Introduction to the theory of fiber bundles and connections I

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Abstract. In lectures 1 and 2 we discuss basic concepts of topology and differential geometry: definition of a topological space and of Hausdorff, compact, connected and paracompact spaces; topological groups and actions of groups on spaces; differentiable manifolds, tangent vectors and 1-forms; partitions of unity and Lie groups. In Lecture 3 we present the concept of a fiber bundle and discuss vector bundles and principal bundles. The concept of a connection on a smooth vector bundle is defined in Lecture 4, together with the associated concepts of curvature and parallel transport; as an illustration we present the Levi-Civita connection on a Riemannian manifold. Finally, in Lecture 5 we define connections on principal bundles and present examples with the Lie groups U(1) and SU(2). For reasons of space the present article only includes Lectures 1, 2 and 3. Lectures 4 and 5 will be published in a forthcoming paper.

Introduction

From the early seventies [12] it became clear that a geometrical understanding of the basic laws of physics could help towards the construction of a unifying theory of the fundamental interactions. For example, we know now that the gauge potentials appearing in the Maxwell theory of electromagnetism and in the Yang-Mills theories of nuclear forces (weak and strong interactions) are nothing but Lie-algebra valued one-forms associated with connections on principal fiber bundles having space-time as the base manifold and an appropriate Lie group (U(1), SU(2) and SU(3) in the above examples) as the fiber space [3], while Einstein theory of gravity deals with the Levi-Civita connection on the bundle of frames of space-time [4]. It should be remarked however that these are classical concepts. In order to go to the quantum theory of a given gauge field one has to consider the space of all gauge field configurations i.e. the space of all connections on the corresponding fiber bundle, to define a measure on this space and to integrate the individual contributions to the quantum probability amplitudes for the processes under study [10].

Motivated by the above remarks we present in these lectures an elementary introduction to the theory of fiber bundles and the connections associated with them. Since basic notions of topology and differential geometry are a previous prerequisite to discuss bundle theory, we devote the first two lectures to these subjects. In Lecture 3 we define the general concept of a fiber bundle and in particular discuss
vector bundles and principal bundles. For reasons of space Lectures 4 and 5, which deal with the theory of connections respectively on vector and principal bundles, will be published in a separate issue.

1. Topology

Let $X$ be a set and $\mathcal{P}(X)$ the power set of $X$ i.e. $\mathcal{P}(X) = \{A \mid A$ is a subset of $X\}$.

**Definition.** A topological space is a pair $(X, \tau)$ with $\tau = \{U_\alpha\}_{\alpha \in J} \subset \mathcal{P}(X)$ such that: i) $\emptyset, X \in \tau$; ii) $\bigcup_{\beta \in K} U_\beta \in \tau$ if $K \subset J$; iii) $\bigcap_{i=1}^N U_i \in \tau$ if $U_i \in \tau$ ($J$ is an indexing set and $N \in \mathbb{Z}$).

$\tau$ is said to be a topology for $X$. If $U \in \tau$, $U$ is called an open set (one writes $U \subset X$). The complement of an open set $U$, $U^c = X - U$ is called a closed set. If $x \in X$ belongs to an open set $U$ one writes $U = U_x$ and says that $U_x$ is a neighborhood of $x$. (In the following a topological space will be called simply a space.)

A cover of $X$ is a collection of open sets such that its union is $X$. A subcover is a subcollection of a cover which by itself is a cover.

A locally finite cover is a cover $U$ such that $\forall x \in X \exists U_x \in \tau \mid U_x$ intersects at most a finite number of open sets of $U$.

A refinement of a cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ is a cover $\mathcal{V} = \{V_\rho\}_{\rho \in M} \mid \forall V \in \mathcal{V} \exists U_\alpha \in \mathcal{U} \mid V_\rho \subset U_\alpha$.

A space is paracompact if any cover has a locally finite refinement.

A closed cover of $X$ is a collection of closed sets such that its union is $X$.

**Definition.** A space $(X, \tau)$ is Hausdorff if $\forall x, y \in X, x \neq y, \exists U_x, U_y \in \tau \mid U_x \cap U_y = \emptyset$.

**Definition.** A space is compact if any cover has a finite subcover. Obviously any compact space is paracompact.

**Definition.** A space is connected if it is not the union of two non-empty disjoint open sets.

Let $(X, \tau)$ be a space, $A \subset X$ and $\tau_A = \{A \cap U, U \in \tau\}$.

**Proposition.** $\tau_A$ is a topology for $A$.

**Proof.** $A \cap \phi = \phi \in \tau_A$; $A \cap X = A \in \tau_A$; if $\{U_\alpha\}_{\alpha \in \Lambda} \subset \tau_A \Rightarrow \bigcup_{\alpha \in \Lambda} U_\alpha = \bigcup_{\alpha \in \Lambda}(V_\alpha \cap A) = (\bigcup_{\alpha \in \Lambda} V_\alpha) \cap A \in \tau_A$ since $\bigcup_{\alpha \in \Lambda} V_\alpha \in \tau$; if $\{U_i\}_{i=1}^N \subset \tau_A \Rightarrow \bigcap_{i=1}^N U_i = \bigcap_{i=1}^N (V_i \cap A) = (\bigcap_{i=1}^N V_i) \cap A \in \tau_A$ since $\bigcap_{i=1}^N V_i \in \tau$.

$(A, \tau_A)$ is called a subspace of $X$ and one writes $A \subset X$.

**Example.** $S^n \subset \mathbb{R}^{n+1}$, $0 \leq n \in \mathbb{Z}$.

**Definition.** $B \subset \tau$ is a basis of $\tau$ if $\forall U \in \tau$, $U = \bigcup B_\gamma$, $B_\gamma \in B$. 
DEFINITION. A metric space is a pair \((X, d)\) where \(X\) is a set and \(d:X \times X \to \mathbb{R}\) is a function called distance which has the following properties:

\[
i) \quad d(x, y) \geq 0, \quad d(x, y) = 0 \text{ iff } x = y; \quad ii) \quad d(x, y) = d(y, x) \quad \text{and} \quad iii) \quad d(x, z) \leq d(x, y) + d(y, z).
\]

A ball of radius \(r > 0\) at \(x_0 \in X\) is the set \(B(x_0, r) = \{y \in X \mid d(y, x_0) < r\}\). \((X, d)\) induces a topological space, the space which has as a basis the set of balls in \((X, d)\). A space \((X, \tau)\) is said to be metrizable if there exists a distance \(d\) in \(X\) such that the induced topology coincides with \(\tau\).

Example. \((\mathbb{R}^n, d)\) with \(d(x, y) = (\sum_{i=1}^{N} (x^i - y^i)^2)^{\frac{1}{2}}\) is the Euclidean \(n\)-dimensional space. \(\mathbb{R}^n\) is a non-compact, paracompact, connected, Hausdorff space.

DEFINITION. Let \(X, Y\) be spaces and \(f:X \to Y\) be a function. \(f\) is continuous if \(\forall x \in X\) and \(\forall V_{f(x)} \in \tau_Y \exists U_x \in \tau_X \mid f(U_x) \subset V_{f(x)}\). Grafically,

![Diagram of continuous function](image)

One writes \(f \in C^0(X, Y) = \text{Map}(X, Y) = M(X, Y)\). Clearly, the composition of continuous functions is continuous.

PROPOSITION. Let \(X\) and \(Y\) be spaces. \(f:X \to Y\) is continuous iff \(f^{-1}(V) \in \tau_X \forall V \in \tau_Y\).

Proof. (Exercise).

DEFINITION. \(f:X \to Y\) is a homeomorphism if it is bicontinuous i.e. if \(f\) is continuous and has a continuous inverse \(f^{-1}\). One writes \(X \overset{f}{\cong} Y\).

NOTE. The category of topological spaces \(\tau\) is the triple \((\text{Obj} (\tau), \text{Mor} (\tau), \circ)\) with \(\text{Obj} (\tau) = \{\text{topological spaces}\}\), \(\text{Mor} (\tau) = \{M(X, Y), X, Y \in \text{Obj} (\tau)\}\), \(\circ\): composition of functions.
NOTE. If Aut(X) = \{f: X \to X \mid X \cong X\} \Rightarrow \text{the pair (Aut(X), o) is a group called the group of automorphisms of X. The identity is \(\text{id}_X: X \to X, \; x \mapsto \text{id}_X(x) = x\).}

**Proposition.** Let \(X, Y\) be spaces and let \(B = \{U \times V \mid U \in \tau_X, V \in \tau_Y\}\). \(\Rightarrow \tau = \{\text{unions of elements of } B\}\) is a topology for \(X \times Y\).

**Proof.** (Exercise)

The space \((X \times Y, \tau)\) is called the topological product of the spaces \(X\) and \(Y\).

**Proposition.** Let \(X\) be a space, \(Y\) a set and \(f: X \to Y\) an onto function. \(\Rightarrow \tau_f = \{V \in \mathcal{P}(Y) \mid f^{-1}(V) \in \tau_X\}\) is a topology for \(Y\).

**Proof.** \(\phi \in \tau_f\) because \(f^{-1}(\phi) = \phi \in \tau_X\); \(Y \in \tau_f\) because \(f^{-1}(Y) = X \in \tau_X\); \(\bigcup_{\alpha \in K} U_{\alpha} \in \tau_f\) if \(\{U_{\alpha}\}_{\alpha \in K} \subseteq \tau_f\) since \(f^{-1}(\bigcup_{\alpha \in K} U_{\alpha}) = \bigcup_{\alpha \in K} f^{-1}(U_{\alpha}) \in \tau_X\) and \(\bigcap_{i=1}^N U_i \in \tau_f\) if \(\{U_i\}_{i=1}^N \subseteq \tau_f\) since \(f^{-1}(\bigcap_{i=1}^N U_i) = \bigcap_{i=1}^N f^{-1}(U_i) \in \tau_X\). QED

\(\tau_f\) is called the quotient topology for \(Y\) with respect to \(f\).

**Definition.** Let \(f, g: X \to Y\) be continuous functions. \(f\) is homotopic to \(g\) \((f \sim g)\) if there exists a continuous function \(F: X \times I \to Y, (x, t) \mapsto F(x, t)\mid F(x, 0) = f(x)\) and \(F(x, 1) = g(x)\) \((I = [0, 1] \subseteq \mathbb{R})\). \(F\) is called an homotopy between \(f\) and \(g\). Graphically,

![Figure 2](image)

**Definition.** Two spaces \(X\) and \(Y\) are of the same homotopy type if there exist continuous functions \(f: X \to Y\) and \(g: Y \to X\) \(| f \circ g \sim \text{id}_Y\) and \(g \circ f \sim \text{id}_X\). One writes \(X \simeq Y\).

**Definition.** \(X\) is contractible if \(X \simeq \{\ast\}\). \((\{\ast\}\) is the space which has only one point.)
Example. If $A \subseteq \mathbb{R}^n$ and $A$ is convex i.e. $\forall x, y \in A, \{\lambda(y - x) + x, \lambda \in [0, 1]\} \subseteq A \Rightarrow A \simeq \{\ast\}$. In particular $\mathbb{R}^n \simeq \{\ast\}$. However, $S^n \equiv \mathbb{R}^n \cup \{\infty\} \not\simeq \{\ast\}$.

Definition. Let $(X, \tau)$ be a space and $A$ a subset of $X$. $x \in X$ is an accumulation point of $A$ if $\forall U_x \in \tau, U_x \cap A \neq \phi$.

The closure of $A$ (denoted by $\bar{A}$) is the set of accumulation points of $A$. Obviously $A \subseteq \bar{A}$.

Proposition. $A$ is closed iff $A = \bar{A}$.

Proof. (Exercise).

Definition. A topological group is a triple $(G, *, \tau)$ where $(G, \ast)$ is a group, $(G, \tau)$ is a space and the composition $*: G \times G \to G, (g, h) \mapsto g \ast h$ and inverse inv: $G \to G, g \mapsto g^{-1}$ functions are continuous. (We shall denote $g \ast h = gh$.)

Definition. A left $G$-space is a triple $(X, G, \mu)$ where $G$ is a topological group, $X$ is a space and $\mu: G \times X \to X, (g, x) \mapsto \mu(g, x)$ is a continuous function with the following properties: i) $\mu(e, x) = x \forall x \in X$ (e is the identity in $G$) and ii) $\mu(g, h, x) = \mu(g, (h, x))\forall g, h \in G, \forall x \in X$. (Notation: $\mu(g, x) = gx \Rightarrow$ for i) and ii) we respectively write $ex = x$ and $(gh)x = g(hx)$.) $\mu$ is onto since $\forall x \in X, (e, x) \in \mu^{-1}(\{x\})$. It is said that $G$ acts on $X$ from the left.

Let $g \in G$ and $\mu_g: X \to X, x \mapsto \mu_g(x) = gx$. Clearly $\mu_g$ is continuous, $\mu_e = id_x$ and $\mu_{g^{-1}} = (\mu_g)^{-1} \Rightarrow \mu_g \in \text{Aut}(X)$. It is easy to verify that $\bar{\mu}: G \to \text{Aut}(X), g \mapsto \bar{\mu}(g) \equiv \mu_g$ is a group homomorphism. In fact, $\bar{\mu}(gh) = \mu_{gh}: X \to X, x \mapsto (gh)x = g(hx) = \mu_g(\mu_h(x)) = \mu_g \circ \mu_h(x)$ i.e. $\mu_{gh} = \mu_g \circ \mu_h \Rightarrow \bar{\mu}(gh) = \bar{\mu}(g) \circ \bar{\mu}(h) \Rightarrow (\bar{\mu}(G), o) \subseteq (\text{Aut}(X), o)$ (denotes subgroup).

The left action of $G$ over $X$ is: effective if $(gx = x \forall x \in X) \Rightarrow g = e$ (i.e. $\forall g \in G, g \neq e \mid gx = x \forall x \in X$); free if $gx = x \Rightarrow g = e$ for any $x \in X$, and transitive if $\forall x, y \in X \exists g \in G \mid y = gx$. Clearly, any free action is an effective action and, if an action is free and transitive, $\forall x, y \in X \Rightarrow \exists g \in G \mid x = (g^{-1})y$ and $\Rightarrow g\neq g'$. Finally, for and effective action $\bar{\mu}$ is a group monomorphism. In fact, $\mu_g = \bar{\mu}(h) \Rightarrow \mu_g(x) = \mu_h(x) \forall x \in X i.e. gx = hx \Rightarrow x = (g^{-1}h)x \Rightarrow g = h$ (contradiction). In particular one then has Ker($\bar{\mu}$) = $\{e\}$.

A right $G$-space is defined in an analogous way. In particular if $(X, G, \nu)$ is a right action, $\bar{\nu}: G \to \text{Aut}(X), g \mapsto \bar{\nu}(g) \equiv \nu_g: X \to X, x \mapsto \nu_g(x) = xg$ is a group anti-homomorphism. In fact, $\bar{\nu}(g\nu'g'') = \nu_g \nu' \nu'' = \nu_g \nu' \nu'' = \nu_{gg'} \nu''$ and therefore $\bar{\nu}(g\nu'g'') = \bar{\nu}(gg') \circ \bar{\nu}(g'')$.

Then $(\bar{\nu}(G), o) \subseteq (\text{Aut}(X), o)$ since for any group anti-homomorphism $f: G \to H, h \mapsto f(h)$ with $f(gg') = f(g')f(g), f(G) \subseteq H (h, \ell \in f(G) \Rightarrow \exists g, k \in G \mid h = f(g), \ell = f(k) \Rightarrow h \ell^{-1} = f(g)(f(k))^{-1} = f(g)f(k^{-1}) = f(k^{-1})g \in f(G)$ since $k^{-1}g \in G$).

