Energy surfaces of algebraic models*

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ABSTRACT. A procedure to study shapes and stability of algebraic models introduced by Gilmore is presented. According to the time dependent variational principle the coherent states, for algebraic models, are appropriate trial wavefunctions. One calculates the expectation value of the Hamiltonian with respect to the corresponding coherent states to study the energy surfaces of the model. Then equilibrium configurations of the resulting energy surface, which depends in general on state variables and a set of parameters, are classified through the catastrophe theory. For one and two-body interactions in the Hamiltonian of the interacting boson model (IBA-1), the critical points are organized through the separatrix. As an example, we apply this separatrix to describe the energy surfaces associated to the dynamical symmetries of the IBA-1, and to the effective hamiltonians of the Ru, Os and Sm isotopes.

RESUMEN. Presentamos un procedimiento establecido por Gilmore para estudiar las formas y estabilidad asociadas a modelos algebraicos. De acuerdo a la formulación variacional dependiente del tiempo de la mecánica cuántica, los estados coherentes son las funciones de prueba apropiadas para modelos algebraicos. En este método se calcula el valor esperado del Hamiltoniano con respecto a los estados coherentes, definiéndose las superficies de energía del modelo. Entonces las configuraciones de equilibrio de las superficies de energía, dependiente en general de variables de estado y un conjunto de parámetros, se clasifican por medio de la teoría de catástrofes. Para interacciones de uno y dos cuerpos en el hamiltoniano del modelo de bosones en interacción (MBI), sus puntos críticos son organizados a través de la separatriz. Como ejemplo, utilizamos esta separatriz para describir las superficies de energía asociadas a las simetrías dinámicas del MBI, y los hamiltonianos efectivos que describen los isótopos del Ru, Os y Sm.

PACS: 21.60.Ev; 21.60.Fw; 03.65.Fd

1. INTRODUCTION

The geometry of algebraic nuclear models can be studied by means of the time-dependent variational principle [1,2]. This formalism provides us with a classical limit of the nuclear model, in particular we are mainly concerned with the static properties of the Hamiltonian function (energy surface) associated to the considered algebraic nuclear model. In

* Work supported in part by project UNAM-DGAPA IN102094.
general these Hamiltonian functions depend on state variables and a set of parameters, then the appropriate mathematical tool to determine the most general behaviour of their equilibrium configurations is the catastrophe formalism [3].

A connection between the interacting boson model [4] and the geometrical approach of Bohr-Mottelson [5] was done by expressing the IBA-1 Hamiltonian in terms of shape variables. This can be achieved by means of the intrinsic boson states defined by [6] or by the corresponding coherent states [2]. Analysis of shape and shape phase transitions in this model have been realized in Ref. [7, 8]. In this work we apply the procedure introduced in Ref. [2] to the interacting boson model, but for the general Hamiltonian of one and two-body central interactions involving $s$ and $d$ bosons [4] and determining its associated separatrix. We show that the equilibrium configurations can be classified by means of two parameters, which are enough to describe the most general energy surface. This analysis generalizes previous works where only transitions between pairs of exact SU(5), O(6) and SU(3) symmetries were considered [4] and transitions between particular isotope nuclei were studied [6]. In the last decade, effective Hamiltonians of the IBA-1 have been used to describe energy spectra and transition probabilities of chains of isotopes and isotones [9, 10]. In particular, the effective Hamiltonians for Ru [11], Os [12] and Sm [12] isotopes were determined, i.e., the best choice of the parameters of the general IBA-1 Hamiltonian that reproduce the corresponding experimental data. Using these effective Hamiltonians we construct their energy surfaces and show how they are organized by the separatrix. This locus of points in the space of control parameters is useful to know: i) How many equilibrium configurations yield the system and ii) If the behaviour of the model around the critical points may or may not be approximated by an harmonic oscillator.

