Introduction to perturbative QCD

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Abstract. The intent of these lectures is to present a pedagogical introduction to perturbative QCD, with an emphasis on calculational techniques rather than formal derivations. First, the standard QCD Lagrangian, dimensional regularization and renormalization are reviewed. Then, deep inelastic scattering in the context of the naive parton model is described, and the QCD-improved parton model is motivated. The Altarelli-Parisi equations for the parton distribution functions are derived, and the experimental extraction of the distribution functions is discussed. The first order QCD radiative corrections to deep inelastic scattering are used to illustrate factorization. Higher order corrections to W and Z production in hadronic colliders are presented.

1. Introduction

This series of lectures is intended as an introduction to perturbative calculations in quantum chromodynamics (QCD) and as an aid to understanding how some experiments interpret their measurements in terms of the QCD-improved parton model. The student is assumed to have some background with Feynman diagrams and perturbative QED. Basic introductions may be found in Refs. [1]-[4]. For introductions to the standard model of strong, weak and electromagnetic interactions and the parton model, see, for example, Refs. [5] and [6]. A particularly good undergraduate level textbook is Ref. [7]. Discussions more specific to QCD are found in Refs. [8]-[12].

The program for these lectures is to begin with a brief review of the QCD Lagrangian, dimensional regularization, and the minimal subtraction renormalization scheme (MS). With these tools, we compute the counter term at order $\alpha_s$ for the quark self energy. Next, we review the renormalization group equations, the running of the coupling constant and the introduction of $\Lambda_{QCD}$. Stepping back from QCD as the theory of strong interactions, we review the naive quark model in deep inelastic scattering and consider some of its successes and defects. We then look at the QCD-improved parton model, and heuristically derive the QED equivalents of the Altarelli-Parisi equations. A discussion of the experimental determination of the quark and gluon distribution functions follows. The property of factorization is illustrated with the order $\alpha_s$ correction to deep inelastic scattering. Finally, higher order corrections to W and Z production are presented.
Introduction to perturbative QCD

<table>
<thead>
<tr>
<th>Particles</th>
<th>$U(1)_Y$</th>
<th>$SU(2)_L$</th>
<th>$SU(3)_C$</th>
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<tbody>
<tr>
<td>Fermions</td>
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<tr>
<td>$(u)_L$, $(d)_L$, $(c)_L$, $(t)_L$</td>
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<td>$(\nu_e)<em>L$, $(\nu</em>\mu)<em>L$, $(\nu</em>\tau)_L$</td>
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<tr>
<td>Spin-1 Bosons</td>
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<tr>
<td>$B_\mu$</td>
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<td>1</td>
<td>1</td>
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<tr>
<td>$W^i_\mu$, $i = 1, \ldots, 3$</td>
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<td>$A^A_\mu$, $A = 1, \ldots, 8$</td>
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<tr>
<td>Spin-0 Bosons</td>
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<td></td>
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<tr>
<td>$(\phi^+, \phi^0)$</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1.1A. $SU(3) \times SU(2) \times U(1)$ charges of the fundamental particles. Electric charge equals $Q = Y/2 + T$ where $T = \pm 1/2$ for $SU(2)_L$ doublets and $T = \pm 1, 0$ for triplets, and zero otherwise.

1.1 The Standard Model

Before we look specifically at QCD, we review the standard model. The standard model is gauge theory based on the gauge group $SU(3) \times SU(2) \times U(1)$. Spontaneous symmetry breaking in the electroweak sector yields finally the electroweak theory. The fundamental particles of the theory are the gauge particles: gluons, $W^\pm$, $Z^0$ and the photon; and the fermions: quarks and leptons. A summary of their transformation properties under the gauge group and their masses (including symmetry breaking) are shown in Tables 1.1A and B.

From Tables 1.1A and 1.1B, we see that the gluon ($g$) field $A_\mu$ and the quarks (e.g., $u$ and $d$) are the only particles that have non-trivial transformations under the color gauge group. Experimentally, fractionally electrically charged particles have not been observed, [15] so our belief is that color is confined: physical particles, like mesons and baryons, are color neutral.

This makes the testing of QCD as the theory of strong interactions a more subtle task than, for example, the electroweak theory. There are several arguments of a numerical nature that favor the $SU(3)$ gauge group. First, $SU(3)$ has complex representations, so the quarks representations (3) are distinguishable from antiquark representation (3). Therefore, color singlet spin-0 mesons can be made from $qq$ pairs and spin-1/2 baryons from $qqq$ triplets. This by itself doesn't fix the gauge group to $SU(3)$, but the calculations of $\pi^0 \rightarrow 2\gamma$ and $R = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow$
\begin{table}[h]
\begin{center}
\begin{tabular}{|c|c|}
\hline
Particles & Masses [GeV] \\
\hline
\hline
Fermions & \\
\hline
\hline
u & $\sim 6 \times 10^{-3}$ \\
d & $\sim 10 \times 10^{-3}$ \\
c & $\sim 1.3$ \\
s & $\sim 200 \times 10^{-3}$ \\
t & $> \mathcal{O}(50)$ \\
b & $\sim 5$ \\
$\nu_i$, $i = e, \mu, \tau$ & 0 \\
e & $0.511 \times 10^{-3}$ \\
$\mu$ & $105 \times 10^{-3}$ \\
$\tau$ & 1.7841 \\
\hline
Bosons & \\
\hline
\hline
$\gamma$ & 0 \\
$W^\pm$ & $81.0 \pm 1.3$ \\
$Z^0$ & $92.4 \pm 1.8$ \\
g & 0 \\
$H^0$ & $> 3.9$ \\
\hline
\end{tabular}
\end{center}
\caption{Particle masses in the spontaneously broken theory. Electromagnetism and the color group are assumed to be unbroken. For light quark masses, see, for example, Ref. [13]. For other masses, see Ref. [14].}
\end{table}

$\mu^+ \mu^-$) distinctly favor quark triplets. To lowest order, the ratio $R$ depends on the sum of the electric charges squared times the number of replicas, that is $N_C = 3$, the number of colors: $R_0 = N_C \sum_i q_i^2$. Above the $b$ quark threshold, this gives $R_0 = 11/3$. Experimentally, the data lie very close to this value. At $\sqrt{s} = 29$ GeV, for example, the MAC collaboration [16] measures $R = 3.96 \pm 0.09$. This agrees well with the theoretical value for $R$ when higher order QCD and electroweak corrections are included. For a summary of the data and the theoretical predictions, see the Particle Data Book [14].

1.2 QCD LAGRANGIAN

Recall in QED, the massless fermion Lagrange density is

$$\mathcal{L} = -\frac{i}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\hat{D} + icA)\psi = -\frac{i}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i\hat{D} \psi ,$$

(1.1)

for $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and fermion $\psi$ with unit charge. Because of the covariant derivative $D_\mu = \partial_\mu + eA_\mu$, when $\psi \rightarrow \exp(i e(x))\psi$, if $A_\mu \rightarrow A_\mu - \partial_\mu \theta(x)$, the Lagrangian is invariant. The single continuous parameter family of transformations represents the group $U(1)$. 

In the $SU(3)$ theory, the fermions transform as a triplet. Under a gauge transformation $\psi \rightarrow \exp(ig \sum_{A=1...8} \theta^A(x) t^A) \psi$ for the parameters $\theta^A(x)$ and the generators of $SU(3)$ in the fundamental representation: $t^A$. For a covariant derivative and field strength tensor defined by

$$\mathbf{(D}_\mu)_{ab} = \partial_\mu \delta_{ab} + g \sum_A (t^A A^A_\mu)_{ab}, \quad a, b = 1 \ldots 3 \tag{1.2}$$

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + ig[\mathbf{A}_\mu, \mathbf{A}_\nu],$$

the Lagrangian with $D \rightarrow \mathbf{D}$ and $F \rightarrow \mathbf{F}$ in Eq. (1.1) is invariant under $SU(3)$ gauge transformations.

Just as in QED, a gauge fixing term is needed to define the gauge boson propagator. We work in covariant gauge: $\partial_\mu A^\mu = 0$. Furthermore, ghost terms are required to remove the unphysical polarization of the gluons in loops. (Ghosts are not needed in QED because the $U(1)$ transformation is Abelian.) A concrete illustration for the necessity of ghosts to preserve unitarity [17] is seen in a comparison of the calculation of $\overline{q}q \rightarrow gg$ matrix element squared and $\overline{q}q \rightarrow \overline{q}q$ matrix element at one loop. The Lagrange density, including a quark mass $m$, now reads

$$\mathcal{L} = -\frac{1}{4} \mathbf{F}^\mu_{\alpha\beta} \mathbf{F}^{\mu\alpha\beta} + \overline{q}(i\mathbf{D} - m)q - \frac{1}{2x}(\partial_\mu A^\mu A^A) + \partial_\beta \eta^A \partial^\beta (D^\alpha_{AB} \eta^B), \tag{1.3}$$

where the sum over $A, B$ is implicit, and $\eta$ is a complex scalar ghost field (with Fermi statistics) with a covariant derivative

$$D^\alpha_{AB} = \partial^\alpha \delta_{AB} + ig(T^C A^A_{AB}), \tag{1.4}$$

in terms of the generators in the adjoint representation. Whether in the fundamental or adjoint basis, the generators satisfy the $SU(3)$ anti-commutation relations

$$[t^A, t^B] = if^{ABC} t^C, \tag{1.5}$$

where $f^{ABC}$ are the structure constants, such that

$$(T^A)_{BC} = -if^{ABC}. \tag{1.6}$$

In multiplicatively renormalizable theories, the bare fields may be written in terms of renormalized field times a renormalization constant. Denote bare fields by subscript 0 and conventionally write, for example, $A_0 = Z^{1/2}_A$. The standard procedure is to write the bare Lagrangian in terms of renormalized fields and products of $Z$-factors, then write the $Z$ factor as, e.g., $Z_3 = (Z_3 - 1) + 1$. The counter term Lagrangian is the difference between the bare Lagrangian and the renormal-
ized Lagrangian. To compute the counter terms, a regularization prescription and renormalization scheme are required.

We choose dimensional regularization to regulate the ultraviolet infinities [18]. An integral

\[ \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4}, \]  

is log divergent in the ultraviolet, however

\[ \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^4}, \]  

is finite for \( n < 4 \). For convenience, we pick \( n = 4 - 2\epsilon \). The logarithmic divergences appear as poles in \( \epsilon \). Dimensional regularization is a convenient scheme because it preserves the identities associated with gauge invariance.

In dimensional regularization, the fields and the coupling constant in the regulated theory pick up fractional dimensions. Recall that the action should be dimensionless, and \( d^4x \rightarrow d^n x \). Then the regulated Lagrange density \( \mathcal{L}_R \) must have dimensions of mass \([M]^n\). From the kinetic terms, we see that \( q \sim [M]^{(n-1)/2} \) and \( A \sim [M]^{(n-2)/2} \). The gauge coupling to the fermions gives \( g \sim [M]^{2-n/2} = [M]^\epsilon \), so the coupling constant cannot be dimensionless in \( n \neq 4 \) dimensions. It is conventional to rewrite the dimensionful coupling constant in terms of a *dimensionless* coupling constant \( g \) times a new (arbitrary) constant \( \mu^\epsilon \), where \( \mu \) has dimensions of mass,

\[ g_0 = g \mu^\epsilon Z_g. \]  

The bare and renormalized quantities are related by

\[ A_0 = Z_3^{1/2} A, \quad m_0 = Z_m m, \]

\[ q_0 = Z_2^{1/2} q, \quad \lambda_0 = Z_3 \lambda, \]

\[ \eta_0 = Z_3^{1/2} \eta, \quad g_0 = Z_g g \mu^\epsilon. \]  

Now the bare Lagrangian appears as

\[ \mathcal{L}_0(A_0, q_0, \eta_0, m_0, g_0, \lambda_0) = \mathcal{L}(A, q, \eta, m, g \mu^\epsilon, \lambda) + \partial \mathcal{L}(A, q, \eta, m, g \mu^\epsilon, \lambda), \]  

(1.11)
with the counter term Lagrangian

\[ \partial L = -\frac{1}{4}(Z_3 - 1)(\partial_\alpha A_\beta - \partial_\beta A_\alpha)^2 + i(Z_2^F - 1)\bar{q}\gamma_\mu q - (Z_2^F Z_m - 1)\bar{q}m q \]

\[ + (\tilde{Z}_3 - 1)\partial_\alpha \eta^{At} \partial^{\alpha A} + \frac{g\mu_\epsilon}{2}(Z_1 - 1)\epsilon^{ABC} A^B_\alpha A^C_\beta (\partial^{\alpha A}\beta A - \partial^{\alpha A}\beta A) \]

\[ - \frac{(g\mu_\epsilon)^2}{4}(Z_3 - 1)\epsilon^{ABC} \epsilon^{ADE} A^B_\alpha A^C_\beta A^D\alpha A^E - g\mu_\epsilon(Z_1^F - 1)\bar{q}\gamma^C A^C q \]

\[ + ig\mu_\epsilon(\tilde{Z}_1 - 1)\partial_\alpha \eta^{Bt}(T^C A^C_\alpha)_{BA} \eta^A \]  

(1.12)

The Z's not defined in Eq. (1.10) are products of Z's appearing together in the counter terms. Since \( g_0 = g \mu^{C} Z_q \) is the same coupling constant for all gauge couplings, the Z's are related by

\[ \frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_1^F}{Z_2} = \frac{Z_4}{Z_1} \]  

(1.13)

This is the QCD generalization of the QED Ward identity \( Z_1 = Z_2 \). The Z's are computed order by order in perturbation theory. The minimal subtraction prescription is to absorb into \( (Z - 1) \) only the terms containing poles in \( \epsilon \). We proceed in the next section to compute the fermion self energy and therefore \( Z_2^F \) to one loop order. To do this, we use the Feynman rules in Table 1.2 and some of the relations found in the Appendix.