Let $\alpha = (X, G, \mu)$ and $\alpha' = (X', G', \mu')$ be right group actions. $\alpha$ is homomorphic to $\alpha'$ if there exists a pair of functions $(f, \gamma)$ with $f: X \to X'$ continuous and $\gamma: G \to G'$ a group homomorphism such that Diagram 1 commutes, i.e. $f \circ \mu = \mu' \circ (f \times \gamma)$. 
Let \((x, g) \in X \times G\). \(f(\mu(x, g)) = f(xg) = \mu'(f \times \gamma(x, g)) = \mu'(f(x), \gamma(g)) = f(x)\gamma(g)\) i.e. \(f\) "preserves the product of the group times the space". We write \(\alpha \simeq \alpha'\).

**Diagram 1.**

Let \(G' = G\). The action \(\alpha = (X, G, \mu)\) is isomorphic to the action \(\alpha' = (X', G, \mu')\) if \(\alpha \simeq \alpha'\) and \(X \cong X'\). We have Diagram 2, and therefore \(f(xg) = f(x)g\). One writes \(\alpha \cong \alpha'\).

**Diagram 2.**

*Example of right \(G\)-space.* Let \(E\) be a space and \(G\) a topological group. \((E \times G, G, \delta)\) is the right action given by \(\delta: (E \times G) \times G \to E \times G, ((x, g), g') \mapsto \delta((x, g), g') = (x, gg')\) i.e. \((x, g)g' = (x, gg')\). We have: i) \((x, g)e = (x, g)\); ii) \((x, g)(g'g'') = (x, gg'g'')\). Let \((x, g)g' = (x, gg') = (x, g); \Rightarrow gg' = g\) and \(\Rightarrow g' = e\) i.e. we have a free action. The action is not transitive since given \((x, g)\) and \((y, g')\) with \(x \neq y\), \(\not\exists g'' \in G \mid (x, g) = (y, g')g''\). For each \(g \in G\) we define \(\delta_g: E \times G \to E \times G, (x, g') \mapsto \delta_g(x, g') = \delta((x, g'), g) = (x, gg')\). \(\delta_g \in \text{Aut}(E \times G)\) with \(\delta_{g'} \circ \delta_{g''}(x, g) = \delta_{g''}(x, gg') = (x, g(g''g'')) = \delta_{g''}(x, g)\) i.e. \(\delta_{g''} \circ \delta_{g'} = \delta_{g''g'}. \delta: G \to \text{Aut}(E \times G), g \mapsto \delta(g) \equiv \delta_g\) is then an anti-homomorphism of groups \((\delta(g', g'') = \delta(g'') \circ \delta(g'))\), and \(\delta(G) \subseteq \text{Aut}(E \times G)\).

Let \((X, G, \nu)\) be an action. \(x \in X\) and \(G_x \equiv \{g \in G \mid xg = x\}\). It is easy to
show that \( G_x \subseteq G \). In fact, if \( g, h \in G_x \), \( x(gh) = (xg)h = xh = x \) and \(.gh \in G_x\); \( e \in G_x \) since \( xe = x \); \( g \in G_x \Rightarrow xg = x \Rightarrow (xy)g^{-1} = x = xg^{-1} \), i.e. \( g^{-1} \in G_x \). \( G_x \) is called the stabilizer or isotropy group of \( x \).

Let \( (X, G, \nu) \) be an action and \( x \in X \). \( X_x \equiv \{xg\}_{g \in G} \) is called the orbit of \( x \) by \( G \). (It is usual the notation \( X_x = [x] = xG \).) It is clear that if \( X_x \cap X_y \neq \emptyset \) \( \Rightarrow \) \( X_x = X_y \). In fact, \( X_x \cap X_y \neq \emptyset \) \( \Rightarrow \exists g, g' \in G \mid xg = yg' \) and \( \therefore x = y(g'g^{-1}) \), \( y = x(gg'^{-1}) \); \( z \in X_x \Rightarrow z = xh = yg'g^{-1}h \) i.e. \( z \in X_y \) and \( \therefore X_x \subseteq X_y \); \( w \in X_y \Rightarrow w = yk = xgg'^{-1}k \) i.e. \( w \in X_x \) and \( \therefore X_y \subseteq X_x \). (From here one obtains that \( xG = yG \) iff \( x = yg \) for some \( g \in G \). In fact, \( x = yg \Rightarrow x \in yG \) and \( \therefore xG \cap yG \neq \emptyset \).

On the other hand, \( xG = yG \Rightarrow \) that \( \forall g \in G \exists g' \in G \mid xg = yg' \); \( \Rightarrow x = yg' \) with \( g'' = g'g^{-1} \in G \). Therefore the action of \( G \) on \( X \) induces an equivalence relation in \( X \), and we have the partition \( X = \bigcup_{x \in X} \{x\} \), the quotient set \( X/G \equiv \{\{x\} \mid x \in X \} \) and the projection \( p : X \rightarrow X/G, x \mapsto p(x) \equiv [x] \). We have then the orbit space \((X/G, \tau_{X/G})\) with \( \tau_{X/G} \equiv \{V \in P(X/G) \mid p^{-1}(V) \in \tau_X \} \). By definition \( p \) is continuous and it is easy to verify that it is also an open function i.e. the image by \( p \) of an open set in \( X \) is open in \( X/G \). In fact, let \( U \in \tau_X \); \( \Rightarrow p(U) = \{yG\}_{y \in U} \) and \( \therefore p^{-1}(p(U)) = \{x \in X \mid p(x) \in p(U)\} = \{x \in X \mid xG = yG \) for some \( y \in U\} = \{x \in X \mid x = yg \) for some \( g \in G \) and some \( y \in U\} = \bigcup_{g \in G} U_g = \bigcup_{g \in G} \nu_g(U) \in \tau_X \) since it is a union of open sets because \( U \in \tau_X \) and \( \nu_g \in \text{Aut} \). From the definition of the topology \( \tau_{X/G} \) we conclude that \( p(U) \in \tau_{X/G} \). Finally, if \( (X, G, \nu) \) is a free action, \( G_x = \{e\} \) and \( xG \equiv G \forall x \in X \); moreover if the action is transitive, there exists the continuous function \( \tau : X \times X \rightarrow G, (x, y) \mapsto \tau(x, y) \equiv g \) such that \( y = \nu(x, g) \). In fact, if \( \nu \) is free \( xg = x \Rightarrow g = e \) and \( \therefore G_x = \{e\} \) for any \( x \in X \); let \( g, h \in G \) with \( g \neq h \) and let \( xg = xh \) with \( x \in X \); \( \Rightarrow x = xh^{-1} \Rightarrow h^{-1} = e \Rightarrow h = g \) which is a contradiction; \( \therefore \{xg\}_{g \in G} \equiv G \) for a transitive action \( \forall (x, y) \in X \times X \exists g \in G \mid y = xg \) and moreover if the action is free, \( g \) is unique. This implies the existence of the function \( \tau \). Its continuity is a consequence of the continuity of \( \nu \).

2. Differentiable manifolds

**DEFINITION.** A real differentiable manifold of dimension \( n \in \mathbb{Z}^+ \) is a pair \((M, \mathcal{A})\) where: i) \( M \) is a locally Euclidean Hausdorff topological space (i.e. \( M \) is Hausdorff and \( \forall x \in M \exists U_x \in \tau \mid \psi : U_x \rightarrow V \cong B(x_0, r) \subseteq \mathbb{R}^n \) is a homeomorphism).

The triple \( c = (U_x, \psi, V) \) is called a chart; \( y \in U_x \) implies that \( \mathbb{R}^n \ni \psi(y) \equiv (x^1(y), \ldots, x^n(y)) \) where \( x^i \) is the \( i \)-th coordinate function of the chart \( c \). One has \( \psi = (x^1, \ldots, x^n) \) and if \( r^i : \mathbb{R}^n \rightarrow \mathbb{R}, (\lambda^1, \ldots, \lambda^n) \mapsto r^i(\lambda^1, \ldots, \lambda^n) = \lambda^i \) is the \( i \)-th coordinate function on \( \mathbb{R}^n \), \( \Rightarrow x^i = r^i \circ \psi \). ii) \( \mathcal{A} = \{c_x\}_{x \in L} \) is an atlas i.e. \( \mathcal{A} \) is a collection of charts that cover \( M(\bigcup_{a \in L} U_a = M) \) and which are pairwise compatible i.e. \( U_a \cap U_b = \emptyset \) or, if \( U_a \cap U_b \neq \emptyset \), the transition functions \( \psi_{ba} = \psi_b \circ \psi_a^{-1}, \psi_a(U_a \cap U_b) \rightarrow \psi_b(U_a \cap U_b) \) are smooth (differentiable or \( C^\infty \)) functions. Graphically we have Fig. 3.
Examples. $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{H}^n$, $S^n$, $S^1 \times \mathbb{R}$, $T^2 = S^1 \times S^1$, Mö (Möbius band), K (Klein bottle). (It is usual the notation $M = M^n$.)

Proposition. If $M$ is a compact space $\Rightarrow$ an atlas for $M$ has at least two charts.

Proof. (Exercise.)

Examples. $S^2$, $T^n = S^1 \times \cdots \times S^1$ ($n$ factors).

Note. A differentiable manifold is a "generalization of $\mathbb{R}^n$", $\mathbb{R}^n$ being considered as a topological space (locally any manifold is topologically equivalent to $\mathbb{R}^n$) but not as a vector space.

Definition. $f: M \to \mathbb{R}$, $x \mapsto f(x)$ is smooth if $f \circ \psi^{-1}_\alpha: V_\alpha \to \mathbb{R}$ is smooth $\forall \psi_\alpha \in \mathfrak{A}$ (Fig. 4).

We write $f \in C^\infty(M, \mathbb{R})$.

Definition. $f: M^m \to N^n$ is smooth if $\forall f \in (C^\infty(N^n, \mathbb{R}))$, $F^*(f) \equiv f \circ F \in C^\infty(M^m, \mathbb{R})$. We write $F \in C^\infty(M^m, N^n)$. Diagram 3 commutes.

Proposition. $(C^\infty(M, \mathbb{R}), +; \mathbb{R}, \cdot)$ is a vector space with $f + g: M \to \mathbb{R}$, $(f + g)(x) = f(x) + g(x)$ and $\lambda \cdot f: M \to \mathbb{R}$, $(\lambda \cdot f)(x) = \lambda f(x)$.

Proof. (Exercise).

Definition. $f: M^m \to N^n$ is a diffeomorphism if it is smooth and has a smooth inverse. One writes $M^m \cong N^n$.

Proposition. $M^m \cong N^n \Rightarrow m = n$. 
NOTE. The category of differentiable manifolds $\mathcal{M}$ is the triple $(\text{Obj}(\mathcal{M}), \text{Mor}(\mathcal{M}), \circ)$ where $\text{Obj}(\mathcal{M}) = \{\text{differentiable manifolds}\}$, $\text{Mor}(\mathcal{M}) = \{C^\infty(M^m, N^n), M^m, N^n \in \text{Obj}(\mathcal{M})\}$ and $\circ$ is the composition of functions. The manifold diffeomorphisms are the isomorphisms of the category.

NOTE. $(\text{Diff}(\mathcal{M}), \circ)$ with $\text{Diff}(\mathcal{M}) = \{F: M \to M, M \cong M\}$ is a group, called the group of diffeomorphisms of the manifold. The identity of the group is $\text{id}_M: M \to M$, $x \mapsto x$.

DEFINITION. An atlas is maximal if it contains all the charts compatible with its charts. It is clear that if $\mathcal{A}$ is an atlas of $M$, $\Rightarrow \mathcal{A}$ determines a maximal atlas $\mathcal{A}^*$.

PROPOSITION. Given $\mathcal{A}$, $\mathcal{A}^*$ is unique.

Proof. (Exercise)
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The pair \((M, \mathcal{A}^*)\) is called a differentiable structure on \(M\).

**Definition.** A tangent vector to \(M\) at \(x \in M\) is a function \(v_x: C^\infty(M, \mathbb{R}) \to \mathbb{R}\) if \(c = (U, \psi, V) \in \mathcal{A}\) and \(x \in U \Rightarrow \exists! (a^1, \ldots, a^n) \in \mathbb{R}^n \mid v_x(f) = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}(f \circ \psi^{-1})|_{\psi(x)}\).

**Definition.** If \(c = (U, \psi = (x^1, \ldots, x^n), V) \in \mathcal{A}\) and \(x \in U\) then \(\frac{\partial}{\partial x^i}|_x\) is the tangent vector to \(M\) at \(x\) \(\forall f \in C^\infty(M, \mathbb{R})\), \(\frac{\partial}{\partial x^i}|_x(f) = \frac{\partial}{\partial x^i}(f \circ \psi^{-1})|_{\psi(x)}\). The \(n\)-uple \((a^1, \ldots, a^n)\) associated with \(\frac{\partial}{\partial x^i}|_x\) is \((0, \ldots, 0, 1, 0, \ldots, 0)\) with 1 in the \(i\)-th place.

**Proposition.** \((T_xM^n, +; \mathbb{R}, \cdot\) where \(T_xM^n = \{\text{tangent vectors to } M^n \text{ at } x\}\), \((v_x + w_x)(f) = v_x(f) + w_x(f)\) and \((\lambda \cdot v_x)(f) = \lambda v_x(f)\) is a real \(n\)-dimensional vector space.

**Proof.** (Exercise)

\(T_xM^n\) is called the tangent space to \(M^n\) at \(x\). It is clear that \(\left\{\frac{\partial}{\partial x^i}|_x\right\}_{i=1}^n\) is a basis of \(T_xM\) i.e. \(v_x = \sum_{i=1}^n a^i(x) \frac{\partial}{\partial x^i}|_x \forall v_x \in T_xM^n\). In the section corresponding to smooth bundles we shall define the smooth vector fields on \(M\) as the sections of the tangent bundle \(TM\) to \(M\) \((TM = \bigsqcup_{x \in M} T_xM = \cup_{x \in M} \{x\} \times T_xM\)\). This means that if \(X\) is a smooth vector field on \(M\), \(\Rightarrow X\) smoothly associates a vector in \(T_xM\) with each \(x \in M\). If \(c\) is a chart in \(\mathcal{A}\), we write \(X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}\) with \(X^i \in C^\infty(U, \mathbb{R})\) and \(X^i(x) = (x, X^i), X_x = \sum_{i=1}^n X^i(x) \frac{\partial}{\partial x^i}\) \(\in T_xM\). If \(f \in C^\infty(M, \mathbb{R})\), \(X(f) \in C^\infty(M, \mathbb{R})\) with \(X(f)(x) = X_x(f)\). Considering the smooth vector fields as functions \(X: C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}), f \mapsto X(f)\), one defines the Lie product \([X, Y]: C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}), f \mapsto [X, Y](f) = X(Y(f)) - Y(X(f))\). It is easy to verify that \((\kappa(M), +, [\cdot, \cdot]; \mathbb{R}, \cdot)\), where \(\kappa(M) = \{\text{smooth vector fields on } M\}\), is a Lie algebra, the Lie algebra of smooth vector fields on \(M\).

**Proposition.** \(v_x \in \text{Der}(C^\infty(M, \mathbb{R}), +, [\cdot, \cdot]; \mathbb{R}, \cdot)\) \((v_x\text{ it is a derivation of the algebra of smooth functions on } M)\) i.e. \(v_x(f + g) = v_x(f) + v_x(g), v_x(\lambda f) = \lambda v_x(f)\) and \(v_x(fg) = v_x(f)g(x) + f(x)v_x(g)\).

