Pseudo-spin Dynamical Symmetry

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Abstract. The origin and consequences of pseudo-spin symmetry in nuclear physics, which is exact for an oscillator potential with one-body orbit-orbit ($v_{ll}$) and spin-orbit ($v_{ls}$) interaction strengths in the ratio $\mu = 2v_{ll}/v_{ls} = 0.5$, are considered. Specifically, the $v_{ls} \approx 4v_{ll}$ condition is shown to be consistent with relativistic mean-field results and a pseudo-LS shell-model coupling scheme. The complementarity of good pseudo-spin symmetry and a residual quadrupole-quadrupole interaction is also explored; in this case, the pseudo-spin picture extends to the pseudo-$SU(3)$ scheme, which applies to superdeformed phenomena.

1. Introduction

The three-dimensional isotropic harmonic oscillator ($H_0$) augmented with the one-body spin-orbit ($l \cdot s$) and orbit-orbit ($l^2$) interactions,

$$\hat{H} = H_0 + v_{ls} \mathbf{l} \cdot \mathbf{s} + v_{ll} l^2,$$

is a good approximation for the nuclear single-particle hamiltonian. The $l^2$ term pushes high angular momentum states down ($v_{ll} < 0$) relative to those with lower $l$ values while the $l \cdot s$ term (coupling spatial and spin degrees of freedom) is required to achieve shell closures ($v_{ls} < 0$) at the magic numbers. Unfortunately, $v_{ls}$ is so large that the $l \cdot s$ term destroys the oscillator $SU(3)$ symmetry for all but light ($A \lesssim 28$) nuclei, rendering it of little value in attempts at unraveling the structure of heavier systems.

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This paper shows that this situation gives way to a much more favorable one, because for medium and heavy \((A \gtrsim 100)\) nuclei, \(v_t \approx 4v_H\) or the Nilsson parameter 
\(\mu = 2v_H/v_t = 0.5\). As a consequence, the level splitting generated by the \(1 \cdot s\) and \(l^2\) interactions can be duplicated by a pseudo-oscillator Hamiltonian plus a pseudo \(l^2\) term, with (at most) a small symmetry-breaking residual pseudo \(1 \cdot s\) interaction \([1-3]\). Since common residual interactions are pseudo-spin scalar operators, a many-particle pseudo \(LS\)-coupled shell-model scheme can be employed, and the basis truncated to leading pseudo-spin symmetries, without losing important physics. In addition, the pseudo-\(LS\) scheme extends to pseudo-\(SU(3)\) when deformation dominates \([4]\).

2. **Spherical Nilsson Scheme**

In the single-particle picture, the pseudo-spin concept means a division of the total particle angular momentum into pseudo \((j = \bar{l} + \bar{s})\) rather than normal \((j = l + s)\) orbital and spin parts, where \(\bar{s} = -s\) (vectors) so \(\bar{l} + \frac{1}{2} = l \mp \frac{1}{2}\). The significance of this simple transformation can be demonstrated by plotting the eigenvalues of \(H\) as a function of \(\mu\). For \(\mu = 0.5\), pairs with \(j = l + \frac{1}{2}\) and \(j = (l + 2) - \frac{1}{2}\) are degenerate for all \(l\) values. Furthermore, the splitting of these degenerate pairs follows a \(\bar{l}(\bar{l} + 1)\) rule where \(\bar{l}\) is the average angular momentum: \(\bar{l} = \frac{1}{2}(l + (l + 2)) = l + 1\). This elementary mapping defines a special \((normal \leftrightarrow pseudo)\) unitary transformation \([5,6]\).