In Sect. 2 we review how a symplectic structure, of a Hamiltonian given in terms of the generators of a Heisenberg-Weyl algebra, can be constructed. In Sect. 3 a brief summary of the IBA-1 is presented. In Sect. 4 as an example of how to construct the separatrix of a function depending of a set of variables and a set of parameters, we study the equilibrium configurations and stability of the potential energy surfaces of the geometrical collective model. Afterwards a similar analysis is carried out by means of the catastrophe formalism to classify the shapes and stability of the most general energy surfaces of the IBA-1. Also the curves in the space of control parameters associated to the IBA-1 dynamical symmetries and to the effective Hamiltonians of Ru, Os and Sm isotopes are shown. Finally a summary of the results is given in the last section.

2. ENERGY SURFACES OF ALGEBRAIC MODELS

In this contribution we study the static properties of the IBA-1. However for its importance we will describe the construction of a symplectic structure, when one considers a complex parametrized trial wavefunction. In this case, the action

$$\int_{t_i}^{t_f} dt \, L(\psi, \psi^*),$$  (1)
is considered with the wavefunction, $\psi(\alpha_1, \alpha_2, \cdots, \alpha_N)$ depending of $N$ complex variables. The Lagrangian is defined in the following form

$$L = \frac{i}{2} \left( \dot{\alpha}_j \langle \psi | (\partial \psi / \partial \alpha_j) \rangle - \dot{\alpha}_j^* \langle \partial \psi / \partial \alpha_j | \psi \rangle \right) \frac{1}{\langle \psi | \psi \rangle} - \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle},$$

(2)

where repeated indices are summed. By introducing the following notation for the normalization and Hamiltonian functions

$$N(\alpha, \alpha^*) = \langle \psi(\alpha) | \psi(\alpha) \rangle,$$

(3a)

$$E(\alpha, \alpha^*) = \frac{\langle \psi(\alpha) | H | \psi(\alpha) \rangle}{N(\alpha, \alpha^*)}.$$

(3b)

The Lagrangian (2) can be rewritten as follows

$$L = \frac{i}{2} \left( \dot{\alpha}_j \frac{\partial}{\partial \alpha_j} - \dot{\alpha}_j^* \frac{\partial}{\partial \alpha_j^*} \right) \ln N(\alpha, \alpha^*) - E(\alpha, \alpha^*).$$

(4)

In this form, it is straightforward to show that variations in the variables $\alpha_k$ and $\alpha_k^*$, induce a transformation in the action that is stationary if the following equations of motion are satisfied

$$i\dot{\alpha}_j C_{lj} = \frac{\partial E}{\partial \alpha_l^*}, \quad -i\dot{\alpha}_j^* C_{lj} = \frac{\partial E}{\partial \alpha_l},$$

(5)

with the definition

$$C_{lj} = \frac{\partial^2 \ln N}{\partial \alpha_l \partial \alpha_j^*}.$$

(6)

By means of the inverse of the matrix given in the last expression, the generalized Poisson brackets can be defined:

$$\{f, g\} (\alpha, \alpha^*) = \left( \frac{\partial f}{\partial \alpha}, \frac{\partial g}{\partial \alpha^*} \right) \left( \begin{array}{cc} 0 & C^{-1} \\ -C^* & 0 \end{array} \right) \left( \frac{\partial g}{\partial \alpha} \right).$$

(7)

It is straightforward to put the equations of motion (5) in terms of the generalized Poisson brackets, i.e.,

$$\dot{\alpha}_k = i \{ E, \alpha_k \}, \quad \dot{\alpha}_k^* = i \{ E, \alpha_k^* \}.$$

(8)

Therefore the energy surfaces (ES) of algebraic models can be determined by means of the coherent states of the associated algebraic structure of the Hamiltonian. As an example, a Hamiltonian written in terms of the generators of a Heisenberg-Weyl algebra is considered, i.e.,

$$H = c_0 b^+ b + c_2 (b^+)^2 + c_1 (b^+ + b),$$

(9)
where the operators $b^\dagger$ and $b$ satisfy standard creation and annihilation commutation relations. Although this Hamiltonian can be solved analytically by means of a Bogoliubov transformation, we use it to illustrate the procedure to construct a symplectic structure and the ES of an algebraic model.

