. (8), except for the sign, when the number of masses in each row is not too small ($\geq 20$); the Bell inequality is then violated.

This model, like the Scalera one, is local: there is no interaction of any kind between the two rows, beyond the identity of the starting impulse which here corresponds to the condition of total spin 0 in the set-up of Eq. (1). The only “hidden” variables are those involved in the randomization of the initial conditions. Since it is not possible to set more than one easy direction on each row, the model belongs to category 2.1 of the preceding section: accordingly the JMA does not hold for it, and the Bell inequality cannot be expected to hold either.
5.2. The harmonic oscillator

Further insight is gained by considering an analytically soluble model. Consider two harmonic oscillators of the same frequency but with different phase angles:

\[ a = a_0 + a_1 \cos(\omega t + \alpha) \]
\[ b = b_0 + b_1 \cos(\omega t + \beta) \]  
(21)

Then

\[ \langle a \rangle = a_0, \quad \langle a^2 \rangle = a_0^2 + \frac{1}{2} a_1^2 \]
\[ \langle b \rangle = b_0, \quad \langle b^2 \rangle = b_0^2 + \frac{1}{2} b_1^2 \]  
(22)

\[ \langle ab \rangle = a_0 b_0 + \frac{1}{2} a_1 b_1 \cos(\alpha - \beta) \]

The averages in (22) are taken either over a complete cycle \( T = 2\pi \), or over an ensemble of uniformly distributed starting times. The harmonic oscillator is known to be ergodic and so these two averages coincide.

The correlation coefficient between the two harmonic oscillators of Eq. (21) is then

\[ \rho_{\alpha\beta} = \cos(\alpha - \beta). \]  
(23)

Except for the sign, this is the quantum correlation (8), and, like it, it violates the Bell inequality.

Since, as noted in section 4, by a simple non-linear transformation the violation may be eliminated, it is of interest to discretise the problem by redefining the variables of Eq. (21) as

\[ a = \begin{cases} +1 & \text{if } a_0 + a_1 \cos(\omega t + \alpha) > 0 \\ -1 & \text{otherwise} \end{cases} \]  
(24)

and similarly for \( b \). In the case most favourable for violation, the correlation coefficient takes the limiting form found by Selleri [39]

\[ \rho_{\alpha\beta} = 1 - \frac{2|\alpha - \beta|}{\pi} \]

The discriminant \( V \) of Eq. (7) now reaches the value 2 but does not exceed it. A
small higher-harmonic term in (24) will make it exceed 2:

\[ a = \begin{cases} 
+1 & \text{if } a_0 + a_1[\cos(\omega t + \alpha) + \epsilon \cos 2(\omega t + \alpha)] > 0 \\
-1 & \text{otherwise}
\end{cases} \]

and similarly for \( b \). Even for \( \epsilon = 0.1 \), a computer simulation has shown that \( V > 2 \) over a considerable region, starting at \( \alpha - \beta = 1.8^\circ, \alpha - \beta' = 30.6^\circ, \alpha - \alpha' = 16.2^\circ \); for small \( \epsilon, V - 2 = \epsilon^2/2 \).

Other small non-linearities have similar effects. A quartic term added to the harmonic-oscillator potential or a term in \( t^2 \) in the phase angles of Eq. (21) lead in analogous fashion to a violation of the Bell inequality.