The single-particle Hamiltonian transforms under this mapping as follows:

\[
H_0 + v_t s \cdot s + v_H l^2 \quad \rightarrow \quad \bar{H}_0 + (4v_H - v_t) \bar{l} \cdot \bar{s} + \bar{v}_H \bar{l}^2 + (h\omega + 2v_H - v_t).
\]  

Since \((h\omega + 2v_H - v_t)\) is a constant, the pseudo form, \(\bar{H} = \bar{H}_0 + \bar{v}_t \bar{l} \cdot \bar{s} + \bar{v}_H \bar{l}^2\), has the same excitation spectrum as the normal one \((H = H_0 + v_t s \cdot s + v_H l^2)\) when \(h\omega = h\omega\), \(\bar{v}_t = (4v_H - v_t)\), and \(\bar{v}_H = v_H\). This transformation is important, because \(\bar{v}_t \approx 4v_H\), so \(\bar{v}_t \approx 0\). Specifically, \(\mu_\nu \approx 0.4\) and \(\mu_\pi \approx 0.6\) \((\nu\) for neutrons and \(\pi\) for protons); this places medium and heavy nuclei close to the exact pseudo-spin limit \((\mu = 0.5)\) of the theory. Indeed, the average value for \(\mu\) is almost exactly 0.5. For these nuclei, the familiar single-particle shell-model Hamiltonian can therefore be replaced by a less familiar, but equivalent, pseudo form which is inherently simpler due to its much smaller spin-orbit term.

The pseudo-spin scheme organizes normal parity \((j = 1/2, 3/2, \ldots, n-1/2)\) levels of the \(n\)-th oscillator shell into a pseudo-oscillator shell with \(\bar{n} = n - 1\). For example, the \((4s_{1/2}, 2d_{3/2}, 2d_{5/2}, 0g_{7/2})\) levels of \(n = 4\) are mapped onto the \((3p_{1/2}, 3p_{3/2}, 1f_{5/2}, 1f_{7/2})\) orbitals of \(\bar{n} = 3\). The \(0g_{9/2}\) level \((j = n + \frac{1}{2})\) defects from the valence space to the shell below, while the \(1h_{11/2}\) level \((j = n + 1 + \frac{1}{2} = n + \frac{3}{2})\) intrudes from the shell above. Unique-parity intruder configurations couple to normal-parity states only through excitations involving pairs of particles.
3. Relativistic Mean-field Results

The pseudo-spin concept may be better understood by comparing an intuitive result for $v_{II}$ with relativistic nuclear mean-field predictions for $v_{ls}$. The origin of the $l^2$ term in $H$ is in the flatness of the mean field in the interior region, as compared with the quadratic oscillator form ($V(r) = \frac{1}{2} M \omega^2 r^2$). In the large mass limit ($A \rightarrow \infty$) the potential approaches that of a spherical well of finite depth. If this spherical well is replaced by one with an infinite depth, the single-particle energies are given by

$$E_{nl} = \frac{\hbar^2}{2M R^2} x_{nl}^2$$

where $M$ is the nucleon mass, $R$ is the radius of the well, and the $x_{nl}$ are zeroes of spherical Bessel functions. These zeroes are approximately given by the result $x_{nl}^2 \approx [(\frac{n}{2} + 1)\pi]^2 - l(l + 1)$. Table I illustrates the dependence of $x_{nl}$ on $l$ for the $n = 4$ case. The results show that the splitting follows an $l(l + 1)$ rule. Therefore,

$$v_{II} = -\frac{\hbar^2}{2M R^2}.$$  (4)

A determination of $v_{II}$ using the Klein-Gordon equation leads to the same conclusion when the kinetic energy is a small fraction of the nucleon mass.

Table I. Zeroes $(x_{nl})$ of spherical Bessel functions and differences of their squares $(x_{n0}^2 - x_{nl}^2)$ compared with the simple $l(l + 1)$ approximation for the $n = 4$ case.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$l$</th>
<th>$x_{nl}/\pi$</th>
<th>$x_{nl}^2$</th>
<th>$x_{n0}^2 - x_{nl}^2$</th>
<th>$l(l + 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>3.000</td>
<td>88.83</td>
<td>0.00</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2.895</td>
<td>82.72</td>
<td>6.11</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>2.605</td>
<td>66.98</td>
<td>21.85</td>
<td>20</td>
</tr>
</tbody>
</table>

