Pseudospin: beyond the conventional explanation

A.L. Blokhin, C. Bahri and J.P. Draayer

Department of Physics and Astronomy, Louisiana State University
Baton Rouge, Louisiana 70803

ABSTRACT. The many-particle helicity operator is found to effect a transformation to the pseudospin basis in heavy nuclei. Estimates based on both the nonrelativistic mean-field and relativistic Dirac-Brueckner approaches show that in the helicity-transformed space the nucleons move in a finite-depth, strongly nonlocal potential with a reduced spin-orbit strength. Since the chirality operation is the limiting case of the helicity operation for massless hadrons, the difference between the standard and helicity-transformed representation disappears in the high-energy chiral limit.

RESUMEN. Se encuentra que el operador de helicidad de muchas partículas efectúa una transformación a la base de pseudoespín en iones pesados. Estimaciones basadas tanto en el campo medio no relativista como en el formalismo relativista Dirac-Brueckner muestran que en el espacio transformado de helicidad los nucleones se mueven en un potencial fuertemente no-local de profundidad finita, con una intensidad espín-órbita reducida. Como la operación de quiralidad es el caso límite de la operación de helicidad para los hadrones sin masa, la diferencia entre la representación común y la transformada de helicidad desaparece en el límite quiral de alta energía.

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1. INTRODUCTION

The pseudo spin-space concept in nuclear theory [1, 2] refers to a division of the single-particle total angular momentum into pseudo \((j = \hat{l} + \hat{s})\) rather than normal \((j = l + s)\) orbital and spin parts. This choice of angular momenta is favored by the observed approximate degeneracy of the pairs of single-particle states \(\{(l-1)j, (l+1)j\downarrow\}\), \(j = l - \frac{1}{2}\) within the major shells of heavy \(A \geq 100\) nuclei, where the spin-orbit interaction is known to be strong [3, 4]. The above approximate degeneracy is reminiscent of the weak splitting of the normal spin-orbit doublets \(\{l_j, l_{j+1}\}\), \(j = l - \frac{1}{2}\) in light \(A \leq 28\) nuclei. Thus, by assigning new \(\hat{l}_j\) and \(\hat{l}_{j+1}\) labels to the \((l-1)_j\) and \((l+1)_{j+1}\) states (for instance, a \(\{d_{5/2}, g_{7/2}\}\) pair acquires \(\{\hat{f}_{5/2}, \hat{f}_{7/2}\}\) “pseudo” labels), these nearly degenerate states can be interpreted as weakly split pseudospin doublets. The pseudo spin-orbit coupling scheme gives rise to the many-particle pseudo-SU(3) theory of heavy deformed nuclei [1, 4, 5]. The decoupling of pseudo space-spin degrees of freedom also gives a possible explanation for the existence of identical superdeformed rotational bands [6].

Good pseudospin symmetry in heavy nuclei, while experimentally well corroborated and successfully used in numerous theoretical applications (see [5] for references), still lacks a sound microscopic explanation. A conventional level of understanding is based on the single-particle Hamiltonian of the oscillator shell model, namely, on the fact that deviations from the oscillator energy spectrum approximately follow a \(2j(j + 1) - l(l + 1)\) dependence, which turns into \(\hat{l}(\hat{l} + 1)\) after the normal \(\rightarrow\) pseudo relabeling. Relativistic
nuclear mean-field estimates have been presented [7] in support of such a dependence in the limit of large nucleon number. Also, a unitary operator has been proposed [8] which acts on the spin and angular variables and accomplishes the normal \( \rightarrow \) pseudo relabeling within a given shell; later this approach was revisited [9] in order to remove the one-shell restriction and resulted in the introduction of the operator which is unitary only within the normal-parity oscillator subspace.

This paper is aimed at going beyond the scope of conventional explanations and is based on the assumption that the pseudospin symmetry, which reveals itself on the single-particle (mean-field) level, has a microscopic origin related to the nature of internucleonic forces. A unitary spin-angular microscopic transformation, different from those mentioned above [8, 9], is shown to fulfill key requirements for the pseudospin transformation when applied to the nucleus as a whole.