**Proof.** (Exercise).

**Definition.** The dual vector space of \(T_xM\), called the cotangent space to \(M\) at \(x\), is the structure \((T^*_xM, +, \cdot; \mathbb{R}, \cdot)\), where \(T^*_xM = \{\omega_x: T_xM \to \mathbb{R}, \omega_x \text{ linear function}\}\) \((= \{1\text{-forms on } T_xM\})\), \((\omega_x + \phi_x)(v_x) = \omega_x(v_x) + \phi_x(v_x)\) and \((\lambda \cdot \omega_x)(v_x) = \lambda \omega_x(v_x)\). (It is usual the notation \(\omega_x(v_x) = (\omega_x, v_x)\)) Clearly \(\dim T^*_xM = \dim T_xM\). In a chart \(c = (U, (x^1, \ldots, x^n), V) \in \mathcal{A}\), \(\{dx^i|_x\}_{i=1}^n\) defined by

\[
dx^i|_x \left( \frac{\partial}{\partial x^j}|_x \right) = \delta^i_j\]
is a basis of $T^*_x M$ for all $x \in U$; if

$$\omega_x = \sum_{i=1}^{n} \omega_i dx^i|_x \in T^*_x M$$

then

$$\langle \omega_x, v_x \rangle = \sum_{i=1}^{n} \omega_i v^i$$

if

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}|_x.$$

As a set, the cotangent bundle to $M$ is

$$T^*M = \coprod_{x \in M} T^*_x M = \bigcup_{x \in M} \{x\} \times T^*_x M.$$

NOTE. Let $V$ be a real $n$-dimensional vector space and let $p \in \{1, \ldots, n\}$. then

$$\Lambda^p(V) = \{p\text{-forms on } V\} \equiv \{f : V \times \cdots \times V \to \mathbb{R}\},$$

$p$-factors

where $f$ is a totally anti-symmetric $p$-linear function. If $\{\phi^i\}_{i=1}^{n}$ is a basis of $V^*$ (the dual space of $V$) then

$$\left\{ \phi^{i_1} \wedge \cdots \wedge \phi^{i_p} \right\}_{1 \leq i_1 < \cdots < i_p \leq n}$$

is a basis of $\Lambda^p(V)$, with

$$\phi^{i_1} \wedge \cdots \wedge \phi^{i_p} = \frac{1}{p!} \sum_{\sigma \in S_p} \phi^{\sigma(i_1)} \otimes \cdots \otimes \phi^{\sigma(i_p)}$$

and $\phi^{i_1} \wedge \cdots \wedge \phi^{i_p}(v(1), \ldots, v(p)) = \frac{1}{p!} \sum_{\sigma \in S_p} \phi^{\sigma(i_1)}(v(1)) \cdots \phi^{\sigma(i_p)}(v(p))$ ($S_p$ is the set of permutations of $p$ elements); if $\{\phi^j\}_{j=1}^{n}$ is dual of $\{e_j\}_{j=1}^{n}$ (basis of $V$), we
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have $\phi^{(i_k)}(v(k)) = v^{(i_k)}$ and 
\[
\phi^{i_1} \wedge \ldots \wedge \phi^{i_p} (v^{(1)}, \ldots, v^{(p)}) = \frac{1}{p!} \sum_{\sigma \in S_p} v^{(i_1)}(1) \ldots v^{(i_p)}(p),
\]
\[
\Lambda^p(V) \ni \phi = \sum_{1 \leq i_1 < \ldots < i_p \leq n} \phi^{i_1} \ldots \phi^{i_p};
\]
\[
\dim(\Lambda^p(V), +; \mathbb{R}, \cdot) = \binom{n}{p}.
\]

One defines $\Lambda^0(V) \equiv \mathbb{R}$.

**Definition.** Let $1 \leq p \leq n, p \in \mathbb{Z}$. $\Lambda^p(T_x M) \equiv \{p\text{-forms on } T_x M\}$. If $c = (U, (x^1, \ldots, x^n), V) \in \mathcal{A}$,
\[
\Lambda^p(T_x M) \ni \omega = \sum_{1 \leq i_1 < \ldots < i_p \leq n} \omega^{i_1} \ldots \omega^{i_p} dx^{i_1}|_x \wedge \ldots \wedge dx^{i_p}|_x.
\]

**Definition.** Let $F: M^m \to N^n$ be a smooth function. The differential of $F$ at $x \in M^m$ is the linear transformation $dF|_x: T_x M^m \to T_{F(x)} N, v_x \mapsto dF|_x(v_x): C^\infty(N^n, \mathbb{R}) \to \mathbb{R}, g \mapsto dF|_x(v_x)(g) = v_x(g \circ F) = v_x \circ F^*(g)$.

**Definition.** Let $(M, \mathcal{A})$ be a differentiable manifold. A partition of unity is a family of smooth functions $\mathcal{F} = \{g_{\alpha}: M \to \mathbb{R}\}_{\alpha \in J}$ such that:

i) $g_{\alpha}(x) \in [0, 1] \forall x \in M, \forall \alpha \in J$;

ii) $\{\text{supp } (g_{\alpha}) \equiv \{x \in M \mid g_{\alpha}(x) \neq 0\}\}_{\alpha \in J}$ is a locally finite closed cover of $M$;

iii) $\sum_{\alpha \in J} g_{\alpha}(x) = 1 \forall x \in M$.

A partition of unity is called subordinate to a cover $\mathcal{U}$ of $M$ if $\forall g_{\alpha} \in \mathcal{F} \exists U_{\beta} \in \mathcal{U} \mid \text{supp } (g_{\alpha}) \subseteq U_{\beta}$. It is easy to verify that if $\mathcal{F} = \{g_{\alpha}\}_{\alpha \in K}$ is a partition of unity in $(M, \mathcal{A})$ subordinate to a cover $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in J}$ of $M$ and $f \in C^\infty(M, \mathbb{R})$, then $f = \sum_{\alpha \in K} f_{\alpha}$ with $f_{\alpha} = g_{\alpha} f \in C^\infty(M, \mathbb{R}), f_{\alpha}(x) = 0 \forall x \notin U_{\beta} \in \mathcal{U}$ for any $U_{\beta} \supset \text{supp } (g_{\alpha})$ and $\forall x \in M$ only a finite number of terms in $\sum_{\alpha \in K} f_{\alpha}(x)$ is different from zero.

**Proposition.** If $(M, \mathcal{A})$ is a differentiable manifold with paracompact topology then any cover $\mathcal{U}$ of $M$ has a subordinate partition of unity $\mathcal{F}$.

**Proof.** See e.g. Kobayashi and Nomizu.

**Note.** Partitions of unity allow to globalize to the whole manifold geometrical objects (vector fields, metrics, connections, etc.) originally defined in a local way i.e. on the open sets of the manifold.
**Definition.** A Lie group is a triple \((G, \ast, \mathcal{A})\) where \((G, \ast)\) is a group, \((G, \mathcal{A})\) is an \(n\)-dimensional differentiable manifold and the composition \(\ast: G \times G \to G, (g, h) \mapsto g \ast h (= gh)\) and inverse \(\text{inv}: G \to G, g \mapsto g^{-1}\) functions are smooth. (Obviously any Lie group is a topological group since any smooth function is a continuous function.)

It is easy to verify that the functions \(L_g: G \to G, h \mapsto L_g(h) \equiv gh\) and \(R_g: G \to G, h \mapsto R_g(h) \equiv hg\) are elements of \(\text{Diff}(G)\). In fact, \(\ast\) smooth implies that \(L_g\) and \(R_g\) are smooth functions \(\forall g \in G\), and \(L_g^{-1} = (L_g)^{-1}, R_g^{-1} = (R_g)^{-1}\). If \(X \in \kappa(G)\) and \(dL_g|_{g'}(X_{g'}) = X_{g g'}\) \((dR_g|_{g'}(X_{g'}) = X_{g'g'})\) \(\forall g, g' \in G\), \(X\) is called a left (right) invariant vector field. Let \(\mathcal{J} \equiv \{X \in \kappa(G) \mid X\) is left invariant\}. \(X \in \mathcal{J} \Rightarrow dL_g|_e(X_e) = X_g\) i.e. \(X_e\) determines \(X\).

**Proposition.** \((\mathcal{J}, +; \mathbb{R}, \cdot)\) is a vector space isomorphic to \((T_eG, +; \mathbb{R}, \cdot)\).

**Proof.** The linearity of \(dL_g|_{g'}\) implies the vector space structure of \(\mathcal{J}\), and it is easy to verify that \(\text{res}: \mathcal{J} \to T_eG, X \mapsto \text{res}(X) \equiv X_e\) is a vector space isomorphism. QED

**Proposition.** \(X, Y \in \mathcal{J} \Rightarrow [X, Y] \in \mathcal{J}\).

**Proof.** See e.g. Bishop and Crittenden.

**Corollary.** \((\mathcal{J}, +, [, ]; \mathbb{R}, \cdot)\) is an \(n\)-dimensional Lie algebra (the Lie algebra of the Lie group \(G\)). (Notice that \(\mathcal{J} \subseteq \kappa(G)\).

**Definition.** Let \(G\) be a Lie group. The exponential function is given by \(\exp: \mathcal{J} \to G, X \mapsto \exp(X) \equiv \gamma^X(1)\) where \(\gamma^X\) is the smooth path \(\gamma^X: \mathbb{R} \to G, t \mapsto \gamma^X(t)\) that satisfies \(\gamma^X(0) = e\) and \(\gamma^X'(t) = X_{\gamma^X(t)}\) (Fig. 5.)

**Figure 5.**
Proposition. It holds: i) \( \exp(tX) = e^{tY}(t) \); ii) \( \exp((t+s)X) = \exp(tX)\exp(sX) \); iii) \( \frac{d}{dt}\exp(tX) = X_{x(t)} \); iv) \( \exp: \mathcal{F} \rightarrow G \) is smooth if \( \mathcal{F} \) is considered a real \( n \)-dimensional differentiable manifold; v) \( d\exp|_0 = \text{id}_G \); vi) \( \exists \mathcal{U}_0 \in \tau_G \mid : a) \exp|_{\mathcal{U}_0} \) is one-to-one, b) \( \exp(\mathcal{U}_0) \in \tau_G \), c) \( \exp|_{\mathcal{U}_0} \): \( U_0 \rightarrow \exp(\mathcal{U}_0) \) is a differentiable manifold diffeomorphism.

Proof. See Bishop and Crittenden.

Definition. An \( n \)-dimensional Riemannian manifold is a pair \((M,g)\) where \( M \) is an \( n \)-dimensional differentiable manifold and \( g \) is a function (called metric or Riemannian metric) given by \( g: M \rightarrow \bigsqcup_{x \in M} \{ \text{non-degenerate positive definite inner products on } T_xM \} \) such that \( X,Y \in \kappa(M) \) then the function \( g(X,Y): M \rightarrow \mathbb{R}, \; x \mapsto g(X,Y)(x) = g_x(X_x,Y_x) \) is smooth on \( M \). \( g \) can also be considered as a function \( g: \kappa(M) \times \kappa(M) \rightarrow C^\infty(M,\mathbb{R}) \). \( g \) is a symmetric bilinear function i.e. \( g_x(v,w) = g_x(w,v) \), \( g_x(\alpha v + \beta w,z) = \alpha g_x(v,z) + \beta g_x(w,z) \); \( \forall \alpha, \beta \in \mathbb{R} \); ii) \( g_x(v,v) \geq 0 \; \forall \; v \in T_xM \). \( g_x(v,v) = 0 \) if \( v = 0 \); iii) if \( g_x(v,w) = 0 \; \forall w \in T_xM \) then \( v = 0 \). \( g \) is \( C^\infty(M,\mathbb{R}) \)-bilinear over the module \( \kappa(M),+; C^\infty(M,\mathbb{R}),. \).

Example. Canonical Riemannian metric on \( \mathbb{R}^n \). \( T_x\mathbb{R} = \mathbb{R}^n \) i.e. \( T_x\mathbb{R}^n \ni v_x = (v_x^1, \ldots, v_x^n) \) and \( g_x^2(v_x,w_x) = \sum_{i=1}^n v_x^i w_x^i \).

If \( c = (U, \psi = (x^1, \ldots, x^n), V) \in \mathcal{A} \), and \( \lambda \in \mathcal{B} \) then \( g_\lambda \) is determined by its matrix

\[
(g_\lambda)_{ij} = g_x \left( \frac{\partial}{\partial x_i} \bigg|_x, \frac{\partial}{\partial x_j} \bigg|_x \right).
\]

(It is usual the notation \( (g_\lambda)_{ij} = g_{ij}(x) \).) Then we have

\[
g_x(X_x, Y_x) = g_x \left( \sum_{i=1}^n X^i(x) \frac{\partial}{\partial x_i} \bigg|_x, \sum_{j=1}^n Y^j(x) \frac{\partial}{\partial x_j} \bigg|_x \right) = \sum_{i=1}^n g_{ij}(x) X^i(x) Y^j(x).
\]

For each \( x \in M \), \( g_x \) induces \( \psi_x: T_xM \rightarrow T_x^*M, v_x \mapsto \psi_x(v_x): T_x^*M \rightarrow \mathbb{R}, \; w_x \mapsto \psi_x(v_x)(w_x) = g_x(v_x,w_x) \) with inverse \( \psi_x^{-1}: T_x^*M \rightarrow T_xM, \; \omega_x \mapsto \psi_x^{-1}(\omega_x)(v_x) = \langle \omega_x, v_x \rangle \) for all \( \omega_x \in T_x^*M \) \( \omega_x = \psi_x(w_x) \) implies that \( g_x(\psi_x^{-1}(\omega_x)(v_x), v_x) = \psi_x(w_x)(v_x) = g_x(w_x,v_x) \), then \( g_x(\psi_x^{-1}(\psi_x(\omega_x)), \omega_x) = 0 \; \forall \; v_x \in T_xM \); so \( \psi_x^{-1}(\psi_x(\omega_x)) = \omega_x \).

\[
T_xM \cong T_x^*M.
\]

l.s.
The matrix of \( \psi_x \) coincides with the matrix of \( g_x \). In fact, defining
\[
\psi_x \left( \frac{\partial}{\partial x^i} \bigg|_x \right) = \sum_{j=1}^{n} (\psi_x)_{ij} dx^i \bigg|_x
\]
we have
\[
\psi_x \left( \frac{\partial}{\partial x^i} \bigg|_x \right) \left( \frac{\partial}{\partial x^j} \bigg|_x \right) = \sum_{j=1}^{n} (\psi_x)_{ij} dx^j \bigg|_x \left( \frac{\partial}{\partial x^k} \bigg|_x \right)
= \sum_{j=1}^{n} (\psi_x)_{ij} \delta^j_k = (\psi_x)_{ik}
= g_x \left( \frac{\partial}{\partial x^i} \bigg|_x \right) \left( \frac{\partial}{\partial x^k} \bigg|_x \right) = g_{ik}(x).
\]

\( \psi_x \) induces an inner product in \( T^*_x M \) given by \( g^*_x : T^*_x M \times T^*_x M \to \mathbb{R} \), \( (\omega_x, \eta_x) \mapsto g^*_x(\omega_x, \eta_x) \equiv g_x(\psi_x^{-1}(\omega_x), \psi_x^{-1}(\eta_x)) \). The matrix of \( g^*_x \) is given by \( (g^*_x)^{ij} \equiv g^*_x(dx^i|_x, dx^j|_x) \). (The usual notation is \( (g^*_x)^{ij} \equiv g^{ij}(x) \).) It is easy to verify that as matrices, \( g^*_x = g_x^{-1} \) and therefore \( \sum_{j=1}^{n} g^{ij}(x)g_{jk}(x) = \delta^i_k \). In fact,
\[
(g^*_x)^{ij} = g_x(\psi_x^{-1}(dx^i|_x), \psi_x^{-1}(dx^j|_x))
= g_x \left( \sum_{k=1}^{n} (\psi_x^{-1})^{ik} \frac{\partial}{\partial x^k} \bigg|_x \right) \sum_{l=1}^{n} (\psi_x^{-1})^{jl} \frac{\partial}{\partial x^l} \bigg|_x \right)
= \sum_{k,l=1}^{n} (\psi_x^{-1})^{ik}(\psi_x^{-1})^{jl} g_x \left( \frac{\partial}{\partial x^k} \bigg|_x \right) \left( \frac{\partial}{\partial x^l} \bigg|_x \right)
= \sum_{k,l=1}^{n} (g_x^{-1})^{ik}(g_x^{-1})^{jl} = \sum_{k=1}^{n} (g_x^{-1})^{ik} \delta^i_k = (g_x^{-1})^{ij}.
\]

Let \([a, b] \subseteq \mathbb{R}\) and let \((M, A)\) be a differentiable manifold. \( \gamma : [a, b] \to M \) is a smooth path in \( M \) if \( \exists \epsilon > 0 \mid \tilde{\gamma} : (a - \epsilon, b + \epsilon) \to M \) is a smooth function between differentiable manifolds \( ((a - \epsilon, b + \epsilon), M) \) and \( \tilde{\gamma}|_{[a,b]} = \gamma \).