1.3 Self-Energy Diagram

In this section we calculate \( Z_2^F \). See Marciano [19] for a more complete discussion. We compute the Feynman graph in Fig. 1.2 in \( n = 4 - 2\epsilon \) dimensions, with \( m = 0 \), but taking the fermions slightly off shell in the spacelike region. We do the latter to avoid mixing the ultraviolet singularities (which go into the counter term) with the infrared singularities.

The dressed fermion propagator may be written as

\[ iS_f = \frac{i}{\not{p} - m - \Sigma(p)} = \frac{iZ_2^F}{\not{p} - m} = \frac{i}{\not{p} - m} + \frac{i}{\not{p} - m}(-i\Sigma(p))\frac{i}{\not{p} - m} + \cdots \]  

(1.14)
Table 1.2

\[ \delta^{AB} \left[ -g^{\alpha \beta} + (1 - \lambda) \frac{p^\alpha p^\beta}{p^2 + i\epsilon} \right] \frac{i}{p^2 + i\epsilon} \]

\[ \delta^{ab} \frac{i}{p - m + i\epsilon} \]

\[ \delta^{AB} \frac{i}{p^2 + i\epsilon} \]

\[ -ig(t^A)_{\alpha \gamma}^{\alpha} \]

\[ -gf^{ABC} q^\alpha \]

\[ -gf^{ABC} [g^{\alpha \beta} (p - q)^\gamma + g^{\beta \gamma} (q - r)^\alpha + g^{\gamma \alpha} (r - p)^\beta] \]

\[ -ig^2 (f^{XAC} f^{XBD} (g_{\alpha \beta} g_{\gamma \delta} - g_{\alpha \delta} g_{\beta \gamma}) + f^{XAD} f^{XBC} (g_{\alpha \beta} g_{\gamma \delta} - g_{\alpha \gamma} g_{\beta \delta}) + f^{XAB} f^{XBD} (g_{\alpha \gamma} g_{\beta \delta} - g_{\alpha \delta} g_{\beta \gamma})) \]

QCD Feynman rules, with \( A, B, C, D = 1...8 \) and \( a, b, c = 1...3 \). In the three-gluon vertex, all of the momenta are in-going.
\[
-i\Sigma(p) = (-i g \mu')^2 \sum_{A,b} t^A_{ab} t^A_{bc} \int \frac{d^n k}{(2\pi)^n} \times \\
\left( \gamma^\alpha \frac{i(p-k)}{(p-k)^2 + i\eta} \frac{i}{k^2 + i\eta} \right) \left( -g^{\alpha\beta} + (1 - \lambda) \frac{k^\alpha k^\beta}{k^2 + i\eta} \right).
\]

First, we evaluate the color sum using the identity [20]

\[
\sum_A t^A_{ab} t^A_{cd} = \frac{1}{2} \left( \delta_{ad} \delta_{bc} - \frac{1}{N} \delta_{ab} \delta_{cd} \right),
\]

which here gives an overall factor of \( C_F \delta_{ac} = (N_C^2 - 1)/(2N_C) \delta_{ac} \). To simplify the discussion, we split up the integrand into two pieces so that

\[
-i\Sigma(p) = (g \mu')^2 C_F \delta_{ac} (\Sigma'_1 + \Sigma'_2)
\]

where

\[
\begin{align*}
\Sigma'_1 &= \int \frac{d^n k}{(2\pi)^n} (-g^{\alpha\beta}) \gamma^\alpha \frac{i(p-k)}{(p-k)^2 + i\eta} \frac{1}{k^2 + i\eta}, \\
\Sigma'_2 &= \int \frac{d^n k}{(2\pi)^n} (1 - \lambda) \frac{k}{(p-k)^2 + i\eta} \frac{1}{k^2 + i\eta}.
\end{align*}
\]

We focus our attention on \( \Sigma'_1 \) and merely will quote the result for \( \Sigma'_2 \). The procedure is the following. First, using the gamma matrix identities of the Appendix,
evaluate the contraction of the gamma matrices, then Feynman parameterize the denominator

$$\Sigma_1 = \int \frac{d^nk}{(2\pi)^n} \int_0^1 dz \frac{(n-2)\gamma - k}{k^2 + z((p - k)^2 - k^2) + i\eta^2}. \quad (1.19)$$

Because the integrals are finite for nonzero $\epsilon$, we may shift the momentum $k = Q + z\mu$ and change the order of integration to get

$$\Sigma_1' = \int_0^1 dz \int \frac{d^n l}{(2\pi)^n} \frac{(n-2)\gamma (1 - z) - Q}{[Q^2 - C + i\eta]^2}; \quad (1.20)$$

$$C = z(1 - z)(-\mu^2).$$

Now integrate over $Q$ (again, see the Appendix), using also the fact that terms odd in $Q$ vanish, to get

$$\Sigma_1' = (n - 2) \frac{i}{(16\pi^2)^{n/4}} \frac{\Gamma(2 - n/2)}{\Gamma(2)} \int_0^1 dz \psi(1 - z) C^{n/2 - 2}$$

$$= (2 - 2\epsilon) \frac{i}{(16\pi^2)} \left(\frac{4\pi}{-\mu^2}\right)^\epsilon \frac{\Gamma(\epsilon) \Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \int_0^1 dz (1 - z)^{1 - \epsilon} z^{-\epsilon}. \quad (1.21)$$

Now integrate over the Feynman parameter $z$ to get the result in terms of $\Gamma$-functions (using the identity that $\Gamma(x + 1) = x\Gamma(x)$)

$$\Sigma_1' = \frac{i}{16\pi^2} \psi \left(\frac{4\pi}{-\mu^2}\right)^\epsilon \frac{\Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \left[ \frac{1}{\epsilon (1 - 2\epsilon)} \right]. \quad (1.22)$$

The procedure for $\Sigma_2'$ is the same, and after the final integration, we get

$$\Sigma_2' = (1 - \lambda) \frac{i}{16\pi^2} \psi \left(\frac{4\pi}{-\mu^2}\right)^\epsilon \frac{\Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \left[ \frac{1}{\epsilon (1 - 2\epsilon)} \right]. \quad (1.23)$$

This gives finally for the unrenormalized self energy

$$-i\Sigma(p) = \frac{ig^2}{16\pi^2} \psi C_F \delta_{ac} \left(\frac{4\pi\mu^2}{-\mu^2}\right)^\epsilon \frac{\Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \left[ \frac{1}{\epsilon (1 - 2\epsilon)} \right] \lambda. \quad (1.24)$$

At this stage we are free to specify our renormalization scheme. One convention is the minimal subtraction scheme (MS) where only the poles in $1/\epsilon$ are absorbed into the counter term, however, one may choose to include finite pieces as well. In a calculation to all orders, a physical quantity cannot depend on the renormalization scheme choice. In a finite order calculation, the discrepancy between theoretical predictions using two different renormalization schemes is of next higher order [21].
\[ Z_3 = 1 + \left(\frac{g^2}{16\pi^2}\right)(1/\epsilon) \left[ N_C \left( \frac{13}{6} - \frac{\lambda}{2} \right) - 4n_f T_R/3 \right] \]

\[ Z_1 = 1 + \left(\frac{g^2}{16\pi^2}\right)(1/\epsilon) \left[ N_C \left( \frac{17}{12} - 3\frac{\lambda}{4} \right) - 4n_f T_R/3 \right] \]

\[ Z_4 = 1 + \left(\frac{g^2}{16\pi^2}\right)(1/\epsilon) \left[ N_C \left( \frac{2}{3} - \lambda \right) - 4n_f T_R/3 \right] \]

\[ Z_3' = 1 + \left(\frac{g^2}{16\pi^2}\right)(1/\epsilon) \left[ N_C \left( \frac{3}{4} - \lambda/4 \right) \right] \]

\[ Z_1' = 1 - \left(\frac{g^2}{16\pi^2}\right)(1/\epsilon) \left[ N_C \lambda/2 \right] \]

\[ Z_2^F = 1 - \left(\frac{g^2}{16\pi^2}\right)(1/\epsilon) \left[ C_F \lambda \right] \]

\[ Z_1^F = 1 - \left(\frac{g^2}{16\pi^2}\right)(1/\epsilon) \left[ N_C \left( \frac{3}{4} + \lambda/4 \right) + C_F \lambda \right] \]

\[ Z_4 = 1 - \left(\frac{g^2}{16\pi^2}\right)(1/\epsilon) \left[ N_C \frac{11}{6} - 2n_f T_R/3 \right] \]

Table 1.3. Renormalization constants at one loop in the MS renormalization scheme. Here \( N_C = 3, C_F = (N_C^2 - 1)/(2N_C) = 4/3 \) and \( T_R = 1/2 \).

Here we choose to use the MS renormalization scheme. We absorb only the \( 1/\epsilon \) times its \( \epsilon \)-independent coefficient into \( Z_2^F \), so

\[ Z_2^F (\text{MS}) = 1 - \frac{g^2}{16\pi^2} C_F \frac{\lambda}{\epsilon} . \]  

(1.25)

We could instead choose the modified minimal subtraction scheme \( \overline{\text{MS}} \), where constants independent of \( \epsilon \) are also subtracted. In the \( \overline{\text{MS}} \) scheme, \( 1/\epsilon \) (times its coefficient) is absorbed into the counter term where

\[ \frac{1}{\epsilon} = \frac{1}{\epsilon} + \log 4\pi - \gamma . \]  

(1.26)

These constants come from the often occurring combination (as in the self energy) of

\[ \frac{1}{\epsilon} (4\pi)^\epsilon \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} = \frac{1}{\epsilon} + \log 4\pi - \gamma + O(\epsilon) . \]  

(1.27)

For \( \lambda = 1 \) and \( \Sigma(p) = p\delta_{ac}\Sigma(p^2) \), we find for the renormalized self energies in the
two schemes:

\[
\Sigma_{\text{MS}}(p^2) = -\frac{g^2 C_F}{16\pi^2} \left( \log 4\pi - \gamma + \log \left( \frac{\mu^2}{-p^2} \right) \right);
\]

(1.28)

\[
\Sigma_{\text{MS}}(p^2) = -\frac{g^2 C_F}{16\pi^2} \log \left( \frac{\mu^2}{-p^2} \right).
\]

It is often convenient to compute loop corrections in the Landau gauge where \( \lambda = 0 \), because \( \Sigma(p^2) \) vanishes at order \( \alpha_s = g^2/4\pi \).

The one loop results for the renormalization constants in the MS scheme are shown in Table 1.3. One can verify that the relations between the \( Z \)'s in Eq. (1.13) are satisfied.

2. The running coupling and renormalization group

We have seen that dimensional regularization necessitates the introduction of an arbitrary mass scale which we have called \( \mu \). Physical quantities cannot depend on the value of \( \mu \), and this invariance generates the renormalization group equations.