The coherent state is defined by the action of the raising generator on the vacuum state $|0\rangle$

$$|\alpha\rangle = \exp(\alpha b^\dagger)|0\rangle.$$  \hfill (10)

The Baker-Campbell-Hausdorff formulas can be used to calculate the overlap of two coherent states:

$$N(\alpha, \alpha^*) = \exp(\alpha \alpha^*).$$  \hfill (11)

Then the symplectic structure is determined by the Hamiltonian function

$$\mathcal{E}(\alpha, \alpha^*) = \frac{\langle \alpha | H | \alpha \rangle}{\langle \alpha | \alpha \rangle} = \left\{ c_0 \alpha \alpha^* + c_2 (\alpha^2 + \alpha^{*2}) + c_1 (\alpha + \alpha^*) \right\},$$  \hfill (12)

and the generalized Poisson brackets

$$\{f, g\} = \left\{ \frac{\partial f}{\partial \alpha} \frac{\partial g}{\partial \alpha^*} - \frac{\partial f}{\partial \alpha^*} \frac{\partial g}{\partial \alpha} \right\}.$$  \hfill (13)

Asking the reality condition $\alpha = \alpha^* = \rho$, the Hamiltonian function or energy surface takes the form

$$\mathcal{E}(\rho) = (c_0 + 2c_2)\rho^2 + 2c_1 \rho.$$  \hfill (14)

Therefore the energy surface is a function depending of one variable and two parameters, and the catastrophe formalism can be used. However for this system the result is too simple and in Sect. 4 we prefer to study, as an example, the potential energy surfaces of the collective model through the catastrophe theory.

3. THE INTERACTING BOSON MODEL

In 1975 this model [4] was introduced to describe collective properties of even-even nuclei far from closed shells through the interactions between two kinds of bosons, one with angular momentum $L = 0$ (the s-boson) and another with angular momentum $L = 2$ (the d-boson). The six possible boson states give rise to a U(6) group structure. The bosons represent pairs of fermions, the s-boson reflects the strong pairing attraction of identical nucleons whereas the d-boson is a result of the weaker $J = 2^+$ attraction [4]. Therefore nuclei are pictured as systems of s and d bosons, whose number is equal to half the number
of the valence nucleons, the core being considered inert. When a shell is more than half full, hole-pairs are counted instead of particle-pairs.

The most general one and two body Hamiltonian that conserves the total number of bosons is

\[ H_{\text{IBA}} = \varepsilon_s N_s + \varepsilon_d N_d + \sum_{L=0,2,4} \frac{c_L}{2} \sqrt{2L + 1} \left( [d^\dagger \times d^\dagger]^L \times [\bar{d} \times \bar{d}]^L \right)^0 \]

\[ + \frac{v_2}{\sqrt{2}} \left( [d^\dagger \times d^\dagger]^2 \times [\bar{d}]^0 + s^\dagger [d^\dagger \times [\bar{d}^\dagger]^2]^{[0]} \right) \]

\[ + \frac{v_0}{2} \left( [d^\dagger \times d^\dagger]^0 s^2 + s^\dagger [\bar{d} \times [\bar{d}]^0] \right) \]

\[ + u_2 s^\dagger s [d^\dagger \times \bar{d}]^0 + \frac{u_0}{2} s^\dagger^2 s^2, \] (15)

where the sets of boson operators \( s, s^\dagger \) and \( d^\dagger_\mu, d_\mu \) satisfy the following, different from zero, commutators

\[ [s, s^\dagger] = 1, \quad [d_\mu, d^\dagger_{\mu'}] = \delta_{\mu, \mu'}. \] (16)

Now, we construct the coherent state of a six dimensional harmonic oscillator, following the procedure indicated in the previous section. However in this case the associated group is compact and then we restrict the exponential to only one term of the Taylor series expansion,

\[ |N, \bar{a} \rangle = \left( s^\dagger + \sum_{\mu} \alpha_\mu d^\dagger_\mu \right)^N |0\rangle. \] (17)

