Quark Models of Nuclear Matter

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Abstract. Nuclear matter is modeled directly in the constituent quark coordinates. A many-body string-flip potential is used which confines quarks, allows hadrons to separate and is symmetric in all quark coordinates. Variational Monte Carlo results are presented for some ground state properties vs. density. A percolation phase transition from nuclear to quark matter is observed which is characterized by a dramatic rearrangement of strings.

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I. Introduction

In this paper we study some simple constituent quark models of nuclear matter. Here, uniform nuclear matter is modeled directly in the quark coordinates. Variational Monte Carlo results will be presented for a variety of ground state properties at different densities.

The motivation for this study is threefold. First, we wish to examine the role of quark substructure in nuclear physics. For example, how are hadronic properties such as the nucleon form factor modified in the medium. By studying simple models, we hope to identify nuclear observables which may be sensitive to hadronic substructure.

Second, we will use these models (in latter papers) to search for qualitatively new modes of excitation of many quark systems. These modes may be present in nuclei but they are not present in either single hadrons or hadronic models of nuclei. For example, there may be collective excitations of several quarks in a nucleus. These ‘quark giant resonances’ could involve coherent density, spin, flavor or color oscillations of several
quarks. In addition, the mixing between ‘quark’ like excitations, such as the Nucleon to Delta transition, and ‘hadronic’ excitations, such as the Gamow Teller resonance can be examined.

One can also study the transition from nuclear to quark matter as the density increases. For example, how sharp is the transition? How does the color susceptibility change with density? What are useful observables which may provide signatures of the transition?

Finally, these simple models provide a first step towards developing “realistic” many quark models for nuclear matter, finite nuclei and few nucleon systems. By realistic we mean a model which fits both the (low lying) excitation spectrum and form factors of the nucleon and the (low energy) NN phase shifts. Such models may be useful in several areas of nuclear physics. For example, CEBAF experiments could be compared to model predictions. One might then gain insight into when a virtual photon couples to ‘individual’ quarks or ‘complete’ hadrons.

In section II we describe a string-flip potential model which confines quarks into hadrons and allows these hadrons to separate without long range van der Waals forces. In this model, Horowitz et al. [1] presented some many-body results for simple two quark “nucleons” in one dimension. Hadron-hadron scattering was discussed in Refs. [2,3]. Gardner et al. [4] constructed equivalent hadronic theories for this quark model.

In three dimensions, the only previous results for uniform matter are those of Horowitz and Panoff [5,6] and Watson [7]. Watson has calculated variational energies for a very small system of quarks and antiquarks. Horowitz et al. calculated (for a model with two quark nucleons) the quark-quark correlation function and from it the static structure factor and nonrelativistic coulomb sum [6].

In section III we describe how quarks are to be grouped into hadrons (in the model) by determining the arrangement of confining strings. We consider three different simple models: two quark hadrons (which we refer to as model 2Q), quark-antiquark hadrons (model QQ) and three quark nucleons (model 3Q). Section IV presents variational Monte Carlo results for energies, wave functions and quark correlation functions for various densities. Discussion and conclusions are in section V. This talk is an abbreviated version of our results. Full results are presented in Ref. [8].

II. Basic Quark Model.

We will use a simple string-flip potential [1,2] which (a) confines quarks, (b) is explicitly symmetric in all quark coordinates and (c) allows cluster separability without long range van der Waals forces. In this paper we try to use the simplest such model that can describe the basic quark core of each hadron.

2.1 Many quark potential.

We start by presenting a two quark version of the model which is very simple and still has non-trivial substructure. We generalize to three quarks below. An isolated hadron is
described by a hamiltonian $h_{12}$,

$$h_{12} = -\frac{\nabla_i^2}{2M} + \frac{\nabla_j^2}{2M} + v(r_{12}).$$

(1)

Here, the quark mass is $M$ and $v(r)$ is a confining potential,

$$\lim_{r \to \infty} v(r) \to \infty.$$  

(2)

For simplicity, we chose a harmonic oscillator for $v(r)$,

$$v(r) = \frac{k}{2} r^2.$$  

(3)

A linear potential is expected to give similar results. However, Eq. (3) will somewhat simplify wave functions (as we see below).

If a two body potential is assumed for the many quark system then Eq. (2) will imply long range van der Waals forces between hadrons. These forces, which only fall off as a power law (in $r$), are not observed. Furthermore, they could have very large effects in nuclear matter.