Our example of a perturbative series in \( \alpha \) is the quantity \( R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-) \). This measurable quantity can be computed theoretically as an expansion in \( \alpha = g^2/4\pi \) and it equals

\[
R = 3 \sum_i g_i^2 (1 + c_1 \alpha + c_2 \alpha^2 + \cdots),
\]

(2.1)

where \( c_1, c_2, \text{etc.} \) depend on \( \mu, \alpha \) and \( s \), the center of mass energy squared. The results for \( c_1 \) appear in Ref. [22], for \( c_2 \), in Ref. [23], and for \( c_3 \) in Ref. [24]. For a fixed bare coupling constant and fixed \( \epsilon \), \( g \) depends on \( \mu \) by \( g\mu^\epsilon Z_g = g'\mu'^\epsilon Z'_g \), so we are interested to know how the explicit \( R \) dependence on \( \mu \) is compensated by the implicit dependence of \( g \) (or \( \alpha \)) on \( \mu \). Since \( R \) is dimensionless, it can depend only on \( t = \log s/\mu^2 \) and \( \alpha: R(t, \alpha) \). The independence of \( R \) from \( \mu \) implies

\[
\frac{dR}{d\log \mu^2} = \left( -\frac{\partial}{\partial t} + \frac{\partial \alpha}{\partial \log \mu^2} \right) R(t, \alpha) = 0.
\]

(2.2)

By introducing the running coupling constant \( \alpha(s) \) such that, for \( \alpha = \alpha(\mu^2) \),

\[
t = \int_{\alpha}^{\alpha(s)} \frac{dx}{\beta(x)},
\]

(2.3)

it is possible to show that \( R(t, \alpha) = R(0, \alpha(s)) \) is a solution to Eq. (2.2). So in computing \( R \), computed as a perturbative series in \( \alpha \), the replacement of \( t \to 0 \) and \( \alpha \to \alpha(s) \) satisfies the renormalization group equation. There is a generalization to
account for wave function renormalization factor dependence on $\mu$, but before we consider this more complicated case, let us first look generally at the $\beta$ function itself.

The $\beta$ function, defined to be

$$\beta(\alpha) \equiv \frac{\partial \alpha}{\partial \log \mu^2} = \frac{g}{4\pi} \frac{\partial}{\partial \mu} \frac{g}{\mu} \ ,$$

(2.4)

is for fixed $g_0$ and $\epsilon$, where $g_0 = g \mu^\epsilon Z_{\gamma}$. The corrections to $g$ that depend on $\mu$ are of order $g^3$, so the first term in the expansion of $\beta$ in terms of $\alpha$ is at second order in $\alpha$:

$$\beta(\alpha) = -b \alpha^2 (1 + b' \alpha + \cdots) \ .$$

(2.5)

Gross and Wilczek [25] and Politzer [26] computed $b$ and found it positive. Therefore, if one begins with a sufficiently small value of $\alpha$, then for a process characterized by an energy scale $Q$, $\alpha(Q^2)$ is a monotonically decreasing function of $Q^2$ and perturbation theory gets better and better.

Substituting the expansion of $\beta$ into Eq. (2.3), we get

$$bt = \frac{1}{\alpha(t)} - \frac{1}{\alpha} + b' \log \left( \frac{\alpha(t)}{\alpha} \right) + \mathcal{O}(\alpha) \ ,$$

(2.6)

where $t = \log \frac{Q^2}{\mu^2}$. Here we note that $\alpha(t = 0) = \alpha(Q^2 = \mu^2) = \alpha$. To first approximation, we can set $b'$ and subsequent coefficients to zero and solve for $\alpha(t)$ to get

$$\alpha^{(1)}(t) = \frac{\alpha}{1 + abt} = \frac{1}{b \log(Q^2/L_{\!LL}^2)} \ ,$$

(2.7)

where we have chosen to define

$$\log \left( \frac{\mu^2}{L_{\!LL}^2} \right) \equiv \frac{1}{b \alpha} \ .$$

(2.8)

This is called the leading log approximation because the expansion in $\alpha$ of $\alpha(t)$ is a series $\alpha(1 + \sum_i c_i(\alpha t)^i)$, that is, there are as many powers of $t$ as there are of $\alpha$.

The next approximation is to include $b'$

$$\alpha^{(2)}(t) = \left[ b \left( \frac{1}{ab} + t - \frac{b'}{b} \log(\alpha(t)/\alpha) \right) \right]^{-1}$$

$$\simeq \alpha_0(t) \left( 1 - \frac{b'}{b} \alpha_0(t) \log(\log(Q^2/L^2)) + \mathcal{O}(\alpha^3) \right) ,$$

(2.9)
where now our conventional definition of $\Lambda$ is through

$$\log \left( \frac{\mu^2}{\Lambda^2} \right) \equiv \frac{1}{b_0} + \frac{b'}{b} \log b \alpha \quad (2.10)$$

and $\alpha_0(t)$ is defined

$$\alpha_0(t) = \frac{1}{b \log(Q^2/\Lambda^2)} \quad (2.11)$$

that is, $\alpha_0(t)$ has the same form as the leading log expression, but a different value of $\Lambda$. Notice that the form of Eq. (2.9) is such that not only the leading logs are retained in the expansion, but also subleading logs.

Before continuing on to a sketch of the derivation of the value of $b$, we make a few comments on $\Lambda$ and its sensitivity to various choices. For a more complete discussion and general review, see Duke and Roberts [27]. The first comment is that with the introduction of $\Lambda$, we have traded the fundamental parameter $\alpha = \alpha(t = 0)$ for a different fundamental parameter $\Lambda$. Unfortunately, $\Lambda$ is sensitive to various choices of schemes. We begin with the observation that $\Lambda_{LL} \neq \Lambda$, in fact, $\Lambda^2 = \Lambda_{LL}^2 \exp(-b'/b \log b \alpha)$. Furthermore, a shift in $\Lambda_{LL}$ is comparable to terms that have been neglected, so calculations beyond the leading log are essential for pinning down the parameters of the theory.

Next, $\Lambda$ is renormalization scheme dependent. In the $\overline{\text{MS}}$ and $\text{MS}$ respectively, the renormalization constants are

$$Z_{g}^\text{MS} = 1 - \frac{1}{\epsilon} \frac{b}{2};$$
$$Z_{g}^\overline{\text{MS}} = 1 - \alpha \left( \frac{\log 4\pi - \gamma}{\epsilon} \right) \frac{b}{2} \quad (2.12)$$

Since the bare coupling constant and $\epsilon$ are fixed, the renormalized $\alpha$'s computed for the two schemes differ:

$$\alpha_{\overline{\text{MS}}} = \alpha_{\text{MS}}(1 + \alpha_{\text{MS}}b(\log 4\pi - \gamma)) \quad (2.13)$$

including only the lowest order correction. From the definition of, for example, $\Lambda_{LL}$ in Eq. (2.8),

$$\Lambda_{LL}^\text{MS} = \Lambda_{LL}^\overline{\text{MS}} e^{(\log 4\pi - \gamma)/2} = 2.66\Lambda_{LL}^\text{MS} \quad (2.14)$$

Not an insignificant factor!

Finally, we shall see that $b = b(n_f)$, so regardless of which choice for $\Lambda$, there is an $n_f$ dependence to $\Lambda$. Conventionally what is quoted is the 4-flavor value. As an
example, note that

$$\Lambda_{\text{MS}}^{(4)} = 260 \text{ MeV} \Rightarrow \Lambda_{\text{MS}}^{(5)} = 175 \text{ MeV} ;$$

$$\Lambda_{\text{MS}}^{(4)} = 160 \text{ MeV} \Rightarrow \Lambda_{\text{MS}}^{(5)} = 100 \text{ MeV} .$$ (2.15)

Values for $\Lambda_{\text{MS}}^{(4)}$ with $\alpha(Q^2)$ defined in Eq. (2.9) are measured in a variety of $e^+e^-$, hadron-hadron and deep inelastic scattering experiments. The particle data book central value [14] and range from a combination of experiments is $\Lambda_{\text{MS}}^{(4)} \approx 200^{+150}_{-80}$ MeV, whereas from deep inelastic scattering alone, a value of $\Lambda_{\text{MS}}^{(4)} \approx 238 \pm 43$ MeV is quoted.

2.1 THE $\beta$ FUNCTION

For the computation of the $\beta$ function, we follow Altarelli [8]. Consider the quark-gluon vertex correction in Fig. 2.1. According to the renormalization constants defined in Eqs. (1.7) and (1.12), the unrenormalized vertex, through one loop, is

$$\Gamma_{U}^{qqG} = g \left[ 1 + \alpha \left( \frac{\mu^2}{-P^2} \right) \left( \frac{B^{qqG}}{\epsilon} + \Lambda^{qqG} + \mathcal{O}(\epsilon) \right) \right]$$

$$= Z_2^FZ_3^{1/2} \Gamma_{R}^{qqG} ;$$

(2.16)
where

$$\Gamma_{qqG} = g \left[ 1 + \alpha \left( B_{qqG} \log \left( \frac{\mu^2}{-p^2} \right) + A_{qqG} \right) \right], \quad (2.17)$$

and $A_{qqG}$, $B_{qqG}$ are constants independent of $\mu$. To compute the $\beta$ function, we solve the renormalization group equation for $\Gamma_{qqG}$. Because there are external quark and gluon lines, the generalization of Eq. (2.2) is

$$\frac{d}{d \log \mu^2} \Gamma_{qqG} = Z_F^2 Z_3^{1/2} \left[ \frac{\partial}{\partial \log \mu^2} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \frac{d \log Z_F^2}{d \log \mu^2} + \frac{1}{2} \frac{d \log Z_3}{d \log \mu^2} \right] \Gamma_{qqG} = 0. \quad (2.18)$$

Define the anomalous dimensions $\gamma_q$ and $\gamma_G$ by

$$\gamma_q = \frac{d \log Z_F^2}{d \log \mu^2} \equiv \gamma_q^{(1)} + \cdots$$

$$\gamma_G = \frac{d \log Z_3}{d \log \mu^2} \equiv \gamma_G^{(1)} + \cdots \quad (2.19)$$

Then, combining Eqs. (2.17–19), we get that the lowest order contribution is

$$g\alpha B_{qqG} + \beta(\alpha) \frac{2\pi}{g} + g\alpha \gamma_q^{(1)} + \frac{1}{2} g\alpha \gamma_G^{(1)} = 0. \quad (2.20)$$

The first term in the expansion of $\beta$ [Eq. (2.5)] is

$$b = 2 \left( B_{qqG} + \gamma_q^{(1)} + \frac{1}{2} \gamma_G^{(1)} \right). \quad (2.21)$$

To compute the anomalous dimensions, we may use the quark self energy and gluon polarization functions. $\Gamma_{qR}^{(2)} \sim (1 - \Sigma_R(p^2))$ satisfies the renormalization group equation

$$\left[ \frac{\partial}{\partial \log \mu^2} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma_q \right] \Gamma_{qR}^{(2)} = 0. \quad (2.22)$$

The expansion of $\beta$ begins at order $\alpha^2$, however $\Sigma_R$ begins at order $\alpha$, so $\beta$ drops out of Eq. (2.22) in lowest order. From our result in Eq. (1.28), we get

$$\gamma_q^{(1)} = -\frac{1}{4\pi} C_F. \quad (2.23)$$
A similar computation of $\Gamma_{GR}^{(2)}$ yields

\[
\gamma_{G}^{(1)} = \frac{1}{4\pi} \left[ N_C \frac{5}{3} - \frac{4}{3} n_f T_R \right].
\]  

(2.24)

Now it remains to compute $B_{qqG}$. We will not do the computation here, rather we compute the color factors for Figs. 2.1b and 2.1c. Labeling the incoming gluon charge by $A$ and external quark colors by $a$ and $b$, the diagram in 2.1b yields a color factor of

\[
C_{1b} = \sum_{B,C,d} t_{ac}^B t_{cd}^B t_{db}^B = \frac{1}{2N_C} t_{ab}^A
\]

\[
= \frac{1}{2} t_{ab}^A \left[ \frac{N_C^2 - 1}{N_C} - N_C \right] = t_{ab}^A (C_F - N_C/2).
\]

(2.25)

We have used Eq. (1.16) to simplify the color sum. Similarly,

\[
C_{1c} = i \sum_{B,C,c} f_{ABC} t_{ac}^B t_{cb}^C = \sum_{B,C,c} [t_{ab}^A, t_{bc}^B] t_{ac}^B
\]

\[
= \frac{N_C}{2} t_{ab}^A.
\]

(2.26)

The standard result, including all factors, is

\[
B_{qqG} = \frac{1}{4\pi} \left[ \left( C_F - \frac{N_C}{2} \right) + \frac{3N_C}{2} \right].
\]

(2.27)

Consequently,

\[
b = \frac{1}{12\pi} \left( 11N_C - 4n_f T_R \right).
\]

(2.28)

For another more straightforward derivation of $b$, see Ellis’ TASI Lectures [12].