Evaluating the corresponding Eq.(3b) one arrives to the formulae for the energy surface of the model [6,8], i.e.,

\[ E(\beta, \gamma) = N \varepsilon + \beta^2 \left( \frac{\beta^2}{1 + \beta^2} \right) + \frac{N (N-1)}{(1 + \beta^2)^2} (a_1 \beta^4 + a_2 \beta^3 \cos 3\gamma + a_3 \beta^2 + u_2), \] (18)

where it was used that the laboratory variables \( \alpha_\mu \) can be expressed in terms of two intrinsic parameters \( \beta \) and \( \gamma \) plus three Euler angles. Besides as the energy surface is a rotational invariant all the dependence in the Euler angles disappear. The parameters \( a_1, a_2, a_3, \) and \( \varepsilon \) of the Eq. (18) are combinations of those that appear in the IBA-1 Hamiltonian (15):

\[ a_1 = \frac{c_0}{10} + \frac{c_2}{7} + \frac{9c_4}{35}, \] (19a)

\[ a_2 = -\frac{2}{\sqrt{35}} v_0, \] (19b)

\[ a_3 = \frac{1}{\sqrt{5}} (v_0 + u_2), \] (19c)

\[ \varepsilon = \varepsilon_d - \varepsilon_s. \] (19d)
4. SHAPES AND STABILITY OF ENERGY SURFACES

The energy surfaces define functions of state variables and a set of parameters, and the catastrophe theory is used to analyze their equilibrium configurations. This formalism let us organize all the possible shapes of the ES into well defined separated regions of the space of control parameters.

To illustrate how this is done, we consider the potential energy surfaces (PES) of the simplest version of the generalized collective model \[51,\]

\[V(\beta, \gamma; C_2, C_3) = \beta^4 - C_3 \beta^3 \cos 3\gamma + C_2 \beta^2. \tag{20}\]

The equilibrium points are determined by solving the equation \(\nabla V(\beta, \gamma) = 0\). These points are given by \((\beta_0 = 0, \gamma_0 = 0)\) for any values of the parameters and when

\[\beta_c = \frac{3C_3 \pm \sqrt{9C_3^2 - 32C_2}}{8}, \quad \gamma_c = 0, \pi/3. \tag{21}\]

Notice that the function (20) has equilibrium configurations only on the axis \(\gamma_c = 0\) and \(\pi/3\) because the other regions of values of the variable \(\gamma\) are physically equivalent \[51,\] which is due to the \(C_{3v}\) point symmetry of the PES.

The set of degenerate critical points defines a locus in the parameters space which is called the bifurcation set; these sets are obtained through the singularities of the Hessian matrix, \(\mathcal{H}_{ij} = \frac{\partial^2 V}{\partial \beta_i \partial \beta_j}(x_1, x_2)\) with \(x_1 = \beta\) and \(x_2 = \gamma\), where \(\det \mathcal{H} = 0\). However for the variable \(\beta\) it is immediate, from the Eq. (21), that critical points present degeneracy if the control parameters satisfy the expressions

\[9C_3^2 - 32C_2 = 0, \tag{22a}\]

\[C_2 = 0. \tag{22b}\]

If the expression (22a) is satisfied we obtain double degenerate critical points at \(\beta_c = 3C_3/8\), while when (22b) is met we get \(\beta_c = 0\) and it degenerates with \(\beta_0 = 0\). The bifurcation set, \(C_3 = 0\) with \(C_2 \leq 0\), for the variable \(\gamma\) is determined from the Hessian matrix, by an examination of its eigenvalues. For the case of the negative \(C_2\)-axis the critical point is given by \((\beta, \gamma) = (\sqrt{-C_2/2}, \gamma)\). Besides, it is straightforward to prove that if \(C_2 = 0\) and \(C_3 = 0\) the critical point is triple degenerate and localized at \(\beta_c = 0\).

There are another sets of points in the parameter space which represent structurally unstable functions, and they are called Maxwell sets. These are curves in the parameter space where the PES take the same value in two or more critical points.