Let \( \gamma : [a, b] \to M \) be a smooth path in \((M, A)\) with \( \gamma = \tilde{\gamma}|_{[a,b]} \) and \( \tilde{\gamma} \in C^\infty((a - \epsilon, b + \epsilon), M) \). The tangent vector to \( M \) at \( \gamma(t) \) \( (t \in [a, b]) \) is given by \( d\tilde{\gamma}|_t(\frac{d}{dt}) \). It is usual the notation \( d\gamma|_t(\frac{d}{dt}) = \dot{\gamma}(t) \). Let \( c = (U, \psi = (x^1, \ldots, x^n), V) \in A \) with
\( \tilde{\gamma}(t) \in U \). Writing \( T_{\gamma(t)} M \ni d\tilde{\gamma}|_t \left( \frac{d}{dt}\right)|_t = \sum_{i=1}^n a^i \left( \frac{\partial}{\partial x^i} \right)|_{\tilde{\gamma}(t)} (a^i \in \mathbb{R}) \) we have

\[
d\tilde{\gamma}|_t \left( \frac{d}{dt}\right)|_t (x^j) = \frac{d}{dt}(x^j \circ \tilde{\gamma})(t) = \sum_{i=1}^n a^i \left( \frac{\partial}{\partial x^i} \right)|_{\tilde{\gamma}(t)} (x^j) = a^j
\]

\( i.e. \)

\[
d\tilde{\gamma}_t(t') = \sum_{i=1}^n \frac{d}{dt}(x^i \circ \tilde{\gamma})(t') \frac{\partial}{\partial x^i}|_{\tilde{\gamma}(t')},
\]

where \( t' \in [a, b] \). If \((M, A)\) is a Riemannian manifold \((M, g)\),

\[
g_{\tilde{\gamma}(t')}(\tilde{\gamma}(t'), \tilde{\gamma}(t')) = \sum_{i,j=1}^n \frac{d}{dt}(x^i \circ \tilde{\gamma})(t') \frac{d}{dt}(x^j \circ \tilde{\gamma})(t') g_{\tilde{\gamma}(t')} \left( \frac{\partial}{\partial x^i}|_{\tilde{\gamma}(t')}, \frac{\partial}{\partial x^j}|_{\tilde{\gamma}(t')} \right)
\]

\[
= \sum_{i,j=1}^n g_{ij}(\tilde{\gamma}(t')) \frac{d}{dt}(x^i \circ \tilde{\gamma})(t') \frac{d}{dt}(x^j \circ \tilde{\gamma})(t') > 0
\]

if \( \dot{\gamma}(t') \neq 0 \) and therefore it exists (in \( \mathbb{R} \))

\[
L(\gamma) \equiv \int_a^b \sqrt{g_{\tilde{\gamma}(t')}}(\dot{\gamma}(t'), \dot{\gamma}(t'))
\]

\[
= \int_a^b \sqrt{g_{ij}(\gamma'(t))} \frac{d}{dt}(x^i \circ \tilde{\gamma})(t') \frac{d}{dt}(x^j \circ \tilde{\gamma})(t').
\]

\( L(\gamma) \) is called the length of the path \( \gamma \).

Let \( x, y \in (M, g) \) and let \( \{ \gamma \} \) the family of smooth paths \([a, b] \to M \mid \gamma(a) = x \) and \( \gamma(b) = y \). The (Riemannian) distance between \( x \) and \( y \) is defined by \( d(x, y) \equiv \inf_{\{ \gamma \}} \{ L(\gamma) \} \). It is possible to show (see e.g., Robertson) that the pair \((M, d)\) is a metric space, and that the (metric) topology induced by \( d \) coincides with the topology \( T_M \) of \( M \) as a differentiable manifold. Since any metrizable space is paracompact (see Robertson) it turns out that any differentiable manifold with a Riemannian metric is paracompact.

**Corollary.** If \((M, A)\) is a differentiable manifold whose topology is not paracompact then \((M, A)\) does not admit a Riemannian metric.

**Definition.** Let \((M^m, A)\) be a differentiable manifold, \((N^n, g)\) a Riemannian manifold and \( \varphi: M^m \to N^n \) a smooth function with \( d\varphi|_x: T_x M^m \to T_{\varphi(x)} N^n \) one-to-one
∀ x ∈ M^m i.e. φ is an immersion (⇒ it holds m ≤ n). The reciprocal image of g by φ in M^m or induced metric in M^m by φ is the smooth function φ*(g)(X, Y): M^m → ℝ, x → φ*(g)(X, Y)(x) = φ*(g)_x(X, Y) = g_{φ(x)}(dφ|^x_x(X), dφ|^x_y(Y)).

PROPOSITION. Any differentiable manifold with paracompact topology admits a Riemannian metric.

Proof. Let A = {(U_α, ψ_α, V_α)}_{α ∈ J} be the atlas on M. ψ_α is a diffeomorphism ∀ α ∈ J and ⇒ dψ_α|^x|: T_x M → T_{ψ_α(x)} ℝ^n = ℝ^n is one-to-one. ⇒ ψ_α*(g^c) is the reciprocal image of g^c by ψ_α in U_α i.e. ψ_α*(g^c)(X, Y) ∈ C^∞(U_α, ℝ), with ψ_α*(g^c)(X, Y)(x) = g_ψ_α(x)(dψ_α|^x_x(X), dψ_α|^x_y(Y)) ∀ X, Y ∈ κ(U_α), ∀ x ∈ U_α. ⇒ we have a Riemannian metric for each U_α ⊂ M (i.e. each pair (U_α, ψ_α*(g^c)) is an n-dimensional Riemannian manifold). The topology of M being paracompact implies that it exists a partition of unity F = {g_β}_{β ∈ K} subordinate to the cover U = {U_α}_{α ∈ J} (U is the cover associated with the atlas A). ψ_α*(g^c)(X, Y) is smooth in U_α and ⇒ g_βψ_α*(g^c)(X, Y) is also smooth since supp (g_β) ⊂ U_α and g_β is smooth. ⇒ it is smooth on M the sum \sum_{β ∈ K} g_βψ_α*(g^c)(X, Y) = g(X, Y) and so (M^n, \sum_{β ∈ K} g_βψ_α*(g^c)) is a Riemannian manifold. (See Fig. 6.) QED

\[\text{\textit{Figure 6.}}\]

COROLLARY. A differentiable manifold admits a Riemannian metric iff its topology is paracompact.
3. Fiber bundles

3a. Continuous bundles

**Definition.** A bundle $\xi$ is a triple $\xi = (E, B, \pi)$ where $E, B$ are topological spaces and $\pi: E \to B$ is a continuous and onto function. $E$ and $B$ are respectively called the **total space** and the **base space** of the bundle. $\pi$ is called the **projection**. The usual representation of the bundle is Diagram 4.

$$
\begin{array}{ccc}
E & \xrightarrow{\pi} & B \\
\end{array}
$$

**Diagram 4.**

$F_b = \pi^{-1}\{b\}$ with $b \in B$ is called the fiber over $b$. Clearly, $b \neq b' \Rightarrow F_b \cap F_{b'} = \emptyset$ and $E = \bigcup_{b \in B} F_b, \forall b \in B, F_b \subseteq E$. ($\bigcup_{b \in B} F_b \cong \bigsqcup_{b \in B} F_b$.)

**Definition.** A **bundle homomorphism** between $\xi$ and $\xi'$ is a pair of continuous functions $(f, g)$ such that Diagram 5 holds.

$$
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\pi & \downarrow & \pi' \\
B & \xrightarrow{g} & B' \\
\end{array}
$$

**Diagram 5.**

i.e. $\pi' \circ f = g \circ \pi$. (We write $\xi \cong^{(f, g)} \xi'$.)

**Definition.** Two bundles $\xi$ and $\xi'$ are **isomorphic** if $(f, \text{id}_B)$ is a bundle homomorphism and $f$ is bijective i.e. it holds Diagram 6. (One writes $\xi \cong \xi'$.)

**Definition.** $\xi = (E, \pi)$ is **trivial** if it exists a space $F$ and a function $f$ such that $\xi \cong \eta \equiv (B \times F, B, \pi_1)$. One has Diagram 7. ($\pi_1: B \times F \to B, (b, f) \mapsto \pi_1(b, f) = b.$)
Diagram 6.

Diagram 7.

$\eta$ is called a product bundle. Obviously any product bundle is trivial; it is enough to choose $f = \text{id}_{B \times F}$.

DEFINITION. A sub-bundle of the bundle $\xi = (E, B, \pi)$ is a triple

$$(\pi^{-1}(U), U, \pi|_{\pi^{-1}(U)})$$

with $U \in \tau_B$ i.e. it is the bundle given in Diagram 8.

Diagram 8.
**Definition.** A section of the bundle $\xi = (E, B, \pi)$ is a continuous function $s: B \to E$ such that $\pi \circ s = \text{id}_B$. (Diagram 9.) $\pi \circ s(b) = \pi(s(b)) = \text{id}_B(b) = b \Rightarrow s(b) \in E$.

**Proposition.** If $\xi$ is trivial then it has at least one section.

*Proof.* Let $\sigma: B \to B \times F$, $b \mapsto \sigma(b) = (b, \chi(b))$ with $\chi: B \to F$ continuous. Then $s = f^{-1} \circ \sigma: B \to E$ is continuous, and $\pi \circ s(b) = \pi(f^{-1}(\sigma(b))) = \pi(f^{-1}(b, \chi(b))) = \pi \circ f^{-1}(b, \chi(b)) = \pi_1(b, \chi(b)) = b$. (Diagram 10.) QED

**Diagram 9.**

**Diagram 10.**

**Definition.** A local section of $\xi$ is a section of a sub-bundle of $\xi$.

**Definition.** A fiber bundle $\xi$ is:

i) A locally trivial bundle $\xi' = (E, B, \pi)$ i.e. there exist a space $F$ (fiber of the bundle) and a cover $\{U_\alpha\}_{\alpha \in J}$ of $B$ such that $\forall \alpha \in J$ the sub-bundle $\alpha = \{(\pi^{-1}(U_\alpha), U_\alpha, \pi|_{\pi^{-1}(U_\alpha)}) \cong (U_\alpha \times F, U_\alpha, \pi_1)\}$ i.e. $\forall \alpha \in J$ one has Diagram 11.

The pair $(U_\alpha, \varphi_\alpha)$ is called a local coordinate system of the bundle and $U = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in J}$ is called the atlas of the bundle.

ii) $\forall \alpha, \beta \in J \mid U_\alpha \cap U_\beta \neq \emptyset$ there exists a continuous function $g_{\beta \alpha}: U_\alpha \cap U_\beta \to \text{Aut}(F)$, $b \mapsto g_{\alpha \beta}(b): F \to F$, $f \mapsto g_{\alpha \beta}(b)(f)$ i.e. one considers Diagram 12 one has $\varphi_\beta \circ \varphi_\alpha^{-1}((U_\alpha \cap U_\beta) \times F) \to (U_\alpha \cap U_\beta) \times F$, $(b, f) \mapsto (\varphi_\beta \circ \varphi_\alpha^{-1})(b, f) = (b', g_{\beta \alpha}(b)(f))$ with $b' = b$ since $\pi_1 \circ (\varphi_\beta \circ \varphi_\alpha^{-1}) = \pi_1$. The functions $\{g_{\beta \alpha}\}$ are called transition
functions of the bundle and the subgroup $G$ of $\text{Aut} (F)$ generated by the set

$$\{g_{\beta \alpha} (b)\}_{\alpha, \beta \in J, b \in U_\alpha \cap U_\beta}$$

is called the structure group of the bundle.

Diagram 12.

Let $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta), (U_\gamma, \varphi_\gamma) \in \mathcal{U}$ with $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. \implies \varphi_\beta \circ \varphi_\alpha^{-1} (b, f) = (b, g_{\beta \alpha} (b)(f)) = \varphi_\beta \circ \varphi_\gamma^{-1} \circ \varphi_\gamma \circ \varphi_\alpha^{-1} (b, f) = \varphi_\beta \circ \varphi_\gamma^{-1} (b, g_{\gamma \alpha} (b)(f)) = (b, g_{\beta \alpha} (b)(g_{\gamma \alpha} (b)(f))) \ \forall f \in F. \implies g_{\beta \alpha} (b) = g_{\beta \gamma} (b) \circ g_{\gamma \alpha} (b).

\[ \gamma = \alpha \implies g_{\beta \alpha} (b) = g_{\beta \alpha} (b) \circ g_{\alpha \alpha} (b) \text{ and so } g_{\alpha \alpha} (b) = \text{id}_F \ \forall \alpha \in J, \ \forall b \in U_\alpha; \]

\[ \beta = \alpha \implies \text{id}_F = g_{\alpha \gamma} (b) \circ g_{\gamma \alpha} (b) \text{ and so } g_{\gamma \alpha} (b) = (g_{\alpha \gamma} (b))^{-1} \ \forall \alpha, \gamma \in J | U_\alpha \cap U_\gamma \neq \emptyset, \forall b \in U_\alpha \cap U_\gamma. \]

Let $b \in U_\alpha$; the restriction to $\{b\}$ of the previous diagram is given by Diagram 13 and one has the homeomorphisms $F_b \cong \{b\} \times \{b\}$ and $F_b \cong \{b\} \times F$.

The fiber bundle is denoted by $\xi = (E, B, \pi, F; \mathcal{U})$ and represented by $F \xrightarrow{\pi} B$. 
Another notation is \( \xi = (E, B, \pi, F; G) \) with the representation

\[
\begin{align*}
G & \quad \phi \quad \pi \\
F & \quad \rightarrow \quad E
\end{align*}
\]

NOTES. a) If \((X, \tau)\) is a space the open-compact topology for \(M(X, X)\) is that which has as a sub-basis the set \(S = \{W(K, U) = \{ f \in M(X, X) \mid f(K) \subseteq U \} \mid K \text{ compact and } U \text{ open in } X\}. \) (If \(B\) is a basis of \(\tau\), a sub-basis is a sub-collection \(S\) of \(B \mid \forall B \in B\) is a finite intersection of elements of \(S\).) \(\text{Aut}(X) \subseteq M(X, X)\).

b) To simplify notation the restrictions of \(\varphi_{\alpha}, \varphi_{\beta}\) and \(\pi\) to \(\pi^{-1}(U_{\alpha} \cap U_{\beta})\) were respectively denoted by \(\varphi_{\alpha}, \varphi_{\beta}\) and \(\pi\).

c) If \((G, \ast)\) is a group and \(S \subseteq G\) it is easy to verify that the pair \((\{\text{finite compositions of elements of } S \cup S^{-1}\}, \ast)\) \(\equiv ((S), \ast)\) is a sub-group of \((G, \ast)\) \((S^{-1} = \{g^{-1} \mid g \in S\}).\) \((S), \ast)\) is called the subgroup of \(G\) generated by \(S\).