For reference, the expression for $b'$ is [28]

\[
b' = \frac{1}{2\pi} \frac{(17N_C - 10n_f T_R)N_C - 6n_f T_R C_F}{11N_C - 4n_f T_R}
\]

(2.29)

Shown in Fig. 2.2 is $\alpha$ as a function of $Q^2$ from the definition of $\alpha$ in Eq. (2.9).
FIGURE 2.2. The strong coupling constant as a function of $Q^2$, following Eq. (2.9).

3. Parton Distribution Functions

In principle, the QCD theory of strong interactions explains how quarks and gluons are bound together into hadrons. In practice, because of difficulties calculating outside of the perturbative regime, calculations of the baryon spectrum, for example, are far from conclusive. Nevertheless, it is an experimental fact that hadrons appear to be composed of constituent particles, generically called partons. We begin this section with a consideration of these partons without specifying the fields that bind them into hadrons. This motivates the introduction of parton distribution functions. After describing some successes, then failures, of the naive parton model, we describe the QCD-improved parton model and comment on its region of applicability. Finally, we sketch the general procedure for extracting distribution functions from deep-inelastic scattering data. For introductions to the parton model in deep inelastic scattering, see, for example, Refs. [5] and [6]. We follow here the presentation of Close [5].

3.1 THE NAIVE QUARK MODEL IN DEEP INELASTIC SCATTERING

To motivate the picture of the proton as comprised of constituent partons, point particles each containing a fraction of the parent proton’s momentum, we will look at the example of electron scattering. First we compute the differential scattering cross section for electron-muon scattering and note the scaling behavior. Then we compare with the general form for inelastic electron-proton scattering in terms of form factors. Early $ep$ experiments exhibited so-called Bjorken-scaling of the form factors consistent with spin-$1/2$ parton constituents, so we write the form factors in terms of parton distribution functions $f_i(x)$, the weighting of the electron-parton...
Figure 3.1. (a) Electron-muon scattering, and (b) electron-proton inelastic scattering.

matrix element squared for a parton of type $i$ with momentum fraction $x$ of the proton momentum.

In Fig. 3.1a and 3.1b, we show the Feynman graphs for $e\mu$ and $ep$ scattering, the only difference being that in $ep$ scattering, the final state is unspecified and may have several particles in addition to the electron (represented by the bold arrow), whereas in the $e\mu$ case, it is simply an elastic scattering. We will work in the lab frame where the proton is stationary and make the definitions

$$l = (E, 0, 0, E') ,$$
$$l' = (E', E' \sin \theta, 0, E' \cos \theta) ,$$
$$P = (M, 0, 0, 0) ,$$
$$q = l - l' .$$

where in Fig. 3.1a, $M = m_\mu$ and in Fig. 3.1b, $M = m_p$. With this, we define the variables

$$s = (l + P)^2 ,$$
$$Q^2 = -q^2 = 2EE'(1 - \cos \theta) ,$$
$$\nu = \frac{q \cdot P}{M} = E - E' ,$$
$$y = \frac{\nu}{E} .$$

(3.1)

(3.2)
Then, for $e\mu$ scattering, the differential cross section in terms of the spin averaged matrix element squared $L_{\mu\nu}^{(e)} L_{\mu\nu}^{(\mu)}$ is

$$\frac{d^2\sigma}{dE'd\Omega_e} = \frac{\alpha^2}{Q^4} \left( \frac{E'}{E} \right) L_{\mu\nu}^{(e)} L_{\mu\nu}^{(\mu)} \frac{1}{2M} \frac{d^3p_4}{2E_4} \delta^4(q + p - p_4)$$

$$= \frac{\alpha^2}{Q^4} \left( \frac{E'}{E} \right) L_{\mu\nu}^{(e)} W_{\mu\nu}^{(\mu)} . \tag{3.3}$$

This works out to be simply

$$\frac{d^2\sigma^{e\mu}}{dE'd\Omega_e} = \frac{4\alpha^2 E'^2}{Q^4} \left( \cos^2 \theta + \frac{Q^2}{2M^2} \sin^2 \theta \right) \delta \left( \nu - \frac{Q^2}{2M} \right) . \tag{3.4}$$

For inelastic scattering, we are interested in a general form for $W_{\mu\nu}$. Since the final hadronic momenta are integrated, the hadronic matrix element squared can depend only on the momenta $q$ and $P$. By writing the most general symmetric two indexed tensor using momenta $q$ and $P$, then applying electromagnetic gauge invariance ($q\mu W_{\mu\nu} = 0$), we can reduce $W$ to the form

$$W_{\mu\nu} = W_1 \left( -g_{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{W_2}{M^2} \left( P - P \cdot q \right) \mu \left( P - P \cdot q \right) \nu . \tag{3.5}$$

We write only the terms symmetric in the indices because the electron matrix element squared is symmetric. In general, $W_i$ may depend on $q^2$ and $P \cdot q$ (in other words, on $\nu$ and $Q^2$). It is a simple algebraic exercise to show that the inelastic electron proton cross section is

$$\frac{d^2\sigma^{e\nu}}{dE'd\Omega_e} = \frac{4\alpha^2 E'^2}{Q^4} \left( W_2 \cos^2 \theta + 2W_1 \sin^2 \theta \right) . \tag{3.6}$$

Then, if the proton were a structureless point particle like the muon, we would have

$$W_2^{pt}(\nu, Q^2) = \delta \left( \nu - \frac{Q^2}{2M} \right) , \tag{3.7}$$

$$2W_1^{pt}(\nu, Q^2) = \frac{Q^2}{2M^2} \delta \left( \nu - \frac{Q^2}{2M} \right) .$$

The combinations $\nu W_2^{pt} \equiv F_2^{pt}$ and $M W_1^{pt} \equiv F_1^{pt}$ depend only on the dimensionless quantity $Q^2/(2M \nu) = Q^2/(2p \cdot q) \equiv x$ and $F_2^{pt}(x) = 2x F_1^{pt}(x)$, independent of scale. From elastic proton scattering, we know, of course, that $W_1 \neq W_1^{pt}$, and it could hardly be the case for inelastic scattering. However, the early SLAC-MIT inelastic proton scattering data [31] do show that $F_2$ is approximately independent of $Q^2$. 

---

The text above is a mathematical discussion on the differential cross section in terms of spin-averaged matrix elements, focusing on elastic and inelastic scattering processes. It involves the use of differential cross sections, matrix elements squared, and inelastic scattering tensors to derive expressions for cross sections and their dependencies on various variables such as energy, momentum, and spin. The text also touches on the independence of certain quantities in elastic scattering and their implications for inelastic processes.
for $Q^2 \approx 1 - 10$ GeV$^2$. Furthermore, experimentally [32] for $Q^2$ between 5 and 15 GeV$^2$, and $x > 0.4$, $F_2(x) = 2xF_1(x)$. The relationship between the $F$'s is called the Callan-Gross relation, and the structure function independence of $Q^2$ is known as Bjorken scaling.

On the basis of the scaling behavior of the form factors, one is led to hypothesize that the hadronic matrix element squared may be represented by an incoherent sum of spin-1/2 parton matrix elements squared appropriately weighted by the parton distribution in the proton. Namely, using $f_i(x_i)$, the distribution of partons of type $i$ with momentum $x_iP$,

$$F_2(x) = \sum_i \int dx_i \ e_i^2 x_i \nu W_2^{pl}(x_i \nu, Q^2) f_i(x_i)$$

$$= \sum_i \int \frac{d\xi}{\mathcal{M}} \ e_i^2 x_i \nu \delta \left(x_i \nu - \frac{Q^2}{2M}\right) f_i(x_i)$$

$$= \sum_i e_i^2 x f_i(x) = 2xF_1(x) . \quad (3.8)$$

The distribution functions can be thought of as parton number densities. This is strictly true only in the infinite momentum frame where the hard scattering time scale is long compared to the time scale of the “measurement” of the parton densities.

For future reference, we write the inelastic $ep$ cross section in terms of $x$ and $y$

$$\frac{d^2\sigma}{dx dy} = \frac{4\pi\alpha^2 s}{Q^4} \left(F_2(x)(1-y) + F_1(x)xy^2\right) . \quad (3.9)$$

The spin-1/2 partons are identified with quarks, and we call $f_i(x)$ the quark distribution functions labeled by their flavor, e.g., $u(x)$, $d(x)$, etc. The functions $F_i$ we shall call structure functions. They can be written in terms of the distributions functions according to the naive parton model. This parton model approach is equivalent to beginning with a matrix element squared for quark-electron scattering, then integrating with the parton distribution functions to find the electron-proton cross section

$$\sigma^{ep}(q, P) = \sum_i \int dx q_i(x) \delta^{eq_i}(q, xP) \quad (3.10)$$

The quark distributions are split into valence and sea quark distributions. The valence quarks in the proton guarantee that $Q(p) = 1$ and that the strong isospin of the proton is $+1/2$. Then for isospin $+1/2$ and charge $+2/3$ up quarks $u$ and isospin $-1/2$, charge $-1/3$ down quarks $d$, we have the sum rule for the valence
quarks that
\[
\int_0^1 dx \left[ \frac{2}{3} u_V(x) - \frac{1}{3} d_V(x) \right] = 1, \tag{3.11}
\]
and similarly, there is an equation for the neutron, the isospin partner of the proton, where \( u_V^n = d_V^p \equiv d_V \) and \( d_V^n = u_V^p \equiv u_V \).

The sea quarks are just the quark-anti-quark pairs created by vacuum fluctuations. For the moment, we assume that \( u_S(x) = \bar{u}_S(x) = d_S(x) = \bar{d}_S(x) = s_S(x) = \bar{s}_S(x) \equiv q_S(x) \) including the third (strange) quark. Then we could write the sum rules above using either \( q_V \) or \( q - \bar{q} \) where \( q = q_V + q_S \) and \( \bar{q} = \bar{q}_S \).

The ratio of \( d\sigma(en)/d\sigma(ep) \) as a function of \( x \) is just \( F_2^{en}/F_2^{ep} \) according to the Callan-Gross relation, and in terms of quark distributions,
\[
\frac{F_2^{en}}{F_2^{ep}} = \frac{u_V + 4d_V + 12q_S}{d_V + 4u_V + 12q_S}. \tag{3.12}
\]

Experimentally [32], [33], one observes that at small \( x \), the ratio is nearly unity, so we believe that the sea quarks dominate at small \( x \). As \( x \) increases, the quantity in Eq. (3.12) decreases, so it appears that the valence quarks begin to have a larger role. If strong isospin symmetry were exact, then \( u_V = 2d_V \) and in the absence of sea quarks, the ratio would equal 2/3. At large \( x \), however, the ratio decreases to approximately 1/4. This can be accounted for by \( q_S \to 0 \) as \( x \to 1 \) and \( d_V \ll u_V \) at large \( x \).

One sign that the naive picture could not be the complete picture comes from considering
\[
F_2^{en} + F_2^{ep} = \frac{5}{3} x \left( u + \bar{u} + d + \bar{d} + 2 \left( s + \bar{s} \right) \right). \tag{3.13}
\]
Momentum conservation tells us that
\[
\sum_i \int_0^1 dx \ x f_i(x) = 1, \tag{3.14}
\]
so in the limit that the strange quarks can be neglected, we would expect that the integral of Eq. (3.13) over \( x \) would equal 5/9. In fact, experimentally [34], the value of the integral is \((0.45) \cdot 5/9\). At this stage, the discrepancy could in principle be accounted for by the strange sea, however implausibly, but it would have disastrous theoretical consequences for the predicted neutrino production of charm. A more satisfactory explanation is that roughly half of the proton momentum is carried not by spin-1/2 quarks, but by spin-1, electrically neutral gluons, as they appear in QCD.