To find these sets: One has to determine the isolated critical points that yield the same value of the energy surface for a given choice of the parameters. First the critical point \((\beta_0 = 0, \gamma_0 = 0)\) is used, in this case the PES take the value zero:

\[\beta^2 \left( \beta^2 - C_3 \beta + C_2 \right) = 0. \tag{23}\]
Solving the last expression one gets the intersections on the $\beta$-axis

$$\beta = \frac{C_3}{2} \pm \frac{1}{2} \sqrt{C_3^2 - 4C_2},$$

and it is clear that between these two cuts there is always an extremum, thus the Maxwell set is defined by the curve

$$C_2 = \frac{1}{4} C_3^2.$$  

This extremum is localized in $\beta_c = C_3/2$ and using (25) it can be checked that it satisfies (21). Second the two critical points (21) are considered, where the PES take a value different from zero, and it is asked

$$V \left( \frac{3C_3 + \sqrt{9C_3^2 - 32C_2}}{8}, 0; C_2, C_3 \right) - V \left( \frac{3C_3 - \sqrt{9C_3^2 - 32C_2}}{8}, 0; C_2, C_3 \right) = 0. \tag{26}$$

This expression implies two conditions on the control parameters, one of them is that the critical points present degeneracy, which is not useful because they must be isolated and the other is the Maxwell set $C_3 = 0$, which is localized in $\beta_c = \sqrt{-C_2/2}$.

Therefore we have constructed the separatrix of the system and it is constituted by the bifurcation and Maxwell sets mentioned above. These curves are displayed in Fig. 1, and they divide the space of control parameters in regions, each one of them characterizing a particular shape yield by the model. One can identify four regions: i) Above the solid parabola, the PES have only one minimum at $\beta = 0$, then describing spherical nuclei. ii) Between solid and dashed parabolae, the PES exhibit two minima, being global that with $\beta = 0$, thus characterizing spherical nuclei with deformed excited states. iii) Between dashed parabola and $C_3$ axis, the PES exhibit two minima, being global that with $\beta \neq 0$, therefore describe deformed nuclei with spherical excited states. iv) Below the $C_3$ axis the PES have one minimum at $\beta \neq 0$, one maximum at $\beta = 0$ and one saddle point, in consequence characterize deformed nuclei. The PES around these critical points can be approximated by an harmonic oscillator. Notice that, the PES can have shape coexistence phenomena between spherical and deformed nuclei when $0 < C_2 < (9/32) C_3^2$. Other region of interest is the $C_2$-axis, for $C_2 > 0$ the PES describe spherical nuclei, while for $C_2 < 0$ the PES characterize $\gamma$-unstable nuclei. When this semiaxis is crossed, there is a transition from oblate to prolate shapes and viceversa.

On the locus of points that define the separatrix, we can have shape transitions and onto degenerate critical points the PES cannot be approximated by quadratic functions. It is important to remark that the PES have a mirror symmetry along the $C_2$ axis, which physically represents transitions from prolate to oblate shapes.

By means of the transformation $\beta = y + C_3/4$ the PES is taken to the canonical form of the Cusp catastrophe, which let us determine the mathematical parameters

$$a = \frac{C_2}{2} - \frac{3C_3^2}{16}, \tag{27a}$$
In terms of these parameters the Eqs. (22) are equivalent to the cusp

\[
\left(\frac{a}{3}\right)^3 + \left(\frac{b}{2}\right)^2 = 0.
\] (28)

Notice that for the cusp catastrophe the condition \(b = 0\) in (27b) implicitly gives the two Maxwell sets calculated before.

Now we study the equilibrium configurations of the energy surface associated to the IBA-1, which is given in Eq. (15). We calculate the critical points by taking the derivatives with respect to \(\beta\) and \(\gamma\) variables. It is straightforward to see that the critical points correspond only to \(\gamma = 0\) (prolate case) or \(\gamma = \pi/3\) (oblate case). We can restrict the range of the variable \(\gamma\) from zero to sixty degrees because the function (15) have the \(C_{3v}\) point symmetry. Therefore we can restrict to the prolate case, without losing generality. The energy surface (15), with \(\gamma = 0\), can be re-written in terms of the following parameters

\[
r_1 = \frac{a_3 - u_0 + w}{2a_1 + w - a_3}, \quad r_2 = -\frac{2a_2}{2a_1 + w - a_3},
\] (29)

where it was defined \(w \equiv \frac{c}{N-1}\) and it takes the form

\[
E(\beta) = \frac{1}{(1 + \beta^2)^2} \left\{ \beta^4 + r_1\beta^2(\beta^2 + 2) - r_2\beta^3 \right\}.
\] (30)
where a renormalization of the energy surface was done and constants terms were neglected. One has to notice that the oblate case can be regained by interchanging \( r_2 \) by \(-r_2\) or equivalently \( a_2 \) by \(-a_2\).