2. MICROSCOPIC PSEUDOSPIN TRANSFORMATION

To incorporate both the single-particle and many-particle aspects of the pseudospin picture, a microscopic operator, responsible for the normal \( \rightarrow \) pseudo transformation, should be of the form

\[
U_{\text{total}} = \prod_{i=1}^{A} U(r_i, p_i, \sigma_i) \tag{1}
\]

where \( r_i \) stand for the position, \( p_i \) for the momentum, and \( \sigma_i \) for Pauli spin matrices of individual nucleons. The structure of \( U(r, p, \sigma) \) is determined by the following general constraints:

a) \( j^2 = UU^{-1} = l^2 + 21 \cdot \sigma + 2 = 2j^2 - 1^2 + \frac{1}{2} \) — transformation rule [8];

b) \([U, J] = 0\) — rotational invariance;

c) \([U, \mathcal{P}] = [U, \mathcal{T}] = 0\) — parity and time-reversal symmetry;

d) \(UU^\dagger = U^\dagger U = 1\) — unitarity and conservation of observables;

e) \([U, \mathcal{P}] = 0\) — translational invariance.

Once constraints a), b), c) and d) are applied, there are only three choices that remain for \( U \):

\[
U = (d \cdot d^\dagger)^{-1/2} d, \quad d = (\cos \theta r_0 p + i \sin \theta r/r_0) \cdot \sigma, \tag{2}
\]

where \( r_0 \) is a characteristic length and, due to the option of rescaling \( r_0 \), the value of \( \theta \) can always be set equal to \( \pm \frac{\pi}{4} \), 0 or \( \frac{\pi}{2} \). The first of these choices yields the operator of Ref. [9], which is specifically designed for oscillator shell-model applications, or its Hermitian conjugate. However, this operator is unitary only within the subspace of normal parity states — it is undefined in the unique parity subspace.

When global unitarity is required, two possibilities remain. The \( \theta = \frac{\pi}{2} \) choice corresponds to the \( U_r = i \sigma \cdot r/r \) operator (henceforth referred to as the \( r \)-helicity) proposed in Ref. [8]. The \( \theta = 0 \) choice is the \( p \)-helicity, \( U_p = \sigma \cdot p/p \). The latter operator is an only one compatible with the constraint of translational invariance and thus consistent with a
realistic many-particle theory. Below we give some additional arguments in favor of this choice.

3. SINGLE-PARTICLE PICTURE

If transformation (1) is responsible for the pseudospin symmetry, then, in addition to the above constraints, it should nearly decouple the spin and orbit valence degrees of freedom in the heavy nuclei. The applicability of mean field approach allows for a reasonable direct check on this by considering the transformed single-particle Hamiltonian and wavefunctions. Corrections for the center-of-mass motion, which are relatively small for heavier nuclei, are not expected to worsen such an argument.

For simplicity a spherically symmetric field is considered. Then the conventional form of the Hamiltonian and wavefunctions is

\[ H = \frac{p^2}{2M} + V(r) + W(r) 1 \cdot \sigma, \]

\[ \psi_{njlm}(r) = i^l R_{njl}(r)(Y_l \otimes \chi)_{jm}, \]  

where \( n \) is a radial quantum number (number of nodes), \( Y_l \) is a spherical harmonic and \( \chi \) is a Pauli spinor (\( s = \frac{1}{2} \)).