One further evidence of the failure of the naive quark model is the violation of scaling in deep inelastic scattering data. The structure functions for deep inelastic
lepton-nucleon scattering, for fixed $x$, show slow variations in $Q^2$ as $Q^2$ ranges between $\sim 5 - 100 \text{ GeV}^2$ [35]. The QCD-improved parton model can account for scaling violations, and it is to this that we turn next.

### 3.2 QCD IMPROVED PARTON MODEL

The idea of the QCD-improved quark distribution functions conceptually originates with the Weizsäcker-Williams effective photon approximation in electrodynamics [36]. The idea of the effective photon approximation is that, say, electron-proton scattering can be approximated by

$$d\sigma(ep \rightarrow eX) \simeq dP_{\gamma \rightarrow e}(z) dz \, d\sigma(\gamma p \rightarrow X).$$

Here, $z$ is the ratio of the photon momentum compared to the initial electron momentum (the fraction of the original momentum carried by the photon). The quantity $dP_{\gamma \rightarrow e}(z)dz$ is the probability of finding the photon carrying fraction $z$ of the parent electron's momentum. The graph with the virtual photon is dominated by small $q^2$, so the virtual photon is almost real ($q^2 \sim 0$).

What we shall show below is that the quantity $dP_{\gamma \rightarrow e}(z)dz$ also depends on $Q$, the characteristic energy scale of the interaction. The effective photon approximation uses QED to compute the probability of an electron splitting into an electron and a photon as a function of $z$ and $Q$, and ultimately, the photon distribution function in the electron. In the QCD improved parton model, we use the theory of QCD to compute the probabilities of quarks splitting into gluons and quarks, hence corrections to the distribution functions $f_i(x)$. To simplify our discussion below, we remain in the electrodynamical case, and generalize later to the non-Abelian case.

We will show that we can define the splitting function $P_{\gamma \rightarrow e}(z)$ such that

$$dP_{\gamma \rightarrow e}(z, Q) = \frac{\alpha}{2\pi} P_{\gamma \rightarrow e}(z)d\tau$$

where $\tau = \log Q^2$. We can write $\gamma(x, \tau)$, the number density of photons with momentum fraction between $x$ and $x + dx$, so that

$$\frac{d\gamma(x, \tau)}{d\tau} = \frac{\alpha}{2\pi} \int_0^1 dy \int_0^1 dz \delta(zy - x)e(y, \tau)P_{\gamma \rightarrow e}(z)$$

where $e(y, \tau)$ is the number density of electrons. Integrating over $z$ gives the elec-
tromagnetic equivalents of the coupled Altarelli-Parisi evolution equations

\[
\frac{d\gamma(x, \tau)}{d\tau} = \frac{\alpha}{2\pi} \int_x^1 \frac{dy}{y} e(y, \tau) P_{\gamma-e} \left( \frac{x}{y} \right),
\]

\[
\frac{de(x, \tau)}{d\tau} = \frac{\alpha}{2\pi} \int_x^1 \frac{dy}{y} \left[ e(y, \tau) P_{e-e} \left( \frac{x}{y} \right) + \gamma(y, \tau) P_{e-\gamma} \left( \frac{x}{y} \right) \right].
\]

(3.18)

We can also write equivalently the positron distribution function.

First we sketch the derivation of the expression for the splitting function \( P_{\gamma-e}(z) \). Because of conservation of momentum, and conservation of probability, we can exhibit relations between splitting functions. We then outline what changes are needed to adapt to the case of QCD.

Now we look at the expression for \( dP_{\gamma-e}(z, Q)dz \) in the limit of massless particles. Altarelli and Parisi did this in the context of QCD using old-fashioned perturbation theory [29]. Another approach is to compare, using the usual Feynman rules,

\[
d\sigma(ep \to eX)
\]

\[
d\sigma(\gamma p \to X)
\]

and extract \( dP_{\gamma-e}(z, Q)dz \) from its definition in Eq. (3.15). The steps are outlined in detail in Berestetskii et al. [4]. Schematically, we can obtain the standard result using

\[
d\sigma(\gamma p \to X) = \frac{1}{2s_{\gamma p}} |V_{\gamma p \to X}|^2 d\Gamma_X,
\]

\[
d\sigma(ep \to e'X) = \frac{1}{2s_{ep}} e^2 |V_{e-e'\gamma}|^2 |V_{\gamma p \to X}|^2 \frac{1}{p_{e'}^4} \frac{d^3 p_{e'}}{(2\pi)^3 2E'} d\Gamma_X.
\]

(3.20)

This gives

\[
dP_{\gamma-e}(z)dz = \frac{s_{\gamma p}}{s_{ep}} \frac{e^2 |V_{e-e'+\gamma}|^2}{p_{\gamma}^4} \frac{d^3 p_{e'}}{(2\pi)^3 2E'}.
\]

(3.21)

The matrix element squared includes the spin average. Use the momentum assign-
\( p_\gamma = \left( [(zP)^2 + q_T^2]^{1/2}; q_T, zP \right) \)  

\( p_{e'} = \left( [(1 - z)^2 P^2 + q_T^2]^{1/2}; -q_T, (1 - z)P \right) \)

expanded to second order in \( q_T / P \) to compute the factors

\[
\frac{s_{\gamma p}}{s_{\gamma p}} = z
\]

\[
p_\gamma^4 \approx \frac{q_T^4}{(1 - z)^2}
\]

\[
\frac{d^3 p_{e'}}{2E'} \approx \frac{\pi}{2} \frac{dz}{(1 - z)dq_T^2}.
\]

We get

\[
dP_{\gamma \rightarrow e}(z)dz = \frac{\alpha}{2\pi} \frac{z(1 - z)}{2} |V_{e \rightarrow e' + \gamma}|^2 \frac{d \log q_T^2}{d z}.
\]

So, the value for the splitting function is

\[
P_{\gamma \rightarrow e}(z) = \frac{z(1 - z)}{2} \frac{|V_{e \rightarrow e' + \gamma}|^2}{q_T^2}.
\]

A few comments are in order, namely, why incorporate the powers of \( q_T \) in the denominator. The reason is that \( V \propto q_T \). This can be seen from angular momentum arguments. A spin 1/2 electron with, say, positive helicity cannot emit a spin 1 particle collinearly \( (q_T \rightarrow 0) \) with a helicity conserving vertex, so the matrix element squared should go like the square of the transverse momentum. Also, just by considering momentum conservation at the vertex for \( A \rightarrow B + C \), one knows that \( P_{B \rightarrow A}(z) = P_{C \rightarrow A}(1 - z) \) so the prefactor includes \( z(1 - z) \).

To get the splitting function \( P_{\gamma \rightarrow e} \), use the momentum assignments from Eq. (3.22) and substitute in

\[
|V_{e \rightarrow e' + \gamma}|^2 = \frac{1}{2} \sum e^\mu e'^\nu \text{tr} \hat{p}_e \gamma_\mu \hat{p}_e \gamma_\nu.
\]
sum, we must use the physical gauge

$$\sum e^{i\mu} e^{i\nu} = \begin{cases} 0 & \text{if } \mu, \nu = 0 \\ \delta^{ij} - p^i p^j / p^2 & \text{otherwise.} \end{cases}$$ (3.27)

This is because we are extracting the initial state radiation. After some arithmetic, we find,

$$|V_{e-e'\gamma}|^2 = 4q^2 \frac{1 + (1 - z)^2}{2z^2(1 - z)}. \quad (3.28)$$

As advertised, the matrix element squared is proportional to $q^2$. Now using Eq. (3.25), we determine

$$P_{\gamma\rightarrow e}(z) = \frac{1 + (1 - z)^2}{z}. \quad (3.29)$$

Momentum conservation at the vertex gives us

$$P_{e\rightarrow e}(z) = \frac{1 + z^2}{1 - z}. \quad (3.30)$$

Eq. (3.30) is valid as long as $z \neq 1$, where the expression is singular. To address the problem of $z = 1$, it is useful to define the distribution $1/(1 - z)_+$ such that

$$\int_0^1 \frac{f(z)}{(1 - z)_+} = \int_0^1 \frac{f(z) - f(1)}{(1 - z)}. \quad (3.31)$$

This makes the integral over the distribution finite. The law of conservation of fermion number is then applied to constrain $P_{e\rightarrow e}(z = 1)$

$$\int dx \left( \frac{de(x, \tau)}{d\tau} - \frac{d\bar{e}(x, \tau)}{d\tau} \right) = 0 \Rightarrow \int_0^1 dz P_{e\rightarrow e}(z) = 0 \quad (3.32)$$

To satisfy Eq. (3.32), add a delta function. It can be shown that

$$P_{e\rightarrow e}(z) = \frac{1 + z^2}{(1 - z)_+} + \frac{3}{2} \delta(1 - z) \quad (3.33)$$

satisfies Eq. (3.32).

Another conservation law regards conservation of momentum. If we consider the
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\[ P_{q\to q}(z) = C_F \left[ \frac{1 + z^2}{(1 - z)_+} + \frac{3}{2} \delta(1 - z) \right] \]

\[ P_{G\to q}(z) = C_F \left[ \frac{1 + (1 - z)^2}{z} \right] \]

\[ P_{q\to G}(z) = TR \left[ z^2 + (1 - z)^2 \right]; \quad TR = \frac{1}{2} \]

\[ P_{G\to G}(z) = 2N_C \left[ \frac{z}{(1 - z)_+} + \frac{1 - z}{z} + z(1 - z) \right] + \delta(1 - z) \frac{11N_C - 4n_fTR}{6} \]

Table 3.1. QCD splitting functions at lowest order.

QCD case (for a general treatment), then momentum conservation is expressed

\[ \int dx x \left( \sum_i \frac{d^2q^i(x, \tau)}{d\tau} + \sum_i \frac{d^2q^i(x, \tau)}{d\tau} + \frac{dG(x, \tau)}{d\tau} \right) = 0, \quad (3.34) \]

where \( i = 1 \ldots n_f \) for \( n_f \) equal to the number of quark flavors. This implies that

\[ \int_0^1 dz \int d\tau \left( P_{q\to q}(z) + P_{G\to q}(z) \right) = 0 \]

\[ \int_0^1 dz \int d\tau \left( 2n_f P_{q\to G}(z) + P_{G\to G}(z) \right) = 0. \quad (3.35) \]

At this point, let us address the question of the extension of the QED calculation to QCD. First and foremost, is that \( P_{G\to G} \neq 0 \) whereas \( P_{\gamma\to \gamma} = 0 \). Second is the fact that there are color factors. As an example, look at \( P_{q\to q} \). Eq. (3.33) will be modified by a factor of 1/3 for a color average times \((g/2)/e)^2 \sum_A \text{tr} \lambda^A \lambda^A \) leading to an overall relative factor of \( 4/3 \cdot \alpha_s/\alpha \), so replace \( \alpha \to \alpha_s \) and \( P_{q\to q}(z) = (4/3)P_{e\to e} \).

A summary of the quark and gluon splitting functions appears in Table 3.1. The coupled Altarelli-Parisi equations [the equivalents of Eq. (3.18)] are

\[ \frac{d}{d\tau} \left( \begin{array}{c} q(x, \tau) \\ G(x, \tau) \end{array} \right) = \frac{\alpha(\tau)}{2\pi} \int_x^1 \frac{dy}{y} \left( \begin{array}{cc} P_{q\to q}(\frac{z}{y}) & P_{q\to G}(\frac{z}{y}) \\ P_{G\to q}(\frac{z}{y}) & P_{G\to G}(\frac{z}{y}) \end{array} \right) \left( \begin{array}{c} q(y, \tau) \\ G(y, \tau) \end{array} \right). \quad (3.36) \]

In QED, where even at low \( Q^2 \), \( \alpha \) is small, to a good approximation, the photon distribution in the electron is

\[ \gamma(x, Q^2) = \frac{\alpha}{2\pi} \log \left( \frac{Q^2}{Q_0^2} \right) P_{e\to \gamma}(x), \quad (3.37) \]
that is, using \(e(y, \tau) = \delta(y - 1)\) and \(\bar{e}(y, \tau) = 0\), we can evaluate the integral over \(z\) and \(\tau\). We integrate \(\tau\) from a reference \(\log Q^2_0\) to \(\log Q^2\), the characteristic scale of the process.