Instead, we consider a many-body generalization of Eq. (1). Perhaps, once color flux tubes are arranged to form a color singlet, the forces largely saturate. Therefore, the potential $v(r)$ is assumed to act only between quarks within a hadron. For a system of $N$ quarks, grouped into $N/2$ hadrons the hamiltonian is,

$$h = \sum_{i=1}^{N} -\frac{\nabla_i^2}{2M} + V,$$

(4)

$$V = \text{Min} \sum_{j=1}^{N/2} v(r_{p_j})^2.$$  

(5)

Here quarks $P^1_j$ and $P^2_j$ are grouped into the $j$-th hadron. The minimum in Eq. (5) is taken over all the $(N-1)!$ ways to group $N$ quarks into $N/2$ hadrons. The hamiltonian in Eqs. (4-5) define the two quark model which we will refer to as model $2Q$.

As an example, consider a system of four quarks. There are three ways to group the quarks into hadrons. Therefore, the potential is the minimum of three terms each of which is the sum of two “strings” $v(r)$, Eq. (3).

$$V(r_1, r_2, r_3, r_4) = \text{Min}(V_1, V_2, V_3),$$

(6)

$$V_1 = v(r_{12}) + v(r_{34}), \quad V_2 = v(r_{13}) + v(r_{24}), \quad V_3 = v(r_{14}) + v(r_{23}).$$  

(7)
This minimum prescription insures that the potential is symmetric in all of the quark coordinates.

Alternatively, one could distinguish two types of quarks (say quarks and antiquarks) and require each pairing to be between a quark and an antiquark. We will call this model QQ. For two quarks (say 1 and 2) and two antiquarks (say 3 and 4) there are now only two allowed pairings.

\[ \nabla(r_1, r_2, r_3, r_4) = \text{Min}(V_2, V_3) \]  

Where \( V_2 \) and \( V_3 \) are defined in Eq. (7) and the pairing \( V_1 \) is no longer allowed.

For a system of \( A \) quarks and \( A \) antiquarks (total of \( N = 2A \) particles) the potential of Eq. (5) is replaced by,

\[ V = \text{Min} \sum_{j=1}^{A} v(r_{jP_j}) \]  

Here the \( j \)th hadron contains quark \( j \) and antiquark \( P_j \) and the minimum is taken over all the \( A! \) ways to group the \( A \) quarks and \( A \) antiquarks into \( A \) hadrons (i.e. over all permutations \( \{P_j\} \)). Note, the number of possible pairings \( A! = (N/2)! \) in the QQ model is much less than the \( (N - 1)!! \) possible pairings in the 2Q model.

In either model, there are no long range van der Waals forces between hadrons. This is because the forces are assumed to saturate with one and only one link going to a given quark. Thus the only interactions between hadrons involve the possibilities that the pairings may change (and the Pauli principle between identical quarks). If one moves a single quark it is possible that all of the pairings can change. Therefore, the potential in Eq. (5) or Eq. (9) is an \( N \) body operator. It can not be reduced to the sum of two body operators.

2.2 Quark pairing.

The confinement of color suggests a basic topological problem for the many quark system. Color flux which leaves a given quark \( i \) must return to another quark or antiquark \( j \). The interesting question is which quark? Let us for a moment focus on the pairing (which quarks \( i \) and \( j \)) and not worry about the exact path the flux takes between the quarks or the precise flux tube energy.

Lattice QCD, presumably, solves this flux tube assignment problem, but at a spectacular cost in computation. On the lattice, many gauge field degrees of freedom are explicitly evolved according to a complex dynamics to determine the arrangement of flux. However, it may be unreasonable to foresee direct lattice QCD calculations for a many quark system. This is both because of the poor accuracy and the lack of insight that may be provided by such complex calculations.

Instead, it may be fruitful to use lattice, strong coupling or other QCD results to determine properties and parameters of a phenomenological model. At present, the exact form of this model is unclear. Indeed, in this paper, we consider a number of different
models and wish to be flexible as to the ultimate form. Nevertheless, we believe most models of the many quark system will need a way to determine which quarks belong to which hadrons. Thus, solving some kind of a quark assignment problem is likely to be general requirement.

For simplicity, we consider heavy quarks and an adiabatic approximation where the gluon flux tubes adjust very quickly to the instantaneous position of the quarks. This motivates the minimum prescription in Eqs. (5) and (9).

For practical calculations, one needs an efficient way of determining the best pairing. The number of possible pairings grows so quickly with increasing \( N \) that a brute force search of all the possibilities is only feasible for small \( N \). However, economists and mathematicians have developed efficient algorithms. (An economist is interested in a related problem, given \( N \) factories and \( N \) raw materials, which do you send where in order to maximize the number of widgets produced.) Burkard and Derigs [9] have implemented algorithms for the linear sum assignment problem (which solves our \( \bar{Q}Q \) pairing) and the sum matching problem (which solves the \( 2Q \) pairing). These algorithms find the best pairing in a time proportional to \( N^3 \). The input to the algorithm is a matrix \( C_{ij} \), which is the potential energy, Eq. (3), if quark \( i \) is paired to quark \( j \).