To find the extrema in \( \beta \), one needs to solve the equation

\[
\beta (r_2 \beta^3 + 4 \beta^2 - 3 r_2 \beta + 4 r_1) = 0.
\] (31)

Afterwards we have obtained the critical points, we follow the procedure described above for the analysis of the PES. Thus through the corresponding Hessian matrix the bifurcation sets are determined, while the Maxwell sets are settled by finding the locus of isolated critical points that yield the same value for the ES.

Then the separatrix of the model is defined by the bifurcation sets: \( r_2 \)-axis, \( r_1 \)-negative semiaxis, and those given by the expressions

\[
r_{11} = -\frac{(9r_2^2 + 16)^{3/2}}{54r_2^2} - \frac{32}{27r_2^2} - 1, \]

\[
r_{12} = \frac{(9r_2^2 + 16)^{3/2}}{54r_2^2} - \frac{32}{27r_2^2} - 1.
\] (32a, 32b)

Together with the Maxwell sets: the negative \( r_1 \)-semiaxis, plus

\[
r_{13}^\pm = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{r_2^2}{2}}.
\] (33)

The separatrix associated to the IBA-1 Hamiltonian is shown in Fig. 2. The dynamical symmetries of the IBA-1 have played an important role in the development of the model, here they are considered to determine the stability properties and shapes of the vibrational, rotational and \( \gamma \)-unstable cases. For the vibrational case one gets \( (r_2, r_1) = (0, 1) \), in the rotational limit there are two possibilities, for prolate and oblate shapes we get \( (r_2, r_1) = \left(\frac{4\sqrt{2}}{3}, -\frac{4}{3}\right) \) and \( (r_2, r_1) = \left(-\frac{4\sqrt{2}}{3}, -\frac{4}{3}\right) \), respectively. In the \( \gamma \)-unstable limit the control parameters take the value \( (r_2, r_1) = (0, -1) \). These points are also displayed in Fig. 2.

When combinations of pairs of Casimir operators of the exact limits U(5), SU(3), and O(6) are studied, it is found that the accessible shapes are localized in straight lines that connect the points mentioned above, which are also displayed in Fig. 2.

Notice that each point \( (r_2, r_1) \) in Fig. 2 is indicating an energy surface. For \( r_2 > 0 \), the separatrix divides the space of control parameters in two main domains: i) When \( r_1 > r_{13}^+ \) the ES characterize spherical nuclei and ii) For \( r_1 < r_{13}^- \) they describe prolate deformed nuclei, whose absolute minimum starts loosing binding for \( r_1 < r_{13}^- \) and this tendency continues for more negative values of \( r_1 \). For \( r_2 < 0 \), the ES characterize spherical and oblate nuclei with similar properties to those indicated above. For the region \( 0 < r_1 < r_{12} \), the ES can present the shape coexistence phenomena between spherical and deformed nuclei. Another region is the \( r_1 \)-axis, for \( r_1 > 0 \) the ES describe spherical nuclei and for \( r_1 > 0 \) the ES characterize \( \gamma \)-unstable nuclei. When this semiaxis is crossed there is a transition between oblate to prolate shapes and viceversa.
FIGURE 2. The separatrix for the IBA-1 Hamiltonian is shown, together with the localization of the points characterizing the energy surfaces of the vibrational, rotational and γ-unstable limits. Straight lines connecting these points are also displayed, indicating transitions between them.

| Table I. Parameters [KeV] used to describe the Ru, Os and Sm isotope chains. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\epsilon$     | $c_0$           | $c_2$           | $c_4$           | $u_0$           | $u_2$           | $v_0$           | $v_2$           |
| Sm 2170.62      | -613.72         | -319.82         | -377.64         | 8.65            | -508.48         | -66.04          | 127.17          |
| Ru 990.20       | -185.40         | -77.40          | -0.40           | 46.60           | -118.51         | -104.20         | 0.00            |
| Os 428.90       | -367.40         | -284.50         | -149.40         | -83.20          | -396.45         | -216.45         | 56.89           |