For the case of the gluon, \(\alpha_s = \alpha_s(\tau)\) rises rapidly as \(Q^2 \to m_p^2\), so the integral is not so trivial. Furthermore, we have only the first order expression for \(P_{G-q}\) and \(\alpha_s\) is becoming large. The non-perturbative regime is taking over in this limit. For this reason, distribution functions are determined experimentally at a particular value of \(Q^2_0 > m_p^2\), then “evolved” in \(Q\) using the coupled Altarelli-Parisi equations.

3.3 HOW ARE DISTRIBUTION FUNCTIONS EXTRACTED

The idea of extracting distribution functions from experimental measurements is to first measure the structure functions \(F_i\) as functions of \(x\), \(y\) and \(Q^2\). Then, using a combination of results from other experiments and some assumptions, separate out the various components from the QCD parton model. There are a variety of parameterizations of the distribution functions [37]-[42]. I follow here essentially the discussion of EHLQ in Ref. [37]. They use the neutrino data from the CDHS experiment [43] taken with narrow band beams of 200 (\(\nu, \bar{\nu}\)) and 300 (\(\nu\)) GeV.

In deep inelastic neutrino scattering in the naive parton model, there are only two structure functions, \(F_2\) and \(F_3\), that is two each for neutrino and anti-neutrino scattering. In terms of the structure functions, the neutrino and antineutrino differential cross sections corresponding to Eq. (3.9) are, for \(F_2 = 2F_1\),

\[
\frac{d^2\sigma^\nu}{dx dy} = \frac{G^2_F M E x}{\pi} \left[ F_2^\nu (x)(1 - y) + F_1^\nu (x)y^2 + F_3^\nu y \left( 1 - \frac{y}{2} \right) \right],
\]

\[
\frac{d^2\sigma^\bar{\nu}}{dx dy} = \frac{G^2_F M E x}{\pi} \left[ F_2^\bar{\nu} (x)(1 - y) + F_1^\bar{\nu} (x)y^2 - F_3^\bar{\nu} y \left( 1 - \frac{y}{2} \right) \right].
\]

In the parton model, for neutrino and anti-neutrino scattering with isoscalar nucleons,

\[
F_2^\nu N = \sum (q + \bar{q}) \quad F_2^\bar{\nu} N = \sum (q + \bar{q})
\]

\[
F_3^\nu N = \sum (q - \bar{q}) + 2s \quad F_3^\bar{\nu} N = \sum (q - \bar{q}) - 2s
\]

where \(q = u, d, s\) in the low momentum region.

Particularly interesting at the outset is \(F_3^{\text{avg}} = (F_3^\nu + F_3^\bar{\nu})/2\) because it is just the sum of the valence distribution functions. \(F_3^{\text{avg}}\) can be extracted by taking the difference of the neutrino and antineutrino differential cross sections. A comparison of charged current interactions for hydrogen and deuterium targets yields \(d\nu/u\nu\) as a function of \(x\), and together with \(F_3^{\text{avg}}\) gives the separate up and down valence contributions.
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Turning now to the sum of the neutrino and antineutrino differential cross sections,

$$\frac{1}{2} \left[ d\sigma^{\nu N} + d\sigma^{\bar{\nu}N} \right] \sim \sum (q + \bar{q}) \left[ (1 - y) + y \right] + 2 s y \left( 1 - \frac{y}{2} \right). \quad (3.40)$$

This yields the $u$ and $d$ sea distributions given from neutrino production of charm, and the valence contributions derived above. It is assumed that $u_s(x) = d_s(x)$.

To obtain the gluon distribution function is more difficult as it cannot be done directly. Instead, we know that from the momentum sum rule, $\int_0^1 dx x [G(x) + \sum (q(x) + \bar{q}(x))] = 1$, so there is a constraint on the integrated gluon distribution. One may postulate a shape and normalization at a reference $Q^2_0$ and check for consistency at higher values of $Q^2$. At larger values of $Q^2$, because the gluon can split into a quark-antiquark pair, an incorrect gluon distribution at $Q^2_0$ will manifest itself as an incorrect theoretical prediction for $q(x, Q^2)$ and $\bar{q}(x, Q^2)$. In Fig. 3.2, it is clear that one of the main differences between various sets of distribution functions is how the gluon is parameterized.

Uncertainties arise from both experimental and theoretical effects. Of the latter, the largest uncertainties come from radiative corrections, the value of $\Lambda$ and charm quark corrections. Depending on precisely how one treats the radiative corrections, there are different parameterizations of $F_2$. In addition, the gluon distribution is intimately tied to this question because of the momentum sum rule. EHLQ present two different gluon parameterizations for two values of $\Lambda = 200, 290$ MeV. Among the experimental uncertainties, there are target effects, beam flux, energy calibration and acceptance uncertainties. For a thorough discussion, see Tung et al. [44].
4. Factorization

So far, the discussion has implicitly relied on the property of factorization. Even with higher order corrections, we write the deep inelastic cross section as the convolution of the parton cross section with the distribution functions

\[ \sigma^H(q, P) = \sum_i \int dx f_i(x, Q^2) \tilde{\sigma}_i(q, xP, \alpha_s(Q^2)). \] (4.1)

This is essentially writing the cross section as a product of the hard scattering (\(\tilde{\sigma}\)) and soft, non-perturbative (\(f_i\)) pieces. For hadron-hadron scattering, we write

\[ \sigma^{HH}(P_1, P_2) = \sum_{i,j} \int dx_1 \int dx_2 f_i(x_1, Q^2) f_j(x_2, Q^2) \tilde{\sigma}_{ij}(x_1P_1, x_2P_2, \alpha_s(Q^2)) \] (4.2)

The property of factorization tells us:

1. Infrared singularities associated with incoming parton lines can always be absorbed into the distribution functions in a consistent way.

2. The distribution functions are independent of the process. The \(f_i\)'s in eqn. 4.2 are the same as in eqn. 4.1.

For proofs of factorization, see Refs. [45]. A sketch of the main points of the proof may be found in Ref. [46]. We content ourselves here with the example of factorization in deep inelastic scattering, at order \(\alpha_s\) in the MS scheme, calculated by Altarelli et al. [47]. We follow Ref. [47] closely below, emphasizing the pole structure rather than the finite corrections themselves.

4.1 Deep Inelastic Scattering Example

Fig. 4.1 shows the graphs contributing to the \(\alpha_s\) corrections to deep inelastic scattering. They come from the interference of the two graphs in Fig. 4.1a and the absolute squares of Fig. 4.1b and Fig. 4.1c. We are interested in the inclusive \(\gamma^*p\) cross section, so we integrate over the final parton momenta.

There are several types of singularities associated with these diagrams. There are, of course, the ultraviolet singularities in the loop graphs which are cancelled by counterterms. We may choose to do the calculation in the Landau gauge where the fermion self energy corrections vanish at one loop [19]. For the real graphs, there are singularities associated with soft and collinear gluons (in the massless limit). Fig. 4.2 shows the potential danger of the collinear gluon.

Since the gluon is on shell, with \(k^2 = 0\), and the incident quark is massless, the intermediate quark momentum is \(k'^2 = (p - k)^2 = -2p_0 k(1 - \cos \theta)\), which vanishes in the \(\cos \theta \to 1\) (collinear) limit, so the intermediate quark propagator diverges. This corresponds to the \(y \to 1\) limit for \(y \equiv (1 + \cos \theta)/2\). We should note,
FIGURE 4.1. The $\alpha_s$ corrections to deep inelastic scattering.

FIGURE 4.2. Diagram for $qg^* \rightarrow qg$. 
however, that collinear gluons are already accounted for in the parton distribution functions. For example, we computed the splitting of a quark into a quark plus gluon with negligible transverse momenta. What we will find is that the singularity will multiply the quark splitting function times the lowest order cross section (evaluated at a fraction of the incident quark momentum). This is absorbed with the naive parton distribution function to yield the QCD improved distribution function.

One should remark that the measured structure functions $F_i(x, Q^2)$ are finite. In terms of the parton model, the $F_i$’s are expressed in terms of the parton distribution functions, which are calculated to have perturbative infinite contributions. Therefore, there is some kind of “physical” regularization in the proton due to bound state effects, which we cannot compute. Our procedure here is to compute in perturbation theory (using dimensional regularization to regulate all infinities) and put all of the wave function (low energy) effects in the definition of the parton distribution functions, which, in perturbation theory, include infinite pieces. We then use the measured distribution functions, appropriately evolved using the Altarelli-Parisi equations, in the numerical calculations.

Following Altarelli et al. [47] we start with the naive quark model and QCD perturbation theory. It is convenient to define $\mathcal{F}_i$ in terms of the $F_i$ in Eq. (3.9) such that

$$\mathcal{F}_1 = 2F_1 \quad \mathcal{F}_2 = \frac{F_2}{x}$$

(4.3)

so that at lowest order $\mathcal{F}_2 = \sum_i e_i^2(q_i(x) + \bar{q}_i(x))$, where $i$ sums over quark flavors. The Callan-Gross relation appears as $\mathcal{F}_1 = \mathcal{F}_2$. We rewrite $\mathcal{F}_2$ at lowest order as

$$\mathcal{F}_2 = \int_x^1 dy \sum_i e_i^2(q_i(y) + \bar{q}_i(y)) \delta \left(1 - \frac{x}{y}\right)$$

(4.4)

Our aim is to compute $\mathcal{F}_2$ through order $\alpha_s$, and we shall define $\tilde{f}_{2q}$ and $\tilde{f}_{2G}$ such that

$$\mathcal{F}_2 = \int_x^1 dy \delta(zy - x) \sum_i e_i^2 \left[(q_i(y) + \bar{q}_i(y))\tilde{f}_{2q}(z, Q^2) + G(y)\tilde{f}_{2G}(z, Q^2)\right].$$

(4.5)

To lowest order, $\tilde{f}_{2q}^{(0)} = \delta(1 - z)$ and $\tilde{f}_{2G}^{(0)} = 0$. The quark contribution is just the (appropriately normalized) matrix element squared of the lowest order graph.
To derive $\tilde{f}_{2q}$ and $\tilde{f}_{2G}$, we write the integrated matrix element squared as

$$W_{\mu\nu}^P = \left(-g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{q^2}\right) W_1^P + \left(p - \frac{q \cdot p}{q^2} q\right)_{\mu} \left(p - \frac{q \cdot p}{q^2} q\right)_{\nu} W_2^P$$  \hspace{1cm} (4.6)

in analogy with Eq. (3.5), where now $p$ is the parton momentum. Further define $z = -q^2/q \cdot p = Q^2/q \cdot p$, then $\tilde{f}_2 = (q \cdot p/z) W_2^P$ and $\tilde{f}_1 = 2W_1^P$. Note that the projections $-g_{\mu\nu}$ and $p_{\mu}p_{\nu}$ give

$$-g_{\mu\nu} W_{\mu\nu}^P = (1 - \epsilon)\tilde{f}_2 - \left(\frac{1}{2} - \epsilon\right) \left(\tilde{f}_2 - \tilde{f}_1\right),$$

$$p^\mu p^\nu W_{\mu\nu}^P = \frac{Q^2}{8z^2} \left(\tilde{f}_2 - \tilde{f}_1\right).$$ \hspace{1cm} (4.7)

At lowest order,

$$W_{\mu\nu}^{(0)p} = PS_1 \cdot \omega_{\mu\nu}^{(0)}$$

$$= PS_1 \cdot \frac{e_q^2}{2} \text{tr} \gamma_{\mu} \gamma_{\nu}. \hspace{1cm} (4.8)$$

The single body phase space equals

$$PS_1 = \frac{d^{n-1}p'}{2E'(2\pi)^{n-1}(2\pi)^n \delta^n(p + q - p')}$$

$$= \frac{2\pi}{2q \cdot p} \delta(1 - z), \hspace{1cm} (4.9)$$

for final quark momentum $p'$. First note that $p^\mu p^\nu \omega_{\mu\nu}^{(0)} = 0$, so the Callan-Gross relation is satisfied. The evaluation of $g_{\mu\nu} \omega_{\mu\nu}^{(0)}$ together with the phase space factor gives $\tilde{f}_{2q}^{(0)}$ proportional to $\delta(1 - z)$, where the prefactors go into the normalization of the cross sections.