### 2.3 Three Quark Nucleons.

We now generalize the quark pairing ideas to three quark nucleons. Consider equal numbers of red, green and blue quarks. We require each red quark to be paired with one and only one blue and one and only one green quark. Likewise, each blue is paired to one red and one green quark, etc. Furthermore, we will determine the optimal blue-green, red-blue and red-green pairings independently. This allows us to use the linear sum assignment problem algorithm which worked for the \( \bar{Q}Q \) model. We simply call the algorithm three times.

Our three quark potential for \( N=3A \) quarks is,

\[
V = \text{Min} \left\{ \sum_j^A v(|{r}_j^r - {r}_{L_j}^r|) \right\} + \text{Min} \left\{ \sum_j^A v(|{r}_j^b - {r}_{M_j}^b|) \right\} + \text{Min} \left\{ \sum_j^A v(|{r}_j^g - {r}_{N_j}^g|) \right\}. 
\]  

(10)

Here \( r_j^r \) is the coordinate of the j-th red quark (likewise \( r_j^b \) the j-th blue and \( r_j^g \) the j-th green quark). The first minimum is over all permutations \( L_j \) of green quarks, while the second minimum is over permutations \( M_j \) and the third over permutations \( N_j \). Note, the three minima are determined independently. Equation (10) along with the Hamiltonian, Eq. (4), and the string energy, Eq. (3), define our three quark model (model 3Q).

The model 3Q is symmetric in all of the quarks and in the colors. Furthermore, it allows three quark clusters to separate without van der Waals forces. However, there is no guarantee that quarks will only be paired in groups of three. It is also possible to have clusters with 6, 9, 12, etc. quarks.

If red quark one is paired to blue quark two and this to green quark three, there is no guarantee that green three will be paired to red one. Instead, the loop could involve any multiple of three quarks before it closes. We chose to interpret the six quark loop pairing...
as a "hidden color" state where the six quark system is in a color singlet but the wave function may not simply separate into two local three quark color singlets.

III. VARIATIONAL MONTE CARLO CALCULATIONS.

We now describe our variational wave function and Monte Carlo procedure for calculating ground state properties. However, first we mention some units. One can measure all energies in units of the oscillator frequency \( \sqrt{\frac{k}{M}} \), and all lengths in terms of the oscillator length \( (Mk)^{-1/4} \). Therefore, we are free to adopt units,

\[
k = M = 1.
\]

(11)

It is a simple matter to scale our results according to your chosen values for \( k \) and \( M \).

3.1 Variational wave function.

A very simple variational wave function is,

\[
\Psi = \exp\{-\lambda V\} \Phi.
\]

(12)

Here \( V \) is the full many-body potential, \( \lambda \) a variational parameter and \( \Phi \) a free Fermi gas Slater determinant. For the 2Q model \( \Phi \) involves only one component. Note, for simplicity, all of our models assume spin zero (fermion) quarks. For the QQ model we take \( \Phi \) to be a simple product of quark and antiquark determinants, while the 3Q model has a simple product of red, blue and green determinants.

Equation (12), with an appropriate choice of \( \lambda \), is exact for an isolated hadron. This is a very convenient feature of the harmonic potential in Eq. (3). Thus, Eq. (12) has the correct low density limit. Equation (12) is also exact in the high density limit. Simply setting \( \lambda = 0 \) recovers a free Fermi gas. Finally, at intermediate density, Eq. (12) provides excellent energies in one space dimension for the 2Q model [1].

Equation (12) should also provide a first orientation for the three dimensional many quark system. The many-body potential term builds in correlations that insure the quarks cluster into hadrons. The variational parameter \( \lambda \) determines the amount of clustering with \( \lambda^{-1/2} \) providing a length scale for quark confinement. We expect this length scale to increase as the density increases (see below).

3.2 Monte Carlo Simulations.

In order to minimize finite size effects, we evaluate the string potential in Eq. (3) using periodic boundary conditions. Therefore, one can either pair directly to a quark or pair to the periodic image of the quark. [We arbitrarily assume that if the quark is unpaired then so is its image.]

Also, to minimize finite size effects, we use a number of quarks \( A \) which corresponds to a closed shell Fermi gas. However, model 2Q clearly requires an even \( N \). Therefore, we
adopt antiperiodic boundary conditions for the wave function. We expect the difference between antiperiodic and periodic boundary conditions to be small. [Note, for periodic boundary conditions closed shells have odd \( A \).]

The calculations use standard Metropolis Monte Carlo. A trial move is generated for a single quark and the optimal pairings redetermined in order to evaluate \( V \) and the wave function. Then, the move is accepted or rejected in standard fashion. This procedure is repeated for the rest of the \( N \) quarks to complete a single Metropolis sweep.

### IV. RESULTS.

We now present results for the 3Q model. We caution that our results could suffer from limitations in the variational wave function Eq. (12) and, of course, limitations in our simple quark model.