NOTE. A fiber bundle is a kind of generalization of the cartesian (topological) product of spaces, since locally any fiber bundle is equivalent to a product.

DEFINITION. A fiber bundle \(\xi = (E, B, \pi, F; U)\) is trivial if there exists a homeomorphism \(\varphi: E \rightarrow B \times F \mid \xi' = (E, B, \pi) \cong (B \times F, B, \pi_1)\).

PROPOSITION. Any fiber bundle \(\xi\) has local sections.

Proof. \(\xi\) is locally trivial and by the previous proposition any sub-bundle \(\alpha\) has at least one section.

QED

PROPOSITION. If \(\xi = (E, B, \pi, F; U)\) is a fiber bundle and \(B\) is a contractible space then \(\xi\) is trivial, i.e. there exists a homeomorphism \(\varphi\) such that Diagram 14 holds.

Proof. See e.g. Eguchi et al.

Example. \(F \rightarrow E \rightarrow \mathbb{R}^n\) is a trivial bundle \(\forall E, F\) i.e. \(\exists \varphi \mid \text{it holds Diagram 15}\).

DEFINITION. Let \(K = \mathbb{R}\) or \(\mathbb{C}\) and \(n \in \mathbb{Z}^+\). An \(n\)-dimensional \(K\)-vector bundle is a fiber bundle \(\xi = (E, B, \pi, F; U)\) where \(F = K^n\), each fiber \(F_b\) is an \(n\)-dimensional
topological vector space over $K$ \( (i.e. F_b = V_b^n) \) and if \( b \in U_\alpha \) with \( (U_\alpha, \varphi_\alpha) \in U_\alpha \), \( F_b \xrightarrow{\varphi_\alpha|F_b} \{b\} \times K^n \) is a vector space isomorphism.

(A topological vector space over $K$ is a structure \((V, +; K, \cdot; \tau)\) where \((V, +; K, \cdot)\) is a vector space, \((V, \tau)\) is a topological space and the operations $+$ and $\cdot$ are continuous functions.)

**Proposition.** The transition functions of a vector bundle take values in the linear automorphisms of the fiber.

*Proof.* Consider Diagram 16.

PICTURE OF DIAGRAM 16.
The composition \( \varphi_\beta \mid_{F_\beta} \circ \varphi_\alpha \mid_{F_\alpha}^{-1} : \{ b \} \times K^n \to \{ b \} \times K^n \), \( (b, f) \mapsto (b, g_\beta(b)(f)) \) is a vector space isomorphism. Then \( \forall b \in U_\alpha \cap U_\beta \), \( g_\beta(b) : K^n \to K^n \) is a linear automorphism of \( K^n \) i.e. \( g_\beta(b) \in \text{LAut}(K^n) \cong \text{GL}(n, K) \). \hfill \text{QED}

**Proposition.** Any vector bundle admits the existence of local frames. (A local frame of an \( n \)-dimensional vector bundle \( \xi \) is a collection of \( n \) linearly independent sections of a sub-bundle of \( \xi \).)

**Proof.** Let \( (U, \varphi) \in \mathcal{U} \), \( i \in \{1, \ldots, n\} \) fixed and \( \sigma_i : U \to K^n \) a continuous function with \( \sigma_i(x) \neq 0 \ \forall x \in U \). (Fig. 7 and Diagram 17.)

\[ \sigma_i(x) \text{ is a vector in } K^n. \sigma_i \text{ induces the continuous function } s_i : U \to U \times K^n, \]
\[ x \mapsto s_i(x) = (x, \sigma_i(x)) \text{, and } s_i \text{ induces } e_i : U \to \pi^{-1}(U), \text{ } x \mapsto e_i(x) = \varphi^{-1} \circ s_i(x) = \varphi^{-1}(x, \sigma_i(x)). e_i(x) \in F_x = V_x^n, \text{ in fact, } \pi_{|x^{-1}(U)}(e_i(x)) = \pi_{|x^{-1}(U)} \circ \varphi^{-1}(x, \sigma_i(x)) = \pi_1(x, \sigma_i(x)) = x \text{ i.e. } \pi_{|x^{-1}(U)} \circ e_i = \text{id}|U. \]

Since \( \varphi|_{F_x} : V_x^n \to \{ b \} \times K^n \) is a vector space isomorphism, \( \sigma_i(x) \neq 0 \) implies \( e_i(x) \neq 0 \ \forall x \in U \) i.e. \( e_i \) is a nowhere-zero local section of \( \xi \). Clearly, it is possible to define \( n \) continuous functions \( \sigma_1, \ldots, \sigma_n : U \to K^n \) such that for each \( x \in U \), \( \{ \sigma_i(x) \}_{i=1}^n \) is a set of \( n \) linearly independent vectors. Through the isomorphism \( \varphi|_{F_x} \) one obtains \( n \) linearly independent vectors \( \{ e_i(x) \}_{i=1}^n \) in \( F_x^n \). The collection of continuous functions \( \{ e_i \}_{i=1}^n \) is called a local frame of \( \xi \). Obviously, for each \( (U, \varphi) \in \mathcal{U} \) one can choose a local frame of \( \xi \). \hfill \text{QED}

Let \( s : U_\alpha \to \pi^{-1}(U_\alpha) \) be a local section of \( \xi \). We can write \( s = \sum_{i=1}^n s_i \varepsilon_i \), \( x \mapsto s(x) = \sum_{i=1}^n s^i(x) \varepsilon_i(x) \in V_x^n \) with \( s^i \in M(U_\alpha, K) \) and \( \varepsilon_i = \varphi_\alpha^{-1} \circ s_i \). One has \( s(x) = \sum_{i=1}^n s^i(x) \varphi_\alpha^{-1}(x, \sigma_i(x)) \) and since \( \varphi_\alpha^{-1} \) is linear, \( s(x) = \varphi_\alpha^{-1}(x, \sigma_\alpha(x)) \) with \( \sigma_\alpha(x) = \sum_{i=1}^n s^i(x) \sigma_i(x) \). If \( x \in U_\alpha \cap U_\beta \) we can also write \( s(x) = \varphi_\beta^{-1}(x, \sigma_\beta(x)) \) and then \( (x, \sigma_\beta(x)) = \varphi_\beta \circ \varphi_\alpha^{-1}(x, \sigma_\alpha(x)) = (x, g_\beta(x)(\sigma_\alpha(x))) \) i.e. \( \sigma_\beta(x) = g_\beta(x) \sigma_\alpha(x) \) where we identified \( \text{LAut}(K^n) \) with \( \text{GL}(n, K) \).

**Proposition.** An \( n \)-dimensional \( K \)-vector bundle is trivial if it admits a global frame (i.e. a collection of \( n \) linearly independent global sections).

**Proof.** Exercise.
NOTE. Construction of fiber bundles from “glueing data”. It is easy to verify that the data: i) base space $B$; ii) $\{U_\alpha\}_{\alpha \in J}$: cover of $B$; iii) fiber $F$ and iv) transition functions

$$\{g_{\beta\alpha}: U_\alpha \cap U_\beta \to \text{Aut}(F)\}_{\alpha, \beta \in J, \alpha \neq \beta}$$

determine a unique fiber bundle $\xi = (E, B, \pi, F, \mathcal{U})$. The bundle construction goes through the following steps:

1) One defines the space $(\hat{E}, \tau_{\hat{E}})$ where

$$\hat{E} = \bigsqcup_{\alpha \in J} U_\alpha \times F = \bigcup_{\alpha \in J} U_\alpha \times F \times \{\alpha\} = \left\{ (b, f, \alpha) \right\}_{(b, f, \alpha) \in \hat{E}}$$

and each $U_\alpha \times F \times \{\alpha\}$ has the product topology.

2) Define the relation $r$ in $\hat{E}$: $(b, f, \alpha) r (b', f', \alpha')$ iff $b = b'$ and $f' = g_{\alpha', \alpha}(b)(f)$. $r$ is an equivalence relation $\sim$.

3) $\sim$ in $\hat{E}$ induces the total space of the bundle, $(E, \tau_E)$, where

$$E = \hat{E}/\sim = \left\{ [(b, f, \alpha)] \right\}_{(b, f, \alpha) \in \hat{E}}$$

with $[(b, f, \alpha)] = \{(b, f', \alpha) \in \hat{E} \mid f' = g_{\alpha', \alpha}(b)(f)\}$ and $\tau_E$ is the quotient topology with respect to the projection $p: \hat{E} \to E$, $(b, f, \alpha) \mapsto p(b, f, \alpha) = [(b, f, \alpha)]$, $\tau_E = \{V \in \mathcal{P}(E) \mid p^{-1}(V) \in \tau_{\hat{E}}\}$. (With this topology, $p$ is continuous and open.)

4) One defines the projection $\pi: E \to B$, $[(b, f, \alpha)] \mapsto \pi([(b, f, \alpha)]) \equiv b$.

5) The atlas of the bundle is given by $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in J}$ with $\varphi_\alpha^{-1}: U_\alpha \times F \to \pi^{-1}(U_\alpha)$, $(b, f) \mapsto \varphi_\alpha^{-1}(b, f) \equiv [(b, f, \alpha)]$. One obtains Diagram 18 since $\pi|_{\pi^{-1}(U_\alpha)} \circ \varphi_\alpha^{-1}(b, f) = \pi|_{\pi^{-1}(U_\alpha)}([(b, f, \alpha)]) = b = \pi_1(b, f)$.

Diagram 18.
6) The structure group $G$ of the bundle is generated by the set $S$ of the images of the transition functions.

**Example.** Möbius band ($\mathbb{R} \to \text{Mö} \xrightarrow{\tau} S^1$, non-trivial real 1-dimensional vector bundle) and Cylinder ($\mathbb{R} \to C = S^1 \times \mathbb{R} \to S^1$, real 1-dimensional trivial vector bundle).

**Data:** i) $B = S^1$. ii) $\{U_1, U_2\}$: cover of $S^1$, $U_1 \cup U_2 = S^1$, $U_1 \cap U_2 = A \cup B$ with $A \cap B = \phi$; $U_i \cong \mathbb{R}^1$. (Fig. 8.) iii) $F = \mathbb{R}$. iv) Mö and C distinguish from each other in the transition functions. For the Möbius band one defines $g_{\beta_0}: U_\alpha \cap U_\beta \to \text{LAut}(\mathbb{R})$ with $g_{11}: U_1 \to \text{LAut}(\mathbb{R})$, $b \mapsto g_{11}(b) = \text{id}_\mathbb{R}$, $g_{22}: U_2 \to \text{LAut}(\mathbb{R})$, $b \mapsto g_{22}(b) = \text{id}_\mathbb{R}$, $g_{12}: A \cup B \to \text{LAut}(\mathbb{R})$.

$$b \mapsto \begin{cases} 
\text{id}_\mathbb{R}, & b \in A \\
-\text{id}_\mathbb{R}, & b \in B 
\end{cases}$$

$g_{21} = g_{12}^{-1} = g_{12}$. For the cylinder, $g_{\beta_0}(b) = \text{id}_\mathbb{R} \forall b \in U_1, U_2$. (id$_\mathbb{R} : \mathbb{R} \to \mathbb{R}$, $\lambda \mapsto \lambda$; $-\text{id}_\mathbb{R} : \mathbb{R} \to \mathbb{R}$, $\lambda \mapsto -\lambda$. Notice that

$$\text{LAut}(\mathbb{R}) \cong \text{GL}(1, \mathbb{R}) = (\mathbb{R}^*, \cdot)$$

with $\rho(\text{id}_\mathbb{R}) = 1$, $\rho(-\text{id}_\mathbb{R}) = -1$). Then:

1) $\hat{E} = \left\{ \left( b, x, \alpha \right) \right\}_{\alpha \in \{1, 2\}} \subset \left\{ \left( b, x, 1 \right), \left( c, y, 2 \right) \right\}_{b \in U_1 \cap c \in U_2 \cap x, y \in \mathbb{R}} = U_1 \times R \times \{1\} \cup U_2 \times \mathbb{R} \times \{2\}$. 

(See Fig. 9.)

2) $(b, x, \alpha) r (b', y, \beta)$ iff $b = b'$ and $y = g_{\beta_0}(b)(x)$.

3a) Möbius band: if $b \in A$, $[(b, x, 1)] = \{(b, x, 1), (b, x, 2)\} = \{o_1, o_2\}$; if $b' \in B$, $[(b', x, 1)] = \{(b', x, 1), (b', -x, 2)\} = \{\ast_1, \ast_2\}$, $[(b, y, 2)] = \{(b, y, 2), (b, -y, 2)\}$.
and if $S^1 \ni b \notin U_1 \cap U_2$, $[(b'', x, 1)] = \{(b'', x, 1)\} = \{\varnothing_1\}$ if $b'' \in U_1$ and $[(b''', x, 2)] = \{(b''', x, 2)\} = \{\varnothing_2\}$ if $b''' \in U_2$. A graphical representation of the total space $\mathbb{M} = E/\sim$ is shown in Fig. 10.

Figure 10.

3b) Cylinder: if $b, b' \in A \cup B$, $[(b, x, 1)] = \{(b, x, 1), (b, x, 2)\} = \{\varnothing_1, \varnothing_2\}$, $[(b', x, 1)] = \{(b', x, 1), (b', x, 2)\} = \{\varnothing_1, +\varnothing_2\}$, and if $S^1 \ni b \notin A \cup B$, $[(b'', x, 1)] = \{(b'', x, 1)\} = \{\varnothing_1\}$ if $b'' \in U_1$ and $[(b''', x, 2)] = \{(b''', x, 2)\} = \{\varnothing_2\}$ if $b''' \in U_2$. A graphical representation of the total space $C = E/\sim$ is given in Fig. 11.

4) See graphical representation in Fig. 12.

5) See graphical representation in Fig. 13 for the Möbius band, and in Fig. 14 for the cylinder.

6) $G = \langle (\{\text{id}_\mathbb{R}, -\text{id}_\mathbb{R}\}, \circ) \rangle \cong \mathbb{Z}_2$ for the Möbius band, and $G = \langle (\{\text{id}_\mathbb{R}\}, \circ) \rangle \cong 0$ for the cylinder.

(As smooth bundles, $\mathbb{M}$ and $C$ are 2-dimensional differentiable manifolds, $\mathbb{M}$ is non-orientable and $C$ is orientable, $\mathbb{R}$ is 1-dimensional and $\mathbb{Z}_2$ has dimension zero. The notation for the bundles is $\mathbb{R}^1 \rightarrow \mathbb{M} \rightarrow S^1$ and $\mathbb{R}^1 \rightarrow C \rightarrow S^1$.)