Next, consider the strength of the spin-orbit coupling. Starting with the usual Dirac equation (with only the time component of the scalar and vector potentials taken into account) and using a non-relativistic reduction of the relativistic mean-field theory, the spin-orbit interaction is given by

$$V_{ls} = \frac{\hbar^2}{2M} \frac{2}{r \, dr} \left( \frac{1}{1 - B \rho / \rho_0} \right) l \cdot s.$$  (5)

In this expression, $\rho$ and $\rho_0$ are respectively the nucleon density at radius $r$ and the nuclear matter density. The dimensionless quantity $B$ in (5) is related to the strength of the scalar and vector coupling constants. The spin-orbit strength $v_{ls}$ can be obtained from the average of $V_{ls}$ over the region inside radius $R$,

$$v_{ls} = -\frac{\hbar^2}{2M R^2} \frac{6B}{1 - B}.$$  (6)
In determining this result, the fact that $\frac{d\phi}{dx}$ vanishes everywhere, except near the surface of the nucleus, has been used.

It follows from equations (4) and (6) that the ratio

$$\mu = \frac{2v_{||}}{v_{ls}} = \frac{1 - B}{3B}$$

is independent of mass number. Furthermore, to obtain $\mu = 0.5$ requires $B = 0.4$. In the simplest version of the theory, $B = \frac{1}{2}(B_s + B_v)$ with its scalar ($i = s$) and vector ($i = v$) components given by $B_i = g_i^2 \rho_0/\mu_i^2 M c^4$ where $\mu_i$ and $g_i$ respectively denote meson masses and coupling constants. Using this expression for $B$, the Nambu–Jona-Lasinio (NJL) model [7,8] — which in its modern form starts with massless quarks and generates hadron masses out of the vacuum by spontaneous symmetry breaking, and which has also been used to predict the coupling constants and masses appearing in a relativistic nuclear field theory — gives the result $\mu = 0.686$ shown in Table II. As also shown in the table, results for the original Walecka model [9] and a derivative coupling model due to Zimanyi and Moszkowski [10] — which gives a more realistic equation of state for nuclear matter; which includes the effect of nucleon recoil; and when extended to include exchange correlations — likewise yield reasonable results for $\mu$.

**Table II.** Comparison of $\mu = 2v_{||}/v_{ls}$ values for various relativistic mean field theories. Exact pseudo-spin symmetry requires $\mu = 0.5$. Results given are for $\rho_0 = 0.16$ nucleons/fm$^3$ and a nuclear binding energy of -16 MeV.

<table>
<thead>
<tr>
<th>Theory</th>
<th>$B_s$</th>
<th>$B_v$</th>
<th>$B$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NJL$^a$</td>
<td>0.339</td>
<td>0.316</td>
<td>0.327</td>
<td>0.686</td>
</tr>
<tr>
<td>Walecka$^b$</td>
<td>0.487</td>
<td>0.368</td>
<td>0.427</td>
<td>0.447</td>
</tr>
<tr>
<td>Zimanyi$^b$</td>
<td>0.252</td>
<td>0.088</td>
<td>0.344</td>
<td>0.635</td>
</tr>
</tbody>
</table>

$^a) B = \frac{1}{2}(B_s + B_v);$ $^b) B = \frac{2}{3}B_s + 2B_v$ (including recoil and exchange effects).