A procedural point requires a comment. Some authors \([40,41]\) have argued for the use of the covariance \( \text{cov}(a,b) = \langle ab \rangle - \langle a \rangle \langle b \rangle \) in calculating \( V \) instead of the correlation coefficient \( \rho(a,b) = \text{cov}(a,b)/[\text{var}(a) \cdot \text{var}(b)]^{\frac{1}{2}} \), where \( \text{var}(a) = \text{cov}(a,a) \). The covariance, however, has dimensions and therefore changes with scale changes; thus using the real values \( \pm \frac{1}{2} \) for the spins would either make even the quantum covariance satisfy the Bell inequality or require a right-hand side \( 2[\text{var}(a) \cdot \text{var}(b)]^{\frac{1}{2}} = \frac{1}{2} \) instead of 2 in the inequality, Eqs. (7) and (20). The dimensionless correlation coefficient is therefore to be preferred. A related problem appears in a paper by Barut and Meystre \([42]\). They treat the classical three-dimensional harmonic oscillator in a very "anschaulich" way, and find that it violates the Bell inequality even in a discretised version; they attribute this to a difference between the quantum and the classical normalizations. In fact their correlation coefficient covers the range \((-1,1)\), as it should, and their normalization factor is similar in origin to the one mentioned above for the spin case, which is eliminated by the artifice of using eigenvalues \( \pm 1 \) for the spin of a spin \(-\frac{1}{2}\) particle.

The harmonic oscillator, seen as a one-dimensional system, clearly falls in category 2.1 of the classification above: an oscillator cannot simultaneously have two phase angles \( \alpha \) and \( \alpha' \). But a two-dimensional rotor may have an unlimited number of projections into harmonic oscillators, all of them simultaneously measurable; a laboratory set-up that realises such a scheme is easily imagined. However, the function \( q \) of \( \phi \) (here the angle \( \alpha \)) has no unique inverse, for to each \( q \) there correspond two values of \( \alpha \), with

\[ \alpha_1 + \alpha_2 = 2\pi - 2\omega t \]

Thus this is a case in category 2.2.2. It illustrates, moreover, that restructuring the model describing the system may move a case from category to category.
Three general comments may be made before concluding.

1. As was shown in section 3, reordering a set of experimental data in such a way that quadruplets satisfying (6) can unambiguously be extracted and the Bell inequality satisfied is possible only under restricted conditions; it cannot in general be done for the data from four separate runs at different angle settings. Consequently, the Bell inequality places a restriction on the correlation coefficients, and coefficients that vary unrestrictedly may violate it. Once this is realized, it is obvious that it is not the violation but the satisfaction of the inequality that could be thought to need transmitting further information between the $A$ and $B$ measurements of the set-up (1). This is precisely the opposite of what is claimed in order to argue for the superluminal transmission of information [7]; such arguments are therefore suspect from the outset.

2. That two different spin projections cannot be determined on the same particle without mutual interference is, of course, well known. By taking (sometimes only implicitly) the JMA as starting point, the deductions of the Bell inequality systematically ignore this fact, however. This is not in itself to be considered a mistake. When building a theoretical model for a physical situation, many even essential characteristics must of needs be ignored, and the success of the model justifies these oversimplifications a posteriori. But if the model misdescribes a significant aspect, such omission must be revised. This has not so far been done as regards the interpretation of the Bell inequality.

3. The concept of a physical system as a finite part of the universe, circumscribed by the experimental techniques for its preparation and isolation, and represented theoretically by a model, has been seen to play a significant rôle in the perception of what the JMA implies. In particular, the use of more than one system at the same time, with their corresponding models, the transitions between which require rather careful description, is exemplified here. It is the author's belief that this situation is not unique. More than one conceptual difficulty bedevilling the foundations of physics would probably be resolved if due attention were paid to stating just what systems are being used and how transitions between them are to be handled. It is unfortunate that the many cases where the system definition is obvious have led to a tradition in the physics literature of ignoring the entire question; together with the obscurity due to the lack of any philosophical account of the system concept, this has created much needless confusion.

In conclusion, it has been shown that all the derivations here examined of the Bell inequality use the JMA, either directly or by assuming a joint probability distribution. Only a few, on the other hand, need hidden variables or a locality assumption. Thus the JMA stands out as the key assumption. Now for many variable pairs, classical as well as quantum-mechanical, the JMA is not valid. This raises, as we have seen, interesting questions concerning the system concept and that of the corresponding theoretical model; but in no way could it imply that we must either doubt the separability of physical systems when no possible interaction between
them can exist, or else abandon any realist position, such as that defended by Einstein.

Thus a final caveat against drawing overhasty philosophical conclusions before the underlying physics is well understood may not be out of place.

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