Definition. A principal fiber bundle is a sextet $\xi = (P, B, \pi, G; \mathcal{U}; \psi)$ where:

i) $(P, B, \pi, G; \mathcal{U})$ is a fiber bundle with atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in J}$; for each $\alpha \in J$ it holds the local triviality condition given in Diagram 19.
ii) \((P, G, \psi)\) is a right action of \(G\) over \(P\), \(\psi: P \times G \to P\), \((P, g) \mapsto pg\) (\(pe = p\) and \(p(gh) = (pg)h\) \(\forall p \in P\), \(\forall g, h \in G\)) such that \(\forall \alpha \in J\) one has the right action isomorphism \(\pi^{-1}(U_\alpha), G, \psi|_{\pi^{-1}(U_\alpha) \times G} \cong (U_\alpha \times G, G, \delta)\) i.e. it holds Diagram 20 with \(\delta((x, g), g') = (x, g)g' = (x, gg')\).

\[
\begin{align*}
\pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} U_\alpha \times G \\
\pi|_{\pi^{-1}(U_\alpha)} & \quad \pi_1 \\
U_\alpha & \quad \pi^{-1}(U_\alpha) \times G
\end{align*}
\]

Diagram 19.

\[
\begin{align*}
\pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} U_\alpha \times G \\
\varphi_\alpha \times \text{id}_G & \quad \varphi_\alpha \\
(U_\alpha \times G) \times G & \quad \delta \\
\pi_1 & \quad \pi^{-1}(U_\alpha) \times G
\end{align*}
\]

Diagram 20.

iii) \(\pi\) is an open function (i.e. \(\pi(V) \in \tau_B\) if \(V \in \tau_P\)).

iv) The transition functions of the bundle take values in the image of \(G\) by \(\tilde{\mu}: G \to \text{Aut}(G)\), \(g \mapsto \tilde{\mu}(g) = \mu_g: G \to G\), \(g' \mapsto \mu_g(g') = gg' = \mu(g, g')\), where \((G, G, \mu)\) is the left action (free and transitive) of \(G\) over \(G\). \(\mu\) effective implies that \(\tilde{\mu}: G \to \tilde{\mu}(G), g \mapsto \tilde{\mu}(g) = \tilde{\mu}(g)\) is a group isomorphism.

**Proposition.** For each \(g \in G\) one has Diagram 21.

with \(\psi_g(x) = \psi|_{\pi^{-1}(U_\alpha) \times G}(x, g) = xg\) and \(\delta_g(x, g') = \delta((x, g'), g) = (x, g'g)\).

**Proof.** From diagram in ii) of the definition of principal bundle we have \(\varphi_\alpha(\psi|_{\pi^{-1}(U_\alpha) \times G}(x, g)) = \varphi_\alpha(\psi_g(x)) = \varphi_\alpha \circ \psi_g(x) = \delta(\varphi_\alpha \times \text{id}_G(x, g)) = \delta(\varphi_\alpha(x), g) = \varphi_\alpha \circ \psi_g(x) = \delta_g \circ \varphi_\alpha(x)\) i.e. \(\varphi_\alpha \circ \psi_g = \delta_\alpha \circ \varphi_\alpha\). QED

**Proposition.** \((P \times G, P, \pi_1) \cong (P, B, \pi)\) is a bundle homomorphism i.e. it holds Diagram 22.
Proof. From the local triviality condition $\forall x \in P \exists b \in B$ and $\exists (U, \varphi) \in \mathcal{U}$ with $b \in U$ and $x \in F_b \subseteq \pi^{-1}(U) \cong U \times G$. Then $\pi \circ \psi(x, g) = \pi \circ \psi_b(x) = \pi|_{\pi^{-1}(U)} \circ \psi_b(x) = (\pi_1 \circ \varphi_b) \circ \psi_b(x) = \pi_1 \circ (\varphi_b \circ \psi_b)(x) = \pi_1 \circ (\varphi_b \circ \delta_b)(x) = \pi_1 \circ \delta_b(\varphi_b(x)) = \pi_1 \circ \delta_b(b, g') = \pi_1(\delta_b(b, g')) = \pi_1(b, g'g) = b$ and $\pi \circ \pi_1(x, g) = \pi(\pi_1(x, g)) = \pi(x) = b$. Then $\pi \circ \psi = \pi \circ \pi_1$. QED

Let $(x, g) \in P \times G$. Then $\pi \circ \psi(x, g) = \pi(\psi(x, g)) = \pi(x, g)$ and $\pi(\pi_1(x, g)) = \pi(x)$. So, $\pi(x, g) = \pi(x)$ and then $xg \in F_b$ if $x \in F_b \ (b = \pi(x))$ i.e. $xG \subseteq F_{\pi(x)}$. 
Introduction to the theory of fiber bundles and connections

Figure 14.

Diagram 21.

Diagram 22.
COROLLARY. \( \varphi_2^2(xg) = \varphi_2^2(x)g \) with \( \varphi_2 = \pi_2 \circ \varphi_1 \) (\( \pi_2 \) is the projection in the second factor).

Proof. Let \( x \in \pi^{-1}(U_1) \). Then \( U_1 \times G \ni \varphi_1 \circ \psi_1(x) = \varphi_1(xg) = (\pi_1^{-1}(U_1)(xg), \pi_2 \circ \varphi_1(xg)) = (\pi_1^{-1}(U_1)(x), \pi_2 \circ \varphi_1(x)g) \). This implies that \( \pi_2 \circ \varphi_1(xg) = (\pi_2 \circ \varphi_1(x))g \in G \). QED

PROPOSITION. \( G \) acts from the right free and transitively on the fibers of \( \xi \).

Proof. From the local triviality condition for \( \xi \) as a fiber bundle it turns out that \( \forall \alpha \in J \) and \( \forall b \in U_1, \varphi_1|_{F_b} : F_b \to \{b\} \times G \) is a space homeomorphism. On the other hand, as a topological group \( G \) acts free and transitively on itself through the right action \( \varphi : G \times G \to G, (g, g') \mapsto \varphi(g, g') \equiv gg' \equiv g \mapsto g' = e \) and \( \forall g, g' \in G \). \( g' = g \mapsto g' = e \) and \( \forall g, g' \in G \). \( g' = g \mapsto g' \) is free and transitive. \( \delta \) induces \( \delta_0 : F_b \times G \to G, (x, g') \mapsto \delta_0(x, g') = (\varphi_1|_{F_b})(x, g') \). One has: i) \( \delta_0(x, e) = \varphi_1|_{F_b}^{-1}(\varphi_1|_{F_b}(x, e)) = \varphi_1|_{F_b}^{-1}(\varphi_1|_{F_b}(x)) = x \); ii) \( \delta_0(x, g', g'') = \varphi_1|_{F_b}^{-1}(\varphi_1|_{F_b}(x, g', g'')) = \varphi_1|_{F_b}^{-1}(\varphi_1|_{F_b}(x, g', g'')) = \varphi_1|_{F_b}^{-1}(\varphi_1|_{F_b}(x, g', g'')) = \varphi_1|_{F_b}^{-1}(\varphi_1|_{F_b}(x, g', g'')) \). Then \( \delta_0 \) is a right action of \( G \) on \( F_b \) (the continuity of \( \delta_0 \) is a result of the continuity of \( \delta \) and of \( \varphi_1|_{F_b} \)). Let \( \delta_0(x, g') = x \); then \( \varphi_1|_{F_b}(x) = \delta_0(x, g') \) and then \( g' = e \) i.e. \( \delta_0 \) is a free action. Finally, let \( x, y \in F_b \); then \( \exists g' \in G | \delta_0(x, g') = y \). In fact, let \( y = \varphi_1|_{F_b}^{-1}(\delta_0(\varphi_1|_{F_b}(x), g')) \); this implies that \( \delta_0(\varphi_1|_{F_b}(x), g') = \varphi_1|_{F_b}(y) \) and therefore \( g' = \pi_2(\varphi_1|_{F_b}(y)) = (b, \pi_2(\varphi_1|_{F_b}(y))) = (b, \pi_2(\varphi_1|_{F_b}(x), g')) = (b, \pi_2(\varphi_1|_{F_b}(x), g')) \) and therefore \( g' = \pi_2(\varphi_1|_{F_b}(x))^{-1} \pi_2(\varphi_1|_{F_b}(y)) \) i.e. \( \delta_0 \) is transitive. QED

There exists the function \( \tau : P \times P \to G, (x, y) \mapsto \tau(x, y) = (\pi_2(\varphi_1|_{F_2}(y)) \) where \( P \times P \equiv \{(x, y) \in P \times P | x, y \in F_c \) for some \( c \in B \} = \bigcup_{d \in B} F_d \times F_d \subseteq P \times P, b = \pi(x) \) and \( (U_1, \varphi_1) \in \mathcal{U} \) with \( b \in U_1 \).

It is usual to represent a principal fiber bundle in the form \( G \to P \to B \).

PROPOSITION. In a principal fiber bundle \( G \to P \to B \), \( B \cong P/G \).

Proof. Let \( P/G \) be the orbit space defined by the projection \( p : P \to P/G, x \mapsto p(x) = xG \) and let \( \varphi : B \to P/G, b \mapsto \varphi(b) \equiv p(x), x \in F_b \). Let \( x' \in F_b \); \( \exists g \in G | x' = xg \) and so \( p(x') = x'G = (xg)G = x(gG) = xG = p(x) \) i.e. \( \varphi \) is well defined. \( b = \pi(x) \mapsto \varphi \circ \pi = p \) and from \( p \) onto it turns out \( \varphi \) onto. Let \( b, b' \in B \) with \( b \neq b' \) (\( \Rightarrow F_b \cap F_{b'} = \emptyset \)) and let \( x \in F_b \) and \( x' \in F_{b'} \). \( \varphi(b) = \varphi(b') \Rightarrow p(x) = p(x') \) i.e. \( xG = x'G \) which is a contradiction. Therefore \( b \neq b' \Rightarrow \varphi(b) \neq \varphi(b') \) i.e. \( \varphi \) is one-to-one and therefore a bijection. Then \( \pi = \varphi^{-1} \circ p \). We have \( \Rightarrow \) Diagram 23 with \( \pi \) and \( p \) open, continuous and onto, and \( \varphi \) bijective. From the appendix it turns out that \( \varphi \) is a homeomorphism. QED
**Definition.** A principal fiber bundle \((P, B, \pi, G; \mathcal{U}; \psi)\) is trivial if it is trivial as a fiber bundle \((P, B, \pi, G; \mathcal{U})\).

**Proposition.** A principal fiber bundle \(\xi = (P, B, \pi, G; \mathcal{U}; \psi)\) is trivial iff it has a section.

**Proof.** \(\Rightarrow\): If \(\xi\) is trivial then it is trivial as a fiber bundle and any trivial fiber bundle has a section. \(\Leftarrow\): Let \(s : B \to P\) be a section of the bundle \(\xi\). Then, \(\forall b \in B, s(b) \in F_b\). From the free and transitive action of \(G\) on the fibers of \(P\), \(\forall x \in F_b \exists! g \in G \mid x = s(b)g = \psi_g(s(b)) = \psi_g \circ s(b)\). Let \(\varphi : P \to B \times G, x \mapsto \varphi(x) \equiv (\pi(x), g)\) with \(s(b)g = x, b = \pi(x)\). Let \(x' \neq x, x' \in F_b \Rightarrow \varphi(x') = (b, g') \neq (b, g)\) and if \(x' \notin F_b \Rightarrow b' = \pi(x') \neq b\) and therefore \(\varphi(x') \neq \varphi(x)\) i.e. \(\varphi\) is one-to-one. Let \((b, g) \in B \times G\); then \(\varphi(s(b)g) = (\pi(s(b)g), g) = (b, g)\) i.e. \(\varphi\) is onto and therefore a bijection with \(\varphi^{-1}(b, g) = \psi(s(b), g)\). The continuity of \(\varphi^{-1}\) results from that of \(s\) and \(\psi\), and the continuity of \(\varphi\) results from that of \(\pi\) and \(\tau\). Then \(P \cong B \times G\). QED

**Definition.** The principal fiber bundles \(\xi = (P, B, \pi, G; \mathcal{U}; \psi)\) and \(\xi' = (P', B', \pi', G'; \mathcal{U}'; \psi')\) are homomorphic if it exists a triple of functions \((h, f, g)\) such that one has: i) The bundle homomorphism \((P, B, \pi) \xymatrix{ \sim \ar[rr]^-{(f, g)} & & (P', B', \pi')},\) equivalent to Diagram 24.

**Diagram 23.**

**Diagram 24.**
and ii) The right action homomorphism \((P, G, \psi)^{(f, h)} \cong (P', G', \psi')\), equivalent to Diagram 25.

One writes \((P, B, \pi, G; \mathcal{U}; \pi)^{(h, f, g)} \cong (P', B', G'; \mathcal{U}'; \psi')\). Both conditions are combined in Diagram 26.

In particular, \(\xi\) and \(\xi'\) (with \(B' = B\) and \(G' = G\)) are isomorphic if \(\xi^{(\text{id}_G, f, \text{id}_B)} \cong \xi'\) and \(P \cong P'\) is a homeomorphism i.e. one has Diagram 27.
Examples of constructions of principal fiber bundles from “glueing data”

Double covering of the circle or “boundary of the Möbius band” \( \mathbb{Z}_2 \to \partial \tilde{M} \xrightarrow{\pi} S^1 \) (non trivial bundle) and “boundary of the cylinder” \( \mathbb{Z}_2 \to \partial C \xrightarrow{\pi} S^1 \) (trivial bundle).

Data: i) \( B = S^1 \). ii) \( \{U_1, U_2\} \): cover of \( S^1 \). iii) \( F = \mathbb{Z}_2 = \{e, a\} \). \( (\mathbb{Z}_2, *) \) is a topological group with the products \( e * e = e, e * a = a * e = a, a * a = e \). Let \( \text{Aut}(\mathbb{Z}_2, *) = \{\text{id}_{\mathbb{Z}_2}, \mu_e, \mu_a\} \). \( \mu_e : \mathbb{Z}_2 \to \text{Aut}(\mathbb{Z}_2), e \mapsto e, a \mapsto a \). Let \( \mu_a : \mathbb{Z}_2 \to \text{Aut}(\mathbb{Z}_2), e \mapsto \mu_e = \text{id}_{\mathbb{Z}_2}, a \mapsto \mu_a = e \). Then \( \mu_a(\mathbb{Z}_2) = \text{Aut}(\mathbb{Z}_2) \) and \( \mu_e(\mathbb{Z}_2) \cong \text{Aut}(\mathbb{Z}_2) \). iv) \( \partial \tilde{M} \) and \( \partial C \) distinguish from each other in the transition functions.

For \( \partial \tilde{M} \) one has \( g_{11} : U_1 \to \text{Aut}(\mathbb{Z}_2), b \mapsto \text{id}_{\mathbb{Z}_2}, g_{22} : U_2 \to \text{Aut}(\mathbb{Z}_2), b \mapsto \text{id}_{\mathbb{Z}_2} \) if \( b \in A \) and \( b \mapsto \varphi \) if \( b \in B \), \( g_1 = g_2^{-1} \). \( (U_1 \cap U_2 = A \cup B, A \cap B = \emptyset) \) as for the cases \( \partial \tilde{M} \) and \( C \). The structure group is \( \{\{\text{id}_{\mathbb{Z}_2}, \varphi\}\}, \circ \cong \mathbb{Z}_2 \). For \( \partial C \), \( g_{30} (b) = \text{id}_{\mathbb{Z}_2} \forall b \in S^1 \). Then the structure group is \( \{\{\text{id}_{\mathbb{Z}_2}\}, \circ \} = 0 \). The constructions of \( \partial \tilde{M} \) and \( \partial C \) are similar to the corresponding constructions of \( \tilde{M} \) and \( C \), and we leave them as an exercise. The graphical representations of the total spaces are given in Fig. 15 and for the bundles themselves one has the representations in Fig. 16.

(As smooth bundles, \( \partial \tilde{M} \) and \( \partial C \) are 1-dimensional differentiable manifolds.)