For the next order, there are two contributions to $\mathcal{F}_2$. From $q_{\gamma^*}$ scattering we get $\tilde{f}_{2q}^{(1)}$ and from $G_{\gamma^*}$ scattering, we get $\tilde{f}_{2G}^{(1)}$. We look first at gluon emission. Here,

$$\omega_{\mu\nu q}^{(1)} = (g_{\mu}^\epsilon)^2 e_q^2 C_F$$

$$\times \text{tr} \left(\gamma_{\mu} \frac{t_1}{t} \gamma_{\alpha} + \gamma_{\nu} \gamma_{\alpha} \frac{s}{s} \gamma_{\mu}\right) \left(\gamma_{\nu} \frac{t_1}{t} \gamma_{\nu} + \gamma_{\nu} \frac{s}{s} \gamma_{\alpha}\right) \hspace{1cm} (4.10)$$

for $r_1 = p - k$, $r_2 = p + q$ and $s = r_2^2, t = r_1^2$. Two body phase space in $n$ dimensions
is
\[
\text{PS}_2 = \int \frac{d^n p'}{(2\pi)^{n-1}} \int \frac{d^n k}{(2\pi)^{n-1}} \delta(p'^2) \delta(k^2) (2\pi)^n \delta^n(p + q - k - p')
\]
\[
= \frac{1}{2} (2\pi)^{2-n} \int |k|^{n-3} d\Omega_{n-2} \delta((p + q - k)^2)
\]

(4.11)

We may choose to evaluate the phase space integral in the frame where
\[
p = (|p|, 0, \ldots, 0, |p|)
\]
\[
q = (q_0, 0, \ldots, 0, -|p|)
\]
\[
k = (|k|, \ldots, |k| \cos \theta)
\]
in which case, using \( n = 4 - 2\epsilon \) and the identities (A.6) in the Appendix, we find
\[
\int d\Omega_{n-2} \rightarrow \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \int d\theta \sin(\theta)^{1-2\epsilon}
\]
\[
= \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{2}{4^\epsilon} \int dy y^{-\epsilon} (1-y)^{-\epsilon}
\]

for \( y = (1 + \cos \theta)/2 \).

In terms of \( y \) and \( z \), \( s = Q^2(1-z)/z \) and \( t = -Q^2(1-y)/z \). First, \( p\nu \omega_{\mu\nu q}^{(1)} \) yields a term proportional to \( s + Q^2 \), so the evaluation of the phase space integral yields a finite difference for \( \bar{f}_{2q}^{(1)} \) in the \( \epsilon \rightarrow 0 \) limit. Evaluating the \( g_{\mu\nu} \) contraction gives (up to numerical factors)

\[
-g_{\mu\nu} \omega_{\mu\nu}^{(1)} \rightarrow g^\mu \mu \epsilon^2 C_F \left( -s \right)
\]
\[
\frac{t}{s} - \frac{t}{s} + 2 \frac{q^2(s + t - q^2)}{st} + O(\epsilon)
\]

(4.14)

By making the \( s \) and \( t \) substitutions in terms of \( y \) and \( z \), the pole structure is manifest. For example, \( 1/(st) = -z^2/(Q^4(1-z)(1-y)) \). The integral over \( y \) gives a factor of \( 1/\epsilon \) and at \( z = 1 \), there is a \( 1/\epsilon \) coming from the expansion

\[
\frac{z^\epsilon}{(1-z)^{1+\epsilon}} = \frac{-1}{\epsilon} \delta(1-z) + \frac{1}{(1-z)^{\epsilon}} - \epsilon \left( \log(1-z) \right) + O(\epsilon^2)
\]

(4.15)
We just quote the answer for an incoming quark [47] scattering with $\gamma^*$:

\[
j_{2q}^{(1)} \bigg|_{\text{real}} = \frac{\alpha_s}{2\pi} C_F \left( \frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \times \left[ \frac{2}{\epsilon^2} \delta(1-z) - \frac{1}{\epsilon} \left(1 + \frac{z^2}{\epsilon(1-z)} \right) + \frac{3}{2\epsilon} \delta(1-z) + \text{finite terms} \right]
\]  

(4.16)

The first term is the overlap of soft and collinear divergences, while the second comes from the collinear gluon, and the third, from a soft gluon. In terms of the splitting function $P_{qq}(z)$, (see Table 3.1) we may write

\[
j_{2q}^{(1)} \bigg|_{\text{real}} = \frac{\alpha_s}{2\pi} \left( \frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \times \left[ \frac{2C_F}{\epsilon^2} \delta(1-z) + \frac{3C_F}{\epsilon} \delta(1-z) - \frac{1}{\epsilon} P_{qq}(z) + \text{finite terms} \right].
\]  

(4.17)

The first order virtual corrections at order $\alpha_s$ are [47]

\[
j_{2q}^{(1)} \bigg|_{\text{virt}} = \frac{\alpha_s}{2\pi} \left( \frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \delta(1-z) \times \left[ -\frac{2C_F}{\epsilon^2} - \frac{3C_F}{\epsilon} + \text{finite terms} \right],
\]  

(4.18)

so

\[
j_{2q}^{(1)} = \frac{\alpha_s}{2\pi} \left( \frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[ -\frac{1}{\epsilon} P_{qq}(z) + j_{2q}^{(1)}(z) \right]
\]  

(4.19)

where $j_{2q}^{(1)}(z)$ is the sum of the finite terms in Eqs. (4.17) and (4.18).

Here we begin to see the consequences of the factorization theorem. The only singularity remaining from $\gamma^* q$ scattering is the collinear one, which multiplies a quantity independent of the details of the process (deep inelastic scattering), namely, the Altarelli-Parisi splitting function $P_{qq}$. If we had instead computed order $\alpha_s$ corrections to $q\bar{q} \rightarrow \gamma^*$, we would have again found this $1/\epsilon P_{qq}$ singularity [47]. As we shall see below, this will be absorbed into the renormalized, $Q^2$ dependent distribution function.

The $g\gamma^*$ scattering contribution to $\tilde{f}_2$ is similarly derived. The only singularities are collinear. Note that the gluon may interact via an intermediate quark or
antiquark. Consequently, there is a factor of two in the expression [47]

$$f_{2G}^{(1)}|_{\text{real}} = \frac{\alpha_s}{2\pi} \left( \frac{4\pi \mu^2}{Q^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \times$$

$$\left[ -\frac{2}{\epsilon} P_{qG}(z) + 2f_{2G}^{(1)}(z) \right].$$

(4.20)

We now write $\mathcal{F}_2$ including the first order expression

$$\mathcal{F}_2 = \int_0^1 dy \int_0^1 dz \delta(zy - x) \sum_i e_i^2 \left[ [q_i(y) + \bar{q}_i(y)] \left( \delta(1 - z) + \frac{\alpha_s}{2\pi} P_{qq}(z) \left( -\frac{1}{\epsilon} + \gamma - \log 4\pi - \log \frac{\mu^2}{Q^2} \right) + \frac{\alpha_s}{2\pi} f_{2q}^{(1)}(z) \right) \right.$$

$$+ \left. \frac{\alpha_s}{2\pi} P_{qG}(z) \left( -\frac{1}{\epsilon} + \gamma - \log 4\pi - \log \frac{\mu^2}{Q^2} \right) + \frac{\alpha_s}{2\pi} f_{2G}^{(1)}(z) \right] \right].$$

(4.21)

The $\overline{\text{MS}}$ scheme definition of the distribution functions could be

$$q_{\overline{\text{MS}}}(x, Q^2) = q(x) + \int_x^1 \frac{dy}{y} \frac{\alpha_s}{y} \left[ -\frac{1}{\epsilon} + \gamma - \log 4\pi - \log \frac{\mu^2}{Q^2} \right] q(y) P_{qq} \left( \frac{x}{y} \right)$$

$$+ \int_x^1 \frac{dy}{y} \frac{\alpha_s}{y} \left[ -\frac{1}{\epsilon} + \gamma - \log 4\pi - \log \frac{\mu^2}{Q^2} \right] G(y) P_{qG} \left( \frac{x}{y} \right)$$

(4.22)

This is one choice, however, it is not the only choice which reproduces the Altarelli-Parisi equations. In particular, all of the finite terms in (4.21) are independent of $\log Q^2/\mu^2$. Altarelli, Ellis and Martinelli [47] propose instead to define beyond leading order:

$$\mathcal{F}_2(x, Q^2) = \sum_i e_i^2 (q_{\text{DIS}}(x, Q^2) + \bar{q}_{\text{DIS}}(x, Q^2))$$

$$= \sum_i e_i^2 \left[ q_{\overline{\text{MS}}}(x, Q^2) + \bar{q}_{\overline{\text{MS}}}(x, Q^2) \right]$$

$$+ \sum_i e_i^2 \int_x^1 \frac{dy}{y} \frac{\alpha_s(Q^2)}{2\pi} \left[ [q_i(x, Q^2) + \bar{q}_i(x, Q^2)] f_{2q}^{(1)} \left( \frac{x}{y} \right) \right.$$  

$$\left. + 2G(y, Q^2)f_{2G}^{(1)} \left( \frac{x}{y} \right) \right]$$

(4.23)

This is sometimes referred to as the deep inelastic scattering (DIS) factorization. In
this case, the structure function $F_1(x, Q^2)$ is

$$F_1(x, Q^2) = \sum_i c_i^2 \int_0^1 \frac{dy}{y} \left[ \delta \left( \frac{x}{y} - 1 \right) + \alpha_s(Q^2) \left[ \bar{f}_{q1} \left( \frac{x}{y} \right) - \bar{f}_{q2} \left( \frac{x}{y} \right) \right] G(y, Q^2) ight]$$

$$+ q(y, Q^2) + \left[ \alpha_s(Q^2) \left[ \bar{f}_{G1} \left( \frac{x}{y} \right) - \bar{f}_{G2} \left( \frac{x}{y} \right) \right] G(y, Q^2) \right]$$

(4.24)

Theoretically, one advantage of this choice is calculational simplicity. The differences between $\bar{f}_{1q}(z)$ and $\bar{f}_{2q}(z)$ and between $\bar{f}_{1G}(z)$ and $\bar{f}_{2G}(z)$ are finite at any order, since $F_1$, the measured structure function, is finite.

5. Transverse momentum of $W$ or $Z$

The final discussion will focus on the calculation of the $W$ or $Z$ transverse momentum at hadron colliders, through order $\alpha_s^2$ [48], [49]. The analytic results described below also apply to virtual photons, however, we restrict ourselves here to weak boson production. For definiteness, we refer to $W$ production. In hadron colliders, because the partons participating in the hard scattering carry unknown fractions of the longitudinal momenta of the incoming hadrons, only conservation of transverse momentum can be experimentally enforced on the final state momenta. We present here the inclusive distribution of transverse momentum ($q_T$) for a single $W$ produced in proton-antiproton collisions.

The order $\alpha_s^2$ contribution is the next-to-leading order contribution to the $q_T$ distribution. Figure 5.1 shows Drell-Yan production and the first order corrections. Since the partons collide with negligible incident transverse momentum, only when there is a “balancing” quark or gluon will there be transverse momentum for the $W$, so the $q_T$ distribution begins at order $\alpha_s$. Consequently, it is a good process to study QCD, especially as electroweak bosons are fairly unambiguously identifiable in hadron colliders. By including the $\alpha_s^2$ corrections, we reduce the theoretical errors.

The theoretical errors are not completely under control in this second order calculation in the region of small $q_T$. A problem arises because we have more than one scale in the problem, here $M_W$ and $q_T$. In the QCD-improved parton model, the renormalization and factorization scales, generically denoted by $M$, are chosen to reduce or eliminate logs of the form $\log M/Q$. When we have a transverse momentum distribution with $q_T^2 \ll M_W^2$, we have two scales which cannot be simultaneously reduced. Choosing $M = M_W$ leaves us with $\log M_W/q_T$. It can be shown [50] that the $\alpha_s^n$ corrections at small $q_T$ go like

$$\alpha_s^n \frac{1}{q_T^m} \log^m \left( \frac{M_W^2}{q_T^2} \right), \quad m \leq 2n - 1 ;$$

(5.1)
so the expansion in higher orders of $\alpha_s$ is accompanied by large (log) coefficients. Perturbation theory breaks down.