#### 4.1 Energies, Variational Wave Functions.

The optimal \( \lambda \) for the 3Q Model is shown in Fig. 1. Here, there is an abrupt transition near \( \rho = 0.12 \).
Figure 2. Quark configuration and optimal pairing for a low density $\rho = 0.046$ in the nuclear matter phase. This is a sample Metropolis configuration of $N=96$ quarks. Note the periodic boundary conditions of some of the strings. The optimal pairing has 29 three-quark clusters and 1 nine-quark cluster which is mostly along the top.

4.2 Quark Cluster Probabilities.

This transition from nuclear to quark matter is accompanied by a dramatic change in the quark pairing. Fig. 2 shows a typical Metropolis configuration at low density. Here, the quarks are clustered into nucleons and almost all of the pairings involve simple three quark groupings.

In contrast, Fig. 3 shows a typical high density configuration. Now, most of the quarks are paired into long strings with only a few belonging to isolated three quark clusters. The transition is not from three to six quark clusters. Instead, it is a transition from three quark to very large quark clusters which are comparable in size to the simulation volume.

We define a 3 (or 6, 9, ...) quark cluster probability as the probability that a quark will be paired into a ring involving 3 (6, 9, ...) quarks. This probability is shown in Fig. 4 vs. density. Surprisingly, there is very little density dependence within a given phase.

In the nuclear matter phase, ninety percent or more of the probability is in three quark clusters. However, in the high density quark matter phase, the three quark probability is only about one third. Instead, the majority of the probability is in large clusters involving 15 or more quarks. The 6, 9 and 12 quark probabilities are never more than a few percent at any density.

This transition, where the quark strings grow to fill the volume is analogous to a percolation transition seen in some condensed matter systems. Here strings of (for example) electrical conductivity extend throughout the sample.
Finally, our definition of cluster probabilities depends on the (somewhat arbitrary) three quark potential. Thus, these results may be model dependent. Nevertheless, we have actually calculated these cluster probabilities in a dynamical model. This is in contrast to the majority of works which treat cluster (or multiquark bag) probabilities as arbitrary parameters.

V. DISCUSSION AND FUTURE WORK.

In this paper we have developed simple constituent quark models for many-hadron systems. The models (a) confine quarks into hadrons (b) allow the hadrons to separate without van der Waals forces and (c) are explicitly symmetric in all of the quark coordinates. Using Monte Carlo techniques, these models are computationally tractable for a variety of interesting systems. We presented calculations for nuclear matter using a simple variational wave function.

The crucial feature of these models is the need to determine an optimal quark pairing. This arranging of “flux tubes” is perhaps the minimum representation of the extremely complex gluon dynamics. Color flux which leaves a given quark must return to another quark or antiquark. The interesting question is which one? Efficient algorithms exist for determining the optimal quark pairing in these models. Indeed, these algorithms work well despite the very large number of possible pairings.

The goals of this work are threefold. First, we wish to examine the role of nucleon
substructure in nuclear physics. For example, we have calculated quark correlation functions for various densities and seen how these compare to those in a free hadron [8]. These show how nucleons may change in the medium. In the future, one can calculate a variety of observables in the model and see if any of these show clear signatures of nucleon substructure.

Second, one can use the models to search for qualitatively new modes of excitation which may be present in many-nucleon systems but which are not present in single hadrons or hadronic models of nuclei. For example, there may be low energy ring like pairing excitations [8]. Alternatively, there could be collective "quark giant resonances" where quarks from many nucleons oscillate coherently. In the future, one can use variational excited state wave functions, such as the Feynman ansatz [10], to study the properties of these collective modes. In addition, one can study how "quark like" excitations such as the nucleon to Delta transition mix with "nucleon like" excitations such as the Gamow Teller resonance. Finally, the transition from nuclear to quark matter can be studied. Indeed, in the 3Q model we found this transition to be characterized by a dramatic rearrangement of strings.

The third goal is to develop a realistic quark model of many nucleon systems. By adding both short range spin dependent (one gluon exchange) and long range (meson exchange) forces to our confining potential one can extend the model to fit the (low lying) nucleon states in addition to the NN phase shifts. The model could then be directly used to predict experiments at CEBAF and elsewhere. Work is underway on a simulation with the full
spin and isospin degrees of freedom of nuclear matter (and possibly a short range spin dependent force to reproduce the nucleon to Delta splitting.)

There are several further directions. For example, simulations could be made in a semiclassical or classical approximation. This would allow a direct calculation of the full electromagnetic response function. Study of the response function should show when and how a virtual photon couples to individual quarks or full hadrons. In addition, strange quarks could easily be added by assuming our potential is flavor independent. This would allow the study of hypernuclei and strange quark matter.

References