The notation for the bundles is \( (\mathbb{Z}_2)^0 \to (\partial \tilde{M})^1 \xrightarrow{\pi} S^1 \) and \( (\mathbb{Z}_2)^0 \to (\partial C)^1 \xrightarrow{\pi} S^1 \).

**DEFINITION.** Associated fiber bundle. Let \( \xi = (P, B, \pi, G; U = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in J}; \psi) \) be a principal bundle and let \( F \) be any space on which \( G \) acts from the left i.e. there exists a continuous function \( \varphi : G \times F \to F \), \( (g, f) \mapsto \varphi(g, f) = g f \) such that \( \varphi_e = \text{id}_F \) and \( \varphi_{g'g} = \varphi_g \circ \varphi_{g'} \). (In particular, \( F \) can be a topological vector space.) Then \( (P \times F, G, \rho) \) with \( \rho((x, f), g) = (\psi_y(x), \varphi_{g^{-1}}(f)) \) is a right action of \( G \) on \( P \times F \). In fact, \( \rho((x, f), e) = (\psi_y(x), \varphi_e(f)) = (x, f) \) and \( \rho((x, f), gh) = (x, f)(gh) = (x, g(h^{-1}(g^{-1}f))) = (x, g^{-1}f) = h^{-1}g^{-1}h(x, f) = h^{-1}g^{-1}h(x, f) = \rho_h \circ \rho_g \). The continuity of \( \rho \) results from that of \( \psi \) and \( \varphi \). Let \( P \times F/G = \{[(x, f)] \equiv (x, f)G \}_{(x, f) \in P \times F} \) be the orbit space associated with the action \( \rho \). \( [(x', f')] = [(x, f)] \) if \( [(x', f')] \) and \( [(x, f)] \) as subsets of \( P \times F \) have a non-empty intersection i.e it exists \( g \in G : (x', f') = (xg, g^{-1}f) \). \( P \times F/G \) has the quotient topology with respect to the projection \( \pi_G : P \times F \to P \times F/G, \pi_G(x, f) = [(x, f)] \). It is then easy to verify that if one defines \( p([(x, f)]) = \pi(x) \), the Diagram 28 commutes i.e \( p \circ \pi_G = \pi \circ \pi \). In fact, \( p \circ \pi_G(x, f) = p([(x, f)]) = \pi(x) = \pi(\pi_1(x, f)) = p([(x, f)]) = \pi \circ \psi_y(x) \) for any \( g \in G \). Also, since \( \pi(x) \) does not depend on \( f \), \( p([(x, f)]) = p([(x, f')]) \forall f, f' \in F \) and then \( p^{-1}((b)) = \{[(x, f)] \}_{F \in F} \) for any \( x \in F_b \). Therefore, as sets, \( p^{-1}((b)) \cong F_b \). Let \( (U_\alpha, \varphi_\alpha) \in U \) and let \( t \) be the homeomorphism \( t : (U_\alpha \times G) \times F \to U_\alpha \times (G \times F), (b, g, f) \mapsto t((b, g), f) = (b, (g, f)) \). Define \( \phi_\alpha \) through Diagram 29.
\[ \partial \mathcal{M}^c = U \quad \partial C = U \]

FIGURE 15.

\[ F_b = [x, y] \]

FIGURE 16.

It is easy to verify that \( \pi_{\rho}(\pi^{-1}(U_\alpha) \times F) = p^{-1}(U_\alpha) \). In fact,

\[ p^{-1}(U_\alpha) = p^{-1} \left( \bigcup_{b \in U_\alpha} \{b\} \right) = \bigcup_{b \in U_\alpha} p^{-1}(\{b\}) = \bigcup_{b \in U_\alpha} \{[(x, f)]\}_{f \in F} \]
We shall now prove that Diagram 30 commutes and that $\phi_\alpha$ is a homeomorphism. From the definition of $\phi_\alpha$, $\phi_\alpha \circ \pi_\rho(x, f) = \phi_\alpha(\pi_\rho(x, f)) = \phi_\alpha(\{(x, f)\}) = ((\text{id}_{U_\alpha} \times \varphi) \circ t)(\varphi_\alpha \times \text{id}_F)(x, f) = (\text{id}_{U_\alpha} \times \varphi) \circ (\varphi_\alpha \times \text{id}_F)(x, f) = (\text{id}_{U_\alpha} \times \varphi) \circ t((\pi(x), \varphi_\alpha(x)), f) = \text{id}_{U_\alpha} \times \varphi(\pi(x), (\varphi_\alpha(x), f)) = (\pi(x), \varphi(\varphi_\alpha(x), f))$ and then $\pi_1 \circ \phi_\alpha(\{(x, f)\}) = \pi(x) = p|_{p^{-1}(U_\alpha)}((x, f))$ i.e. $\pi_1 \circ \phi_\alpha = p|_{p^{-1}(U_\alpha)}$. Let $A$ be a non-empty subset of $U_\alpha$, since $p|_{p^{-1}(U_\alpha)}$ is onto, $p|_{p^{-1}(U_\alpha)}(A) = \phi_\alpha^{-1}(\pi_1^{-1}(A)) \neq \phi$ and then $\phi_\alpha$ is onto. Let $[(x, f)] \neq [(x', f')]$ with $\phi_\alpha([(x, f)]) = \phi_\alpha([(x', f')])$ i.e.
(\pi(x), \varphi_\alpha^2(x)f) = (\pi(x'), \varphi_\alpha^2(x')f'). If x' \in F_{\pi(x)}, then \exists g \in G \mid x' = xg and then \varphi_\alpha^2(xg)f' = \varphi_\alpha^2(x)gf' which implies (multiplying by (\varphi_\alpha^2(x))^{-1}) f = gf' i.e. f' = g^{-1}f. Then [(x, f)] = [(x', f')], which is a contradiction. So, \phi_\alpha is one-to-one and therefore a bijection. The continuity of \phi_\alpha and \phi_\alpha^{-1} results from that of \pi, \varphi, and \varphi_\alpha^2. Then the structure \((P \times F, G, \pi, \varphi, \varphi_\alpha)\) is a fiber bundle which, for given \varphi, is called the fiber bundle associated with the principal bundle \(\xi\). The usual notation for the bundle is \(F \rightarrow P \times F / G \rightarrow B\).

3b. Smooth bundles

Definitions and theorems for real smooth bundles are obtained from those for continuum bundles through the replacements:

- topological space \rightarrow real differentiable manifold,
- continuous function \rightarrow smooth function,
- homeomorphism \rightarrow diffeomorphism,
- topological group \rightarrow Lie group,
- \(G\)-space \rightarrow \(G\)-manifold.

NOTE. Relation between the dimensions of the differentiable manifolds in a smooth fiber bundle.

Let \(F^n \rightarrow E^n \rightarrow B^n\) be a smooth fiber bundle. The local triviality condition is represented by Diagram 31, where \(U\) and \(\pi^{-1}(U)\) are respectively open submanifolds of \(B^n\) and \(E^n\). (If \((M^n, A)\) is an \(n\)-dimensional differentiable manifold and \(W \in \tau_{M^n}\), then \((W, A_W)\) is a differentiable manifold of the same dimension with atlas \(A_W = \{\tilde{c}_\gamma = (U_\gamma \cap W, \psi_\gamma |_{U_\gamma \cap W}, \psi_\gamma(U_\gamma \cap W))\}_{\gamma \in K}\) if \(A = \{(U_\gamma, \psi_\gamma, \psi_\gamma(U_\gamma))\}_{\gamma \in K}\). One writes \(W \subseteq M^n\). If \(U \subseteq B^n\) then \(\dim U = n_B\) and so, \(\dim U \times F^n = n_B + n_F\). \(\pi^{-1}(U) \subseteq E^n\) \(\Rightarrow\ \dim \pi^{-1}(U) = n_E\) and from the fact that \(\varphi\) is a diffeomorphism, \(n_E = n_B + n_F\).
Diag. 31.

Examples of smooth bundles

1. Tangent bundle. (Real $n$-dimensional vector bundle.) Let $M$ be an $n$-dimensional differentiable manifold. Defining the set $TM$ as

$$TM \equiv \bigsqcup_{x \in M} \{ x \} \times T_x M = \bigsqcup_{x \in M} F_x$$

it is easy to verify that the differentiable structure in $M$ induces a $2n$-dimensional differentiable structure in $TM$. In fact,

$$TM = \left\{ (x, v_x) \right\}_{x \in M}$$

and if $c = (U, \psi = (x^1, \ldots, x^n), V) \in \mathcal{A}$ (the atlas of $M$) and $x \in U$, then $\hat{\psi}:TU \to \hat{\psi}(TU) \subset \mathbb{R}^{2n}$, $(x, v_x) \mapsto \hat{\psi}(x, v_x) \equiv (x^1(x), \ldots, x^n(x), a^1, \ldots, a^n)$ $(v_x = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}|_x)$ is a homeomorphism and $\hat{A} \equiv \{ \hat{c} \equiv (TU, \hat{\psi}, \hat{\psi}(TU)) \}_{c \in \mathcal{A}}$ is a set of compatible charts. One defines the projection $\pi:TM \to M$, $(x, v_x) \mapsto \pi(x, v_x) \equiv \pi_1(x, v_x) = x$, with $F_x = \{ x \} \times T_x M = \pi^{-1}\{ x \}$. $F_x$ is a vector space with $(x, v_x) + (x, w_x) = (x, v_x + w_x)$ and $\lambda(x, v_x) = (x, \lambda v_x)$. Clearly, $\pi$ is a smooth function between the manifolds $(TM, \hat{A})$ and $(M, \mathcal{A})$. The fiber of the bundle is $F = \mathbb{R}^n$ and its diagrammatic representation is $\mathbb{R}^n \to TM \to M^n$.

2. Cotangent bundle. (Real $n$-dimensional vector bundle.)

$$T^*M \equiv \bigsqcup_{x \in M} T^*_x M = \bigsqcup_{x \in M} \{ x \} \times T^*_x M = \bigsqcup_{x \in M} F^*_x.$$ 

$T^*M$ is a $2n$-dimensional differentiable manifold:

$$T^*M = \left\{ (x, \omega_x) \right\}_{x \in M, \omega_x \in T^*_x M}$$
and if \( c = (U, \psi = (x^1, \ldots, x^n), V) \in \mathcal{A} \) and \( x \in U \), \( \phi: T^*U \to \phi(T^*U) \subset \mathbb{R}^{2n} \),

\[
(x, \omega_x) \mapsto \phi(x, \omega_x) = (x^1(x), \ldots, x^n(x), \omega_1, \ldots, \omega_n)
\]

\( (\omega_x = \sum_{i=1}^{n} \omega_i \frac{dx^i}{x}) \) is a homeomorphism, and the induced charts on \( T^*M \) are compatible. One defines the projection \( \pi^*: T^*M \to M, (x, \omega_x) \mapsto \pi^*(x, \omega_x) = x \) with \( F_x^* = \{ x \} \times T^*_x M = \pi^{-1}(\{ x \}) \), \( \pi^* \) is smooth. \( \mathbb{R}^n \) is the fiber of the bundle and its representation is \( \mathbb{R}^n - T^*M \to M^n \).

3. Let \( M \) be a differentiable manifold. One defines the product bundle \( \xi = (M \times \mathbb{R}, M, \pi_1) \) (Diagram 32) with \( \pi_1(x, t) = x \) and then \( \pi^{-1}(\{ x \}) = \{ x \} \times \mathbb{R} \). \( f \in C^\infty(M, \mathbb{R}) \) induces the section \( s_f \in C^\infty(M \times \mathbb{R}), s_f: M \to M \times \mathbb{R}, x \mapsto s_f(x) = (x, f(x)) \). Clearly \( (C^\infty(M \times \mathbb{R}), +; \mathbb{R}, \cdot) \) is an infinite dimensional vector space with the operations \( (s_f + s_g)(x) = (x, f(x) + g(x)) \) and \( (\lambda s_f)(x) = (x, \lambda f(x)) \). In this way, the smooth functions on a differentiable manifold can be considered as the sections of the trivial bundle \( M \times \mathbb{R} \).

\[
\begin{array}{ccc}
M \times \mathbb{R} & \xrightarrow{\pi_1} & M \\
\uparrow & & \downarrow \\
& & \\
& & \\
\end{array}
\]

Diagram 32.

4. Bundle of frames of a differentiable manifold. Let \((M^n, \mathcal{A})\) be an \( n \)-dimensional differentiable manifold and let \( P(M^n) \) be the set

\[
P(M^n) \equiv \bigcup_{x \in M^n} P(T_x M^n) = \bigcup_{x \in M^n} \{ x \} \times P(T_x M^n) = \left\{ (x, r_x) \right\}_{r_x \in P(T_x M^n)}.
\]

(Given an \( n \)-dimensional vector space \( V \), the set of frames of \( V \) is the set of ordered basis of \( V \) i.e the set given by \( P(V) = \{(v_1, \ldots, v_n) \mid \{v_i\}_{i=1}^n \text{ is a basis of } V\} \).)

Define the projection \( \pi: P(M^n) \to M^n, (x, r_x) \mapsto \pi(x, r_x) = x \). \( P(M^n) \) is given the pre-image topology with respect to \( \pi: \tau_{P(M^n)} = \{ P(V) = \pi^{-1}(V) \}_{V \in \tau_{M^n}} \). \( \pi \) is then continuous. We now give to \( P(M^n) \) and \( (n + n^2) \)-dimensional differentiable structure such that \( \pi \) becomes a smooth function. (Under these circumstances, \( \xi^n = (P(M^n), M^n, \pi) \) is a smooth bundle.) Let \( x \in M^n \) and \( c = (U, \psi = (x^1, \ldots, x^n), \psi(U)) \in \mathcal{A} \) such that \( x \in U \). Let \( e: U \to \pi^{-1}(U), x \mapsto e(x) = (x, e_x) \) with

\[
e_x = ((e_x)_1, \ldots, (e_x)_n) \equiv \left( \frac{\partial}{\partial x^1} \bigg|_x, \ldots, \frac{\partial}{\partial x^n} \bigg|_x \right).
\]
Then, \( \hat{\psi}: P(U) \rightarrow \hat{\psi}(P(U)) \subset \mathbb{R}^n \times \text{GL}(n, \mathbb{R}), (x, r_x) \mapsto \hat{\psi}(x, r_x) \equiv (x^1(x), \ldots, x^n(x), g(r_x)) \) with \( g(r_x) \in \text{GL}(n, \mathbb{R}) \) such that \( r_x = e_x g(r_x) \) i.e.

\[
((r_x)_1, \ldots, (r_x)_n) = \left( \sum_{j_1=1}^{n} (e_{x})_{j_1}(g(r_x))_{j_1}^{j_1}, \ldots, \sum_{j_n=1}^{n} (e_{x})_{j_n}(g(r_x))_{j_n}^{j_n} \right)
\]

is an homeomorphism and the set \( \{ \hat{c} = (P(U), \hat{\psi}, \hat{\psi}(P(U))) \} \in \hat{A} = \hat{A} \) is an atlas on \( P(M^n) \). (In \( g^j_i \), \( i \) (\( j \)) denotes a row (column). The general linear group \( \text{GL}(n, \mathbb{R}) \) is a matrix Lie group which as a smooth structure is an open submanifold of the \( n^2 \)-dimensional manifold of \( n \times n \) real matrices \( \mathbb{R}(n) \).) It is then clear that \( e \) is a local section of the bundle \( \mathcal{E} \) since it is a section of the sub-bundle \( (P(U), U, 1_{x^{-1}}(U)) \).

In fact, \( \pi|_{P(U)} \circ e: U \rightarrow U \), \( y \mapsto \pi|_{P(U)}(e(y)) = \pi|_{P(U)}(y, e_y) = y \) i.e. \( \pi|_{P(U)} \circ e = \text{id}_U \).