4. **Pseudo-spin Dynamical Symmetry**

A consequence of good pseudo-spin symmetry is distinct $\hat{S}$ multiplets that are decoupled and ordered. The $\hat{m}$-particle valence spaces ($\hat{m} = \hat{m}_s$ for protons and $\hat{m} = \hat{m}_v$ for neutrons, which occupy different major shells) divide into subspaces with $\hat{S} = 0, 1, 2, 3, \ldots, \hat{S}_{\text{max}}$ for $\hat{m}$ even or $\hat{S} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \hat{S}_{\text{max}}$ for $\hat{m}$ odd with $\hat{S}_{\text{max}} = \min(\frac{1}{2}\hat{m}, \hat{N} - \frac{1}{2}\hat{m})$ where $\hat{N} = \frac{1}{2}(\hat{n} + 1)(\hat{n} + 2)$ is the pseudo-shell degeneracy. If $\hat{n}$ labels $\hat{L}$ multiplicity, the $L\hat{S}$-coupled states $|\Psi_J\rangle = \{|(\hat{\alpha}_s \hat{L}_s, \hat{\alpha}_v \hat{L}_v) L \times (\hat{S}_s, \hat{S}_v) \hat{S}\rangle\}$ with $\hat{S}_s = \hat{S}_{\text{min}}$, $\hat{S}_v = \hat{S}_{\text{vmin}}$ and $\hat{S} = \hat{S}_{\text{min}} = |\hat{S}_{\text{min}} - \hat{S}_{\text{vmin}}|$ are expected to dominate, because realistic interactions favor pseudo-space symmetric (pseudo-spin antisymmetric) configurations. But this truncation to the lowest $\hat{S}_s$, $\hat{S}_v$ and $\hat{S}$ multiplets is usually not enough to reduce the full model space to a reasonable and workable size.
Figure 1. Group structure of the pseudo-spin dynamical symmetry model. The proton and neutrons fill different major shells, and within each of these subshells the structures $\tilde{U}(2\tilde{N}) \supset \tilde{U}(\tilde{N}) \otimes \tilde{SU}(2)$ with $\tilde{U}(\tilde{N}) \supset \tilde{SU}(3) \supset \tilde{SO}_L(3)$ organize the allowed normal-parity configurations according to their pseudo-space deformation. The many-particle dynamics require that the most deformed of these lie lowest and the least deformed highest. The $\tilde{SU}(3)$ strong coupling limit, which is motivated by a deformation reenforcement principle, likewise organizes the combined proton-neutron space according to its deformation. Particles distributed in the unique parity intruder orbitals tend to reenforce this picture.

Fortunately, another symmetry for strongly deformed nuclei can be invoked to effect a further truncation. Just as for light nuclei, $\tilde{SU}(3)$ of the pseudo oscillator which lies between $\tilde{U}(\tilde{N})$ and $\tilde{SO}_L(3)$ organizes states associated with each $\tilde{S}$ multiplet according to deformation [11]. In this case, however, the deformation is realized in terms of the pseudo, and not the normal, space symmetry. Nonetheless, this also gives rise to strongly enhanced $B(E2)$ transition strengths, because the electric quadrupole operators $\tilde{Q}_E$ and $Q_E$ differ very little from one another [12].

Of the various coupling schemes that can be built with these group structures, the $\tilde{SU}(3)$ strong-coupled limit shown in Fig. 1 is the most natural. In the dynamical symmetry limit, when the interaction is expressed solely in terms of group invariants, the corresponding eigenvalue spectrum is given by

$$E\{\tilde{m}_\pi \tilde{S}_\pi(\lambda_\pi, \mu_\pi); \tilde{m}_\nu \tilde{S}_\nu(\lambda_\nu, \mu_\nu); \tilde{\rho}(\lambda, \mu) \tilde{K}(\tilde{L}, \tilde{S})J\} = C_{\tilde{m}_\pi} \tilde{m}_\pi + C_{\tilde{m}_\nu} \tilde{m}_\nu$$
This choice is consistent with a deformation reenforcement principle which recognizes the favored configuration to be the one with maximum overlap of maximally deformed proton and neutron spatial configurations. In (8), $C_2(\lambda, \mu)$ is the second order $SU(3)$ invariant with eigenvalue $\lambda^2 + \mu^2 + 3(\lambda + \mu)$. An explicit form for an operator that has $K^2$ as its eigenvalue is known in the limit $L \ll \sqrt{C_2(\lambda, \mu)}$ [13]. The constants in (8) are related to the effective interaction. For example, the $\chi$’s are given by the strengths of the quadrupole-quadrupole interactions; the $C_S$’s are related to centroid separations of the pseudo-spin multiplets; $C_{\bar{L}}$ is the inertia parameter; $C_K$ determines the band splitting; etc. The $J^2$ term can be replaced by $\bar{L} \cdot \bar{S} = \frac{1}{2}[J^2 - \bar{L}^2 - \bar{S}^2]$ and used to fine tune the placement of the $\bar{L}(\bar{L} + 1)$ bands with respect to $\bar{S}$.