Under the \( r \)-helicity operation (3, 4) transform into

\[ H = U_r H U_r^\dagger \]

\[ = \frac{p^2}{2M} + V(r) - 2W(r) + \frac{\hbar^2}{Mr^2} + \left( \frac{\hbar^2}{Mr^2} - W(r) \right) 1 \cdot \sigma, \]

\[ \psi^{(r)}_{njlm}(r) = U_r \psi_{njlm}(r) \]

\[ = i^l R_{njl}(r)(Y_l \otimes \chi)_{jm}, \]

where \( l \) is determined by the known rule \( l = j \pm \frac{1}{2} \rightarrow \tilde{l} = j \mp \frac{1}{2} \). Note the striking difference between (4), the initial wavefunction, and (7), its \( r \)-helicity transform. Under the transformation the angular part of the wavefunction changes in accordance with the change of angular momentum while the radial function is not affected [8] and, consequently, has an abnormal behavior \( \propto r^l \) instead of \( r^l \) at \( r \rightarrow 0 \). The reason for such a behavior is a noncommutability of the \( r \)-helicity operator with the kinetic energy, which generates \( \frac{1}{r^2} \) corrections to both the central and spin-orbit potentials (see (6)). Moreover, since \( W(r) \) is an attractive potential, the \( r \)-helicity operation can only increase the spin-orbit splitting in absolute value. This is an additional reason why the translationally non-invariant \( r \)-helicity transformation cannot be regarded as an appropriate microscopic pseudospin transformation.
To consider the $p$-helicity case, start from the coordinate representation for the unitary operator $U_p$:

$$U_p = -iK((\hat{l} - \Lambda - 1)^{-1}r(\sigma \cdot \nabla),$$

where $K = \Gamma\left(\frac{i+\Lambda+2}{2}\right)\Gamma\left(\frac{i-\Lambda}{2}\right)\Gamma\left(\frac{i+\Lambda+3}{2}\right)\Gamma\left(\frac{i-\Lambda-1}{2}\right)^{-1}$ is a unitary operator, $\hat{l} = \frac{1}{2}((1 + 4l^2)^{-1/2} - 1)$ is the operator whose eigenvalues are quantum numbers of the orbital momentum, and $\Lambda = r \cdot \nabla = r\partial/\partial r$ is the infinitesimal shear operator. The unitarity of $K$ follows from the conjugation rules $\mathcal{J}t = -\mathcal{J} - 3$, $\mathcal{I}t = i$. Then (3,4) transform into

$$H = U_pHU_p^\dagger$$

$$= \frac{\mathbf{P}^2}{2M} + K(V(r) - 2W(r) - (\hat{l} + 1)\nu(r))K^\dagger + K(\nu(r) - W(r))K^\dagger \cdot \sigma,$$

$$\psi_{njlm}^{(p)}(r) = U_p\psi_{njlm}(r) = i\tilde{R}_{njl}(r)(\Psi_{l0}^\dagger \chi_{jm}),$$

where the function $\nu(r) = (\hat{l} - \Lambda - 1)^{-1}(rV'(r) - (\hat{l} + 2)rW'(r))((\hat{l} + \Lambda + 2)^{-1}$ and $\tilde{R}_{njl}(r) = -\Gamma\left(\frac{i+\Lambda+3}{2}\right)\Gamma\left(\frac{i-\Lambda}{2}\right)\Gamma\left(\frac{i+\Lambda+3}{2}\right)\Gamma\left(\frac{i-\Lambda-1}{2}\right)^{-1}R_{njl}(r)$. In contrast with the previous case, the difference $H_p - H$ is now due entirely to the transformation of the potential energy. Because of the complicated dependence on $\Lambda$ in the r.h.s. of (9), the transformed potential energy operator is strongly nonlocal. Although (9) in its general form does not provide incontrovertible evidence of a reduction in the magnitude of the spin-orbit splitting, the latter is likely to happen at low $\hat{l}$ within the nuclear surface region if an effective value of the $\Lambda$ operator exceeds $\hat{l} - 1$.