The local triviality condition given by Diagram 33

![Diagram 33.](attachment:image.png)

with \( \varphi(x, r_x) \equiv (x, g(r_x)) \) makes \( \mathcal{E} \equiv (P(M^n), M^n, \pi, \text{GL}(n, \mathbb{R}); U) \) with \( U = \{(U, \varphi)\} \) a smooth fiber bundle. We leave it as an exercise to verify that the action \( \mu: P(M^n) \times \text{GL}(n, \mathbb{R}) \rightarrow P(M^n), ((x, r_x), g) \mapsto \mu((x, r_x), g) \equiv (x, r_x g) \) turns \( \mathcal{E} \) into a principal fiber bundle \( \xi = (P(M^n), M^n, \pi, \text{GL}(n, \mathbb{R}); U; \mu) \), called the bundle of frames of the differentiable manifold \( M^n \).

Finally, let us briefly discuss the transition functions and the structure group of the bundle. Let \( (U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta) \in U \) with \( U_\alpha \cap U_\beta \neq \phi \) and let \( (x, g) \in (U_\alpha \cap U_\beta) \times \text{GL}(n, \mathbb{R}) \). Then \( \phi_\beta \circ \phi_\alpha^{-1}(x, g) = \phi_\beta(\phi_\alpha^{-1}(x, g)) = \phi_\beta(x, e^{(\alpha)}_x g) = (x, h) = (x, g_{\beta\alpha}(x) g) \) with \( h \in \text{GL}(n, \mathbb{R}) \) such that \( e^{(\beta)}_x g = e^{(\beta)}_x h \).

\[
e^{(\alpha)}_x = \left( \frac{\partial}{\partial x^{1}_\alpha|_x}, \ldots, \frac{\partial}{\partial x^{n}_\alpha|_x} \right),
\]

\[
e^{(\beta)}_x = \left( \frac{\partial}{\partial x^{1}_\beta|_x}, \ldots, \frac{\partial}{\partial x^{n}_\beta|_x} \right),
\]
and the chain rule
\[
\frac{\partial}{\partial x^i_{\beta}} \bigg|_x = \sum_{k=1}^{n} \frac{\partial}{\partial x^k_{\alpha}} \bigg|_x \frac{\partial (x^k_{\alpha} \circ \psi^{-1}_\alpha)}{\partial (x^i_{\beta} \circ \psi^{-1}_\beta)}_{\psi_\alpha(x)} \Rightarrow h^i_s = \sum_{i=1}^{n} \frac{\partial (x^i_{\beta} \circ \psi^{-1}_\beta)}{\partial (x^i_{\alpha} \circ \psi^{-1}_\alpha)}_{\psi_\alpha(x)} g^i_s
\]
and therefore
\[
(g_{\beta\alpha}(x))_{\beta} = \frac{\partial (x^i_{\beta} \circ \psi^{-1}_\beta)}{\partial (x^i_{\alpha} \circ \psi^{-1}_\alpha)}_{\psi_\alpha(x)}.
\]

The structure group is then given by
\[
\left( \left\{ \frac{\partial (x^i_{\beta} \circ \psi^{-1}_\beta)}{\partial (x^i_{\alpha} \circ \psi^{-1}_\alpha)}_{\psi_\alpha(x)} \right\}_{\alpha,\beta \in \mathcal{J}} \right)_{x \in U_\alpha \cap U_\beta}^{g_r} \subset GL(n, \mathbb{R}).
\]

The representation of the bundle is $GL(n, \mathbb{R}) \to P(M^n) \to M^n$.

5. Associated bundles to the bundle of frames. Let $\xi = (P(M^n), M^n, \pi, GL(n, \mathbb{R}); U; \mu)$ be the bundle of frames of the differentiable manifold $M^n$ and let $\sigma: GL(n, \mathbb{R}) \to GL(m, \mathbb{R})$ be a Lie group homomorphism. Then $(GL(n, \mathbb{R}), \mathbb{R}^m, \tilde{\sigma})$ given by $\tilde{\sigma}: GL(n, \mathbb{R}) \times \mathbb{R}^m \to \mathbb{R}^m$, $(g, x) \mapsto \tilde{\sigma}(g, x) = \sigma(g)x$ is a left $GL(n, \mathbb{R})$-manifold i.e. a left action of $GL(n, \mathbb{R})$ on $\mathbb{R}^m$. In fact, $\tilde{\sigma}(e, x) = \sigma(e)x = e'x = x$ and $\tilde{\sigma}(hg, x) = \sigma(hg, x) = \sigma(h)\sigma(g)x = \tilde{\sigma}(h, \tilde{\sigma}(g, x))$. Then the structure $(\sigma(M^n), M^n, p, \mathbb{R}^n; \tilde{U})$, where:

\[
\sigma(M^n) \equiv P(M^n) \times \mathbb{R}^m / GL(n, \mathbb{R}) = \left( \bigcup_{x \in M^n} P(T_x M^n) \right) \times \mathbb{R}^m / GL(n, \mathbb{R})
\]

\[
\equiv \left\{ (x, r_x) \right\}_{x \in M^n} \times \mathbb{R}^m / GL(n, \mathbb{R})
\]

\[
\equiv \left\{ ((x, r_x), r) \right\}_{x \in M^n \atop r \in P(T_x M^n) / GL(n, \mathbb{R})} = \left\{ ((x, r_x), r)GL(n, \mathbb{R}) \right\}_{x \in M^n \atop r \in \mathbb{R}^m}
\]

\[
\equiv \left\{ [[(x, r_x), r]] \right\}_{x \in M^n \atop r \in \mathbb{R}^m} = \left\{ ((x, r_xg), (\sigma(g))^{-1}r) \right\}_{x \in M^n \atop g \in GL(n, \mathbb{R})}
\]

with $[((x', r'_{x'}), r')] = [[[x, r_x), r]]$ iff $\exists g \in GL(n, \mathbb{R}) \mid ((x', r'_{x'}), r') = ((x, r_xg),$
(\sigma(g))^{-1}r) \Leftrightarrow x', r' = rzg, and r' = (\sigma(g))^{-1}r; p: \sigma(M^n) \rightarrow M^n, \pi(x, rz) = x and therefore

\[ p^{-1}\{x\} = \left\{\left((x, rz), r\right)\right\}_{r \in \mathbb{R}^m} = \left\{\left((x, rzg), (\sigma(g))^{-1}r\right)\right\}_{g \in \text{GL}(n, \mathbb{R})} \]

for any rz \in P(T_xM^n); and \mathcal{U} = \{(U, \phi)\} with (U, \varphi) \in \mathcal{U} and (U, \phi) the local coordinate system given by Diagram 34

\[ \sigma(U) = p^{-1}(U) \]

Diagram 34.

with \phi(\{(x, rz), r\}) = (x, \sigma(\varphi^2(x, rz))r), is the associated bundle with fiber \(\mathbb{R}^m\). \(\mathbb{R}^m\) is a vector space, so the associated bundle is a vector bundle. The usual notation is \(\mathbb{R}^m - P(M^n) \times \mathbb{R}^m / \text{GL}(n, \mathbb{R}) \xrightarrow{p} M^n\). As a differentiable manifold, \(\dim \sigma(M^n) = m + n\).

**Definition.** A smooth vector field \(X\) on a differentiable manifold \(M\) is a section of its tangent bundle i.e. \(X: M \rightarrow TM\) is smooth and \(\pi \circ X = \text{id}_M\). Then \(X(x) = (x, X_x) \in \{x\} \times T_xM = F_x\). Graphically,

\[ TM \xrightarrow{X} M. \]

The set of smooth vector fields on \(M, \kappa(M)\), is usually denoted by \(C^\infty(TM)\).

**Definition.** A differential 1-form \(\omega\) on a differentiable manifold \(M\) is a section of its cotangent bundle i.e. \(\omega: M \rightarrow T^*M\) is smooth and \(\pi^* \circ \omega = \text{id}_M\). Then, \(\omega(x) = (x, \omega_x) \in \{x\} \times T^*_xF_x^* = F^*_x\). Graphically,

\[ T^*M \xrightarrow{\omega} M. \]
The set of differential 1-forms on $M$, $\Omega^1(M)$, is usually denoted by $C^\infty(T^*M)$.

NOTE. Let $c = (U, \psi = (x^1, \ldots, x^n), \psi(U)) \in \mathcal{A}$. Then,

$$C^\infty(TU) \ni X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} : U \to TU, \quad x \mapsto X(x) = \left( x, \sum_{i=1}^n X^i(x) \frac{\partial}{\partial x^i} |_x \right)$$

and

$$C^\infty(T^*U) \ni \omega = \sum_{i=1}^n \omega_i dx^i : U \to T^*U, \quad x \mapsto \omega(x) = \left( x, \sum_{i=1}^n \omega_i(x) dx^i |_x \right),$$

with $X^i, \omega_j \in C^\infty(U \times \mathbb{R})$.

PROPOSITION. Let $\xi = (E, B^s, \pi, \mathbb{R}^n; \mathcal{U})$ be a smooth vector bundle; let $C^\infty(E) = \{ s : B^s \to E, s \in C^\infty(B^s, E), \pi \circ s = \text{id}_B \}$ be the set of its sections. It holds: i) $(C^\infty(E), +; \mathbb{R}, \cdot)$ is an $\infty$-dimensional vector space with the operations $(s + t)(x) = s(x) + t(x), (\lambda \cdot s)(x) = \lambda s(x)$, ii) $(C^\infty(E), +; C^\infty(B^s \times \mathbb{R}), \cdot)$ is a module with $+$ as in i) and $:C^\infty(B^s \times \mathbb{R}) \times C^\infty(E) \to C^\infty(E), (f, s) \mapsto f \cdot s : B^s \to E, x \mapsto f \cdot s(x) = f(x)s(x)$, iii) $(\text{End}(C^\infty(E)), +, \circ, \mathbb{R}, \cdot)$ where $\text{End}(C^\infty(E)) = \{ f : C^\infty(E) \to C^\infty(E), f \text{ lineal} \}$ is an associative algebra with $(f + g)(s) = f(s) + g(s), (\lambda \cdot f)(s) = \lambda f(s)$ and $f \circ g(s) = f(g(s))$, iv) $(\text{End}(C^\infty(E)), +, [\cdot, \cdot], \mathbb{R}, \cdot)$ with $+$ and $\cdot$ as in iii) and $[f, g] = f \circ g - g \circ f$ is a Lie algebra.

Proof. (Exercise).

NOTE. $C^\infty(E) \ni s$ is a generalization of the concept of a smooth vector field. Graphically,

$$\mathbb{R}^n - E \xrightarrow{s} B^s.$$

DEFINITION. Vertical bundle associated with a smooth fiber bundle. Let $\xi = (E, B^s, \pi, F^s; \mathcal{U})$ be a smooth fiber bundle. Let $p \in E, x = \pi(p)$ and $F^s_x$ the fiber through $p$. By definition, $F^s_x$ is a differentiable manifold of dimension $s$. The inclusion $i_x \equiv \text{id}_E |_{F^s_x} : F^s_x \to E, \quad q \mapsto i_x(q) = q$ is a smooth one-to-one function with differential $di_x : T_q F^s_x \to T_q E$ a vector space monomorphism. Then $i_x$ is an embedding of $F^s_x$ in $E$. Then, $F^s_x$ is a submanifold of $E$. Consider $T_p F^s_x$ (see Fig. 17).

It is clear that $di_x |_p (T_p F^s_x)$ is an $s$-dimensional vector subspace of $T_p E$. Define the set

$$V^{n+2s} \equiv \bigsqcup_{p \in E} di_x |_p (T_p F^s_x).$$

$V^{n+2s}$ is an $(n+2s)$-dimensional differentiable manifold, submanifold of $TE$. (Notice
that \( n + 2s < 2(n + s) = \text{dim } TE \). The inclusion \( i_V \equiv \text{id}_{TE}|_{V^{n+2s}} \) is the corresponding embedding. Defining the projection \( \pi_V : V^{n+2s} \to E \), \( (p,v) \mapsto \pi_V(p,v) \equiv p \) which implies that \( \pi_V^{-1}(\{p\}) = \{p\} \times d_{x}[p](T_pF_x^s) \), we have the real vector bundle of dimension \( s \ (V^{n+2s}, E, \pi_V, \mathbb{R}^s; U_V) \) called the vertical bundle associated with the smooth fiber bundle \( \xi \). Notice that \( V^{n+2s} \) is a bundle over \( E \), while \( \xi \) is a bundle over \( B^n \). We have the bundle homomorphism in Diagram 35.

**Example of construction of a smooth principal bundle from “glueing data”: monopole bundle.**

**Data:** i) \( B = S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3 \) with atlas \( A = \{c_+, c_-\} \) where \( c_\pm = (U_\pm, \psi_\pm, V_\pm), U_+ = S^2 \setminus \{(0,0,-1)\}, U_- = S^2 \setminus \{(0,0,1)\}, V_\pm = \mathbb{R}^2, \psi_\pm : U_\pm \to \mathbb{R}^2, (x,y,z) \mapsto \psi_\pm(x,y,z) = \frac{(x,y)}{1 \pm z}. (\psi_\pm \text{ are the coordinate functions on } S^2, \text{ while } x,y,z \text{ are parameters to label the points.}) S \equiv (0,0,-1) \text{ and } N \equiv (0,0,1) \text{ are respectively called the “south” and “north” poles of the sphere; as } p(q) \in U_+(U_-) \text{ “approaches” } S(N), \psi_+(x,y,z)(\psi_-(x,y,z)) \text{ “moves away” from the origin } (0,0) \text{ of } \mathbb{R}^2. \text{ In particular, } \psi_+(N) = \psi_-(S) = (0,0). \text{ From } U_+ \cap U_- = S^2 \setminus \{S,N\} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1, (x,y,z) \neq (0,0,\pm1)\} \text{ one has } \psi_\pm(U_+ \cap U_-) = (\mathbb{R}^2)^* \text{ and defines the restrictions } \varphi_\pm \equiv \psi_\pm|_{U_+ \cap U_-} \text{ i.e. } \varphi_\pm(x,y,z) = \frac{(x,y)}{1 \pm z}. \text{ Replacing } (x,y,z) \text{ by the parameters } (\theta, \varphi) \text{ with } x = \sin \theta \cos \varphi, y = \sin \theta \sin \varphi, z = \cos \theta, \theta \in (0,\pi) \text{ and } \varphi \in \mathbb{R} \text{ one has } \varphi_\pm(\theta, \varphi) = \frac{\sin \theta}{1 \pm \cos \theta}(\cos \varphi, \sin \varphi). \text{ ii) } \{U_+, U_-\}: \text{ cover of}
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References


Resumen. En las lecciones 1 y 2 se discuten conceptos básicos de topología y geometría diferencial, como son las definiciones de espacio topológico; espacios Hausdorff, compactos, conexos, paracompactos; grupos topológicos y acciones de grupos sobre espacios; variedad diferenciable, vectores tangente y 1-formas; particiones de la unidad y grupos de Lie. En la lección 3 se presenta el concepto de haz fibrado y se discuten haces vectoriales y haces principales. En la lección 4 se define el concepto de conexión en un haz vectorial suave y los conceptos asociados de curvatura y transporte paralelo; se ilustra con la conexión de Levi-Civita en una variedad Riemanniana. Por último, en la lección 5 se definen conexiones en haces principales y se ilustran con los grupos de Lie U(1) y SU(2). Por razones de espacio este artículo incluye sólo las lecciones 1, 2 y 3. Las lecciones 4 y 5 serán publicadas en un artículo próximo.