When the pseudo-spin dynamical symmetry picture applies, there are $\bar{L}(\bar{L} + 1)$ bands — one for each pseudo-spin orientation — that differ in total angular momenta ($J = \bar{L} + \bar{S}$) by integer (even-$A$ compared with even-$A$) or half-integer (odd-$A$ with even-$A$) amounts. This seems to affirmatively answer the question: “Whether low-lying collective states having alignment one would occur in a nuclei with rather good pseudo-spin symmetry?” [14]. In considering this matter, it is important to emphasize that the alignment can be either proton or neutron in origin, or a combination. In particular, a consequence of good pseudo-spin symmetry is the prediction of $(2\bar{S} + 1)$ identical $\bar{L}(\bar{L} + 1)$ bands with $J$ values given by $J = \bar{L} - \bar{S}$ in the first, $\bar{L} - \bar{S} + 1$ in the second, $\ldots$, $\bar{L} + \bar{S}$ in the last. Indeed, for the $^{151}$Tl case ($\bar{S} = \frac{1}{2}$) two bands ($J = \bar{L} + \frac{1}{2}$ and $J = \bar{L} - \frac{1}{2}$) have been reported and taken as evidence for the goodness of the pseudo-$SU(3)$ picture [15]. The model further predicts (since on the average $\mu_\pi > 0.5$ and $\mu_\nu < 0.5$) that in odd-$A$ proton nuclei the $J = \bar{L} + \frac{1}{2}$ series should fall below the $J = \bar{L} - \frac{1}{2}$ sequence and vice-versa for odd-$A$ neutron systems.

5. Conclusions

The origin and consequences of the $\mu = 2v_\mu/v_\delta \approx 0.5$ result was examined. Actual estimates for $\mu$ are $(0.60$ and $0.65$) for protons with $(50 < Z < 82$ and $Z > 82$), and $(0.42$ and $0.33$) for neutrons with $(82 < N < 126$ and $N > 126$), respectively. These values ensure that the many-particle extension of the single-particle picture has good total pseudo-spin, so long as the residual interaction is a pseudo-spin scalar operator. Examples include pairing, the surface delta interaction, and $\tilde{Q} \cdot \tilde{Q}$, which generates $\bar{L}(\bar{L} + 1)$ rotational sequences in the decoupled pseudo-spaces. At a more fundamental level, good pseudo-spin symmetry was shown to be consistent with relativistic mean field results for $v_\delta$ and $v_\mu$.

Further consequences of good pseudo-spin symmetry were noted; particularly, the ap-
pearance of identical bands. This is consistent with strong deformation in the pseudo-space part of the many-particle basis, thereby giving rise to $\hat{L}(\hat{L} + 1)$ rotational sequences for each of the $(2S + 1)$ orientations of the pseudo-spin. That these bands yield strongly enhanced $B(E2)$ strengths follows because $Q_E \approx Q_E$. A further prediction is that many additional, strongly deformed bands should be found when the detectors currently under construction come on line.

We have assumed that the particles in the intruder levels do not affect the dynamics in a significant way. This assumption is justified so long as intruder pair alignment (common in normally deformed systems) which results in backbending, accompanied by weakened $E2$ transition strengths, is not observed. The intruder level particles apparently keep an internal structure that changes (at most) slowly as a function of increasing angular momentum of the system. This feature is currently under further investigation.

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