To qualitatively understand the behavior of the $\tilde{R}_{njl}(r)$ function, observe that $U_p$ contains three consecutive operations: a Fourier transform, a switch from $l$ to $\hat{l}$ (see (7)), and an inverse Fourier transform, which altogether determine the mapping for the radial function. This mapping, when applied to the harmonic oscillator functions, shows the following universal asymptotics:

$$\tilde{R}_{njl}(r) \propto \begin{cases} r^{\hat{l}}, & r \to 0, \\ r^{-(\hat{l}+3)}, & r \to \infty. \end{cases}$$

The completeness of the oscillator basis validates (10) for more realistic cases, including that of our consideration. The standard $r^{\hat{l}}$ dependence in the interior region follows because deeply in the bulk of a heavy nucleus the transformed central potential is not expected to deviate significantly from the flat behavior of $V(r)$. The $r^{-(\hat{l}+3)}$ outer region asymptotics means a more diffuse surface (with the nuclear density decreasing as $r^{-6}$). Thus, the $p$-helicity transformation brings the higher diffuseness and strong nonlocal potential to the surface region while performing the normal $\rightarrow$ pseudo relabeling of the angular momenta.
4. Estimates Based on OBEP

A relativistic extension of the Brueckner theory (see [10] for references) provides parameter-free microscopic predictions for both infinite and finite nucleon systems. While the description of nuclear matter is of good accuracy, the gross features of finite (especially lighter) nuclei are reproduced less well but nonetheless much better than in nonrelativistic approaches [11]. For this reason, the results of Dirac-Brueckner nuclear matter calculations are used below for estimating the p-helicity transformed two-body nuclear interaction, as well as the mean field, in heavier nuclei.

For a wide range of densities, including the saturation point, the nucleon-nucleon interaction in the infinite medium is perfectly approximated by a one-boson exchange potential (OBEP) with the boson parameters fit to the Bonn model, and the density-dependent effective nucleon mass \( M^* \) calculated in a self-consistent manner [10]. To a very good approximation, the density-dependent self-consistent field has the same Lorentz structure as the free Dirac Hamiltonian. Consequently, a single-particle Hamiltonian in the medium commutes with the p-helicity, and the helicity transformation does not affect the single-particle energies.

However, the two-body interaction changes dramatically. In the representation of plane-wave Dirac spinor nucleon states, normalized to unity, which is the basis for Dirac-Brueckner calculations, the p-helicity operation is equivalent to \( i\gamma^5 S \) when acting on the right (ket) states. Here \( \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) is the product of Dirac matrices, and \( S \) is a formal operation for switching the sign of the effective mass: \( Sf(M^*, p) = f(-M^*, p) \).

Since the \( \gamma \)-matrices simply switch the sign under the chiral transformation, the OBEP is bilinear in those matrices, the helicity transformation of the OBEP is reduced to changing the sign of \( M^* \) in the momentum representation. This is easily accomplished in the two-nucleon center-of-mass frame (simulating the center-of-mass frame of the nucleus) and produces strongly incident-energy dependent, that is nonlocal, interactions. Because only a rough estimate for the potentials is sought, here these potentials are converted into local approximations by averaging over allowed values of the relative momentum \( q \) with an appropriate distribution of \( q \) at a fixed momentum transfer \( k \). The localized helicity-transformed OBEP in the momentum space converges rapidly in the shortwave region \( (k > 2k_F) \) to the initial potential, averaged with the same distribution. The values of the localized central part of the potential also coincide at \( k = 0 \) before and after the helicity transformation in accordance with the helicity-invariance of the single-nucleon energy in the medium.

The localized estimates for transformed single-particle potentials in the coordinate space, as given in Figs. 1 and 2, are calculated in the first order perturbation theory with respect to \( \delta V(k) \), the localized difference between transformed and initial OBEP. Unperturbed potentials are taken in the standard Woods-Saxon parametrization [12] with the following adjustment for the radial dependence

\[
\left( 1 + \exp \left( \frac{r - R}{a} \right) \right)^{-1} \rightarrow \frac{1}{2} \left( 1 - \text{erf} \left( \sqrt{\frac{3}{2}} \frac{R - r}{\pi a} \right) \right),
\]