For $W$ and $Z$ production, the leading log terms have been summed to give a Sudakov like form factor [51]. The distribution is normalized so that the integral over $q_T$ reproduces the perturbative cross section through order $\alpha_s$, and is required to reproduce the perturbative $q_T$ distribution for sufficiently large $q_T$. It has been shown [52] that for a range of center of mass energies, $(\sqrt{s} = 0.63 - 10 \text{ TeV})$ the summed and first order perturbative results match for $q_T \gtrsim 20 \text{ GeV}$. On this basis, we restrict ourselves to $q_T > 20 \text{ GeV}$ in the second order perturbative calculation.

The earlier partial $\alpha_s^2$ result by Ellis, Martinelli and Petronzio [53] has recently been completed [48], [49]. The procedure is to compute the matrix element squared at the parton level and analytically integrate over the final parton phase space using dimensional regularization to regulate all singularities. The result is then factorized and numerically integrated.

The weak boson couplings are distinguished from gluon or photon couplings by the appearance of $\gamma_5$ couplings. The $\gamma_5$ is not unambiguously defined in $n \neq 4$ dimensions. We [48] have simplified the $\gamma_5$ problem somewhat by omitting graphs which are proportional to quark mass splittings within an electroweak doublet. Furthermore, following Chanowitz et al. [54] we assume:

1. $\{\gamma_\mu, \gamma_5\} = 0$ in $n$ dimensions,
2. $\gamma_5^2 = 1$,
3. $\text{tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) = 4 i e^{\mu\nu\alpha\beta} + \mathcal{O}(\epsilon)$ ambiguity,

and we also assume
4. $\text{tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta)$ is antisymmetric.
FIGURE 5.2. Virtual correction diagrams for $q\bar{q} \rightarrow gW$. The $\alpha_s^2$ corrections come from the interference of $V_i$ with $L_i$ of Fig. 5.1.

FIGURE 5.3. Diagrams for $q\bar{q} \rightarrow ggW$.

FIGURE 5.4. Diagrams for $q\bar{q} \rightarrow q\bar{q}W$. 

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**FIGURE 5.2.** Virtual correction diagrams for $q\bar{q} \rightarrow gW$. The $\alpha_s^2$ corrections come from the interference of $V_i$ with $L_i$ of Fig. 5.1.

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**FIGURE 5.2.** Virtual correction diagrams for $q\bar{q} \rightarrow gW$. The $\alpha_s^2$ corrections come from the interference of $V_i$ with $L_i$ of Fig. 5.1.

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These assumptions, together with gauge invariance arguments, are sufficient for our purposes. For a more complete discussion of $\gamma$'s in this context, together with the results of the omitted diagrams (numerically small contributions), see Gonsalves et al. [49].

The set of diagrams for nonzero $q_T$ at order $\alpha_s^2$ are shown in Figs. 5.2–5.5. In addition to these diagrams are some crossed diagrams, for example, the $qq \rightarrow qW$ diagram which is related to the $q\bar{q} \rightarrow gW$. For the real graphs, the three body phase space reduces to two angular integrals:

$$
\frac{sd\sigma}{dtdu} = \frac{1}{2s} \int d\theta_2 \sin^{-2\epsilon}\theta_2 \int d\theta_1 \sin^{1-2\epsilon}\theta_1 
\sum_{\text{avg}} |M|^2 \frac{S_f}{2^8\pi^4} \left(\frac{4\pi}{1-2\epsilon}\right) \frac{1}{s} \left(\frac{s}{s_2(ut-s_2Q^2)}\right)^\epsilon 
$$

where $s$, $t$ and $u$ are the Mandelstam variables defined by $s = (p_1+p_2)^2$, $t = (q-p_1)^2$ and $u = (q-p_2)^2$, and $s_2$, the invariant mass of the two-parton final state, is $s_2 = (p_1+p_2-q)^2 = (p_3+p_4)^2$. Here, $p_1$ and $p_2$ are incident parton momenta and $q$ is the $W$ momentum. The factor $S_f$ is the statistical factor. The angular integrals are evaluated in the rest frame of the two parton final state.

The parameter choices for the following figures are $m_q = 0$, $N_f = 5$ and the four flavor value of $\Lambda_{QCD} = 260$ MeV in the modified minimal subtraction scheme. We use the particle data book central values [14] of $M_W = 81.8$ GeV and $M_Z = 92.6$ GeV together with the next-to-leading order evolved structure functions of Diemoz, Ferroni, Longo and Martinelli [42]. For a choice of the renormalization scale (equal...
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**Figure 5.6.** Predictions and experiment [55], [56] for $W$ production at $\sqrt{s} = 630$ GeV.

**Figure 5.7.** Predictions for $W$ production at $\sqrt{s} = 1.8$ TeV.
Figure 5.8. Relative contributions of different processes to W production at $\sqrt{s} = 630$ GeV using $M = M_W$. First order contributions are (A) $\bar{q}q \to gW$ and (B) $qg \to qW$ and $\bar{q}g \to \bar{q}W$. Second order contributions are (C) $[\bar{q}q \to gW + ggW] + |F_1 + F_2|^2$, (D) $qg \to qW + qgW$ and $\bar{q}g \to \bar{q}W + \bar{q}gW$, (E) $gg \to q\bar{q}W$, (F) remaining $q\bar{q} \to q\bar{q}W$, (G) $qg \to q\bar{q}W$ and $\bar{q}g \to \bar{q}qW$.

Figure 5.9. Same as Fig. 5.8 with $\sqrt{s} = 1.8$ TeV.
to the factorization scale) of $M_W$ or $M_Z$, we get for proton-antiproton collisions

$$
\sqrt{s} = 1.8 \text{ TeV} \quad \quad \quad \quad \sqrt{s} = 630 \text{ GeV}
$$

$$
M = M_W \quad \quad \sigma(W^+) = 9.68 \text{ nb} \quad \quad \sigma(W^+) = 2.80 \text{ nb} \quad (5.3)
$$

$$
M = M_Z \quad \quad \sigma(Z) = 6.05 \text{ nb} \quad \quad \sigma(Z) = 1.78 \text{ nb}
$$

In Fig. 5.6 we see the $q_T$ distribution (normalized to $q_T \sigma$) at the SPS energy of $\sqrt{s} = 630$ GeV, compared with the UA1 [55] and UA2 [56] data. Fig. 5.7 shows the same distribution at the Tevatron energy of $\sqrt{s} = 1.8$ TeV. To get an idea of the relative importance of the various graphs, we show in Figs. 5.8 and 5.9 the various components, normalized to the full differential cross section, as a function of $q_T$ for SPS and Tevatron energies, respectively. By comparison, it is clear that the Tevatron is a much better probe of the gluon component of the proton than the SPS in this context. Except at the lowest transverse momenta, where the $q\bar{q}$ matrix element is larger, the $qg$ contribution dominates at the Tevatron until $q_T \approx 150$ GeV, where $x$ gets sufficiently large so that the valance quark component dominates the cross section.

The uncertainties in the theoretical calculation at large $q_T$ include the choice of scale, the value of $\Lambda$ and the structure functions. By considering $M$ between $q_T$ and $M_W$, and $\Lambda = 160-360$ MeV, with the Diemoz et al. structure functions, we estimate the following errors: For $\sqrt{s} = 1.8$ TeV, the error in $1/\sigma d\sigma/dq_T$ is is approximately $\pm 10-15\%$ relative to the mean of $M = q_T$ and $M = M_W$. The total cross sections were evaluated at $M = M_W$ and $M = (q_T^2)^{1/2}$. For $\sqrt{s} = 630$ GeV, at low $q_T$ the estimated error is $\pm 10\%$ and at high $q_T$, $\pm 35\%$, again, relative to the mean.

The larger error at the SPS energy for large $q_T$ originates from the fact that $(q_T^2)^{1/2} \approx 10$ GeV at $\sqrt{s} = 630$ GeV, whereas $(q_T^2)^{1/2} \approx 20$ GeV at $\sqrt{s} = 1.8$ TeV. At the Tevatron, at $q_T = 20$ GeV, the normalized distribution $1/\sigma d\sigma/dq_T$ is larger for $M^2 = q_T^2$ than for $M^2 = M_W^2$. For $q_T > M_W$, this is reversed as $\alpha_s(q_T^2) < \alpha_s(M_W^2)$. For the SPS, however, even at $q_T = 20$ GeV, the normalized distribution is larger for $M^2 = M_W^2$ than for $M^2 = q_T^2$, because the normalizing cross section is on the order of 30\% larger for the latter scale (compared with $\sim 15\%$ smaller at the Tevatron). Again, as $q_T$ increases beyond $M_W$, the distribution with $M^2 = q_T^2$ drops more rapidly than with $M^2 = M_W^2$. The discrepancy is compounded by the larger cross section normalization factor.

APPENDIX

1. Gamma matrix properties in $n$ dimensions:

$$
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu \nu} \quad \quad \quad \gamma_\mu \gamma^\mu = n
$$

$$
\gamma_\mu \gamma_\alpha \gamma^\mu = (2-n)\gamma_\alpha \quad \quad \quad \gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu = 4g_{\alpha \beta} + (n-4)\gamma_\alpha \gamma_\beta \quad (A.1)
$$

$$
\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma^\mu = -2\gamma_\gamma \gamma_\beta \gamma_\alpha - (n-4)\gamma_\alpha \gamma_\beta \gamma_\gamma
$$
2. Feynman parameterization:

\[ \frac{1}{a^r b^m} = \frac{\Gamma(r + m)}{\Gamma(r) \Gamma(m)} \int_0^1 dx \frac{x^{r-1}(1 - x)^{m-1}}{[b + x(a - b)]^{r+m}} \]

\[ \frac{1}{a^r b^m c^n} = \frac{\Gamma(r + m + n)}{\Gamma(r) \Gamma(m) \Gamma(n)} \int_0^1 dx x^{r-1} y^{m-1} z^{n-1} \times \]

\[ \frac{1}{a_1^{n_1} a_2^{n_2} \ldots a_k^{n_k}} = \frac{\Gamma(\sum_{i=1}^k n_i)}{\Gamma(n_1) \Gamma(n_2) \ldots \Gamma(n_k)} \int_0^1 \ldots \int_0^1 dz_1 dz_2 \ldots dz_k \times \]

\[ \delta(1 - z_1 - z_2 - \ldots - z_k) \frac{z_1^{n_1-1} z_2^{n_2-1} \ldots z_k^{n_k-1}}{(\sum_{i=1}^k a_i z_i)^{n_1+n_2+\ldots+n_k}} \]

(A.2)

3. Phase space integrals:

\[ \int \frac{d^n Q}{(2\pi)^n (Q^2 - C + i\epsilon)^m} = \int \frac{d^n Q}{(2\pi)^n} \frac{g_{\mu\nu}(Q^2/n)}{(Q^2 - C + i\epsilon)^m} \]

\[ \int \frac{d^n Q}{(2\pi)^n} \frac{(Q^2)^r}{(Q^2 - C + i\epsilon)^m} = \frac{i(-1)^{r-m}}{(16\pi^2)^{n/4}} \frac{(C - i\epsilon)^{-m+n/2}}{\Gamma(r+n/2) \Gamma(m-n/2)} \times \frac{\Gamma(r+n/2) \Gamma(m-r-n/2)}{\Gamma(n/2) \Gamma(m)} \]

(A.3)

4. Useful Gamma function relations:

The beta function \( B(x, y) \) is defined by

\[ B(x, y) = \int_0^1 dt t^{x-1}(1 - t)^{y-1} = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \]

(A.4)

The Gamma functions obey the following:

\[ \Gamma(1 + z) = z \Gamma(z) \]

\[ \lim_{\epsilon \to 0} \Gamma(1 + \epsilon) = 1 - \gamma \epsilon + \frac{\epsilon^2}{2} (\gamma^2 + \frac{\pi^2}{6}) + O(\epsilon^3) \]

(A.5)

where \( \gamma = 0.5772 \ldots \) is the Euler constant.
5. Phase space integrals in \( n = 4 - 2\epsilon \) dimensions:

\[
\int d\Omega_{n-2} = \frac{2\pi^{3/2-\epsilon}}{\Gamma(3/2 - \epsilon)} = \int d\theta_{n-2} \sin^{n-3} \theta_{n-2} \ldots d\theta_1 \tag{A.6}
\]

where

\[
\int_0^\pi d\theta \sin^m \theta = \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})} \sqrt{\pi} \tag{A.7}
\]

\[
\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}
\]

References


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[35] See, for example, p. 115 of Ref. [14].


[50] See, e.g., Reference [8].