Now we apply the results to find the shapes and stability of the Ru, Os and Sm isotopes. For the Ru case, one has the number of valence protons pairs, $N_\pi = 3$. As we consider isotopes with a mass number varying from $A = 98$–$110$, the corresponding number of neutron bosons runs from $N_\nu = 2$ to $N_\nu = 8$. Thus the size of the space in the IBA-1 is determined by the total number of bosons, which is the sum of the numbers of proton and neutron bosons. Next the Os isotopes with $A = 180$ to $A = 194$ are considered, they have $N_\pi = 3$ and $N_\nu$ takes the values from 2 to 11. In the Sm case, one has the number of proton bosons, $N_\pi = 6$, and as we take into account $A = 148$–$160$, the number of neutron bosons runs from $N_\nu = 2$ to $N_\nu = 8$. An analysis of their energy spectra and electromagnetic transitions using the Hamiltonian (15) is made in [11] and [12]. For these isotope chains, the parameters used are presented in Table I. Substituting these parameters into the equations (19) we get the values of $a_1$, $a_2$ and $a_3$, by means of which one can easily construct the corresponding energy surface of each studied isotope.

To find the region of the $separatrix$, Fig. 2, where the different isotopes are localized one calculates the parameters $r_1$ and $r_2$ through the equations (29), as functions of the total number of bosons. Eliminating the total number of bosons in that equation, it is found
Figure 3. Control parameters of $^4\text{Ru}$, $^8\text{Os}$ and $^{16}\text{Sm}$ isotopes. The $^4\text{Ru}$ nuclei are indicated by open circles, the $^8\text{Os}$ by solid points and the $^{16}\text{Sm}$ by crosses. In these three cases the points go down as the total number of bosons increases.

the straight line

$$r_1 = \frac{a_3 - a_1 - u_0/2}{a_2} r_2 + 1.$$  \hspace{1cm} (34)

Notice that for all the cases the straight line has an ordinate equal to one, unless there is not one body terms in the Hamiltonian. This is the case when purely two body interactions are considered belonging to the SU(3) and O(6) symmetries.

For the Ru, Os and Sm, one gets the following parametric equations of the straight line (35), that is

$$r_1 = \frac{990.2 - 146.2(N - 1)}{990.2 + 40.2(N - 1)}, \hspace{1cm} r_2 = 0,$$  \hspace{1cm} (35a)

$$r_1 = \frac{428.9 - 190.9(N - 1)}{428.9 + 42.5(N - 1)}, \hspace{1cm} r_2 = \frac{38.47(N - 1)}{428.9 + 42.5(N - 1)},$$  \hspace{1cm} (35b)

$$r_1 = \frac{2170.62 - 257.80(N - 1)}{2170.62 - 151.40(N - 1)}, \hspace{1cm} r_2 = \frac{85.98(N - 1)}{2170.62 - 151.40(N - 1)},$$  \hspace{1cm} (35c)

respectively. The localization of the points (32) are shown in Fig. 3.

5. Summary

In this contribution we have shown how to use coherent states and the catastrophe formalism to classify the shapes and stability properties of the energy surfaces of algebraic
models. In particular we have constructed explicitly the separatrix for the PES of the geometrical collective model and for the ES of the IBA-1 model. They give us information about the parameter strengths of the models where the equilibrium configurations can be approximated by quadratic forms and where they can not. Also they provide with a classification of the accessible shapes yield by the models.

For the IBA-1 we want to enhance the appearance in the space of control parameters of a double triangle where the vertices characterize the shape properties of the $U(5)$ vibrational, $SU(3)$ rotational, and $O(6)$ $\gamma$–unstable limits. Another result presented in this work is related with the stability properties of the energy surfaces of the Ru, Os, and Sm isotopes. As it can be seen from Fig. 3, the Ru isotope chain represents a transition between the vibrational to $\gamma$–unstable nuclei, for Os is a mixture of the three limits and in the Sm case we have a transition between vibrational to rotational nuclei. Besides in these isotope chains the values of the parameters for which the shape transitions are happening is well defined, that is in the crossings with the separatrix.

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