which allows for a simple analytic Fourier transform along with a quantitative fit. The estimate is done analytically using a Skyrme-type low momentum expansion for \( \delta V(k) \)
and zero-order nuclear density distribution of the same kind as (11) but with a lesser diffuseness [3]. Due to the strong nonlocality of transformed OBEP, the analytic formulas for single-particle potentials are more complicated than in the conventional scheme with Skyrme forces [13]. Basic complications and approximations are the following: a) $d$-waves and $f$-waves of the relative motion make an impact that is on the same order of magnitude as the $s$-waves and $p$-waves; b) $\delta G(k)$, a difference between the localized $G$-matrix in the transformed space and the physical $G$-matrix, coincides with $\delta V(k)$ in the first order because of a small contribution to the $(k > 2k_F)$ region (see previous paragraph); c) instead of choosing a specific distribution for the relative momentum $q$, the $k$-dependence of averaged $q^{-2}$ function (which is responsible for most of the nonlocalities) is set equal to a model function $4(1 - \exp(-\frac{nk^2}{4k_F}))k^2$ which mimics an effect of various distributions in the $k \to 0$ region by using different values of $\eta$ and has correct asymptotics at $k \gg k_F$ (the upper limit $\eta = 9$ corresponds to the Fermi gas distribution in the infinite matter, the lowest possible value $\eta = 1$ is associated with a distribution localized at $q = k_F$ and makes sense for surface effects); d) the ratio of proton and neutron densities is fixed for all $r$, and Coulomb corrections are not considered due to the coarseness of the procedure; e) $M^*$ and $k_F$ are fixed at their saturation point values [10].

Although the single-nucleon potentials shown in the figures are rough local estimates for the strongly nonlocal fields, they display several features that are characteristic of the

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**FIGURE 1.** Localized estimates for the neutron central and spin-orbit potentials of $^{208}$Pb before and after the helicity transformation (continuous lines and shaded areas, respectively). The two curves that define the borders of the shaded areas were determined by using different reasonable approximations for the relative momentum distribution in a finite nucleus ($5 \leq \eta \leq 9$ for the central potential, $1 \leq \eta \leq 3$ for the spin-orbit potential — see the explanation in the text.)
pseudospin symmetry. First, in accordance with Section 3, the transformation preserves the finite depth of the central potential and increases the surface diffuseness. Second, a minimum of the spin-orbit potential, which is located in the surface region in the normal representation, gets shifted deeper into the bulk as a result of the helicity transformation. And from this it follows that the magnitude of the spin-orbit potential in the region where the wavefunctions are localized and which is primarily responsible for the interaction strength, exhibits a dramatic decrease. Also note that the effective *pseudo* spin-orbit interaction of the neutrons is more repulsive than one of the protons — in agreement with experiment [8].

5. Conclusions

The topic of this contribution is the microscopic origin of pseudospin symmetry in heavy nuclei. The many-particle $p$-helicity operator was found to be the only form that generates a proper relabeling of the spin and orbital momenta while satisfying all other necessary general symmetry requirements. This form for the transformation was then shown to
yield wavefunctions with physically reasonable properties and single-particle spin-orbit interaction strengths for protons and neutrons that are significantly less in magnitude as compared to those required in the normal (not pseudo) picture.

The approximate independence of the single-nucleon spectrum in an infinite medium on the helicity transformation and the consistency of the microscopic estimates for the single-particle nuclear potentials with the Dirac-Brueckner calculations was used to connect the pseudospin symmetry to the boson-exchange nature of nucleon-nucleon interactions. Because of the close relation (coincidence in the ultrarelativistic limit) between the helicity and chirality operations, the goodness of pseudospin symmetry can be expected to increase with raising densities (energy per particle) in hadronic systems, and even yield to chiral symmetry in the region of asymptotic freedom. The results also seem to suggest that due to relatively small values of the s-quark and d-quark masses, quark models in a broad energy domain might actually profit from an introduction of the pseudospin basis.

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