Pairing-plus-quadrupole model and nuclear deformation: the role of the spin-orbit interaction*

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ABSTRACT. The pairing-plus-quadrupole model, realized in the framework of the Elliott SU(3) scheme, is used to study the combined effects of the quadrupole-quadrupole, pairing, and spin-orbit interactions on ground state shapes of nuclear systems. Representation mixing induced by the symmetry-breaking pairing and spin-orbit forces is shown to soften the deformation. The angular momentum dependence of the results is discussed.

RESUMEN. Se usa el modelo de apareamiento-más-cuadrupolo, elaborado en el contexto del esquema SU(3) de Elliot, para estudiar los efectos combinados de las interacciones cuadrupolo-cuadrupolo, apareamiento y espín-órbita en la forma de los estados fundamentales de sistemas nucleares. Se muestra que la mezcla de representaciones inducida por las fuerzas (rompedoras de simetría) de apareamiento y espín-órbita suavizan la deformación. Se discute cómo dependen esos resultados del momento angular.

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1. INTRODUCTION

The phenomenon of nuclear deformation has a long and interesting history. It has played a prominent role in the work of N. Bohr, J. Rainwater, D. Hill and J. Wheeler, A. Bohr and B. Mottelson, S. Nilsson and others [1]. To this day, it remains a topic of lively interest as recent studies of phenomena like super- and hyperdeformation, shape coexistence, etc. show. One of the recurring themes, however, is the origin of nuclear deformation: the question about the microscopic mechanism leading to the existence of deformed nuclear states. Within the single-shell nuclear shell model, the long-range neutron-proton $T = 0$ interaction has been suggested to be a source of nuclear deformation [2] and calculations in the pairing-plus-quadrupole model [3], where strong neutron-proton correlations are implicitly introduced via the assumption of equal shape deformations for protons and neutrons, have shown that the quadrupole-quadrupole interaction induces deformation. It is the pairing-plus-quadrupole model (PQM), realized in the framework of the Elliott SU(3) scheme, which will be employed here to investigate the effects of the spin-orbit force on ground state deformations of nuclear systems.

The pairing-plus-quadrupole model (PQM), first introduced by Bohr and Mottelson [4] and Belyaev [5], has been widely used to reproduce both few-particle non-collective and many-particle collective features of nuclei [3]. It incorporates those features that are most important in nuclear mean field theories: the interaction between particles can be summed

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up, in a first approximation, to an average spherical single-particle potential; and long-range particle-hole correlations and short-range particle-particle correlations can be taken into account by a deformation of the field and a pairing potential, respectively [6]. Since the quadrupole-quadrupole term emerges (apart from a constant) as a leading contribution in the multipole expansion of any nuclear long-range potential and pairing correlations are closely associated with the short-range part of the nucleon-nucleon interaction, the inclusion of both forces as residual two-body interactions into a shell-model theory represents a step towards a more realistic picture of the nuclear many-body system.

In a previous study [7] the PQM was used to investigate the competing and complementary features of the two-body pairing and quadrupole-quadrupole interactions and their effects on the shape of nuclear systems. Ground state deformations induced by pairing were shown to be triaxial but rather soft, whereas the quadrupole-quadrupole interaction was confirmed to favor prolate or oblate shapes which are sharp. The study was realized in the framework of the Elliott SU(3) model [8], an algebraic approach to the many-particle shell model which treats the nucleus microscopically as a many-fermion system and makes use of the symmetries of the physical system. A small but complete shell-model space was used in order to ensure that the results could be properly interpreted and generalized. While the quadrupole-quadrupole interaction takes a very simple (diagonal) form in the SU(3) scheme, the expression for the pairing operator has a rather complex structure: pairing — unlike the quadrupole-quadrupole interaction — breaks the SU(3) symmetry.

Another important symmetry-breaking interaction that has been known for a long time is the spin-orbit interaction, which was introduced by Meyer, Haxel, Jensen, and Suess [9] in order to explain shell closures and magic numbers. In any given harmonic oscillator shell, the spin-orbit force mainly affects the largest-j orbital by lowering it energetically. In heavy nuclei, where the effect is so strong that magic numbers deviate from the major shell closures of the harmonic oscillator, the pseudo-spin concept can be applied [10]. In this approach the largest-j level in each major shell is removed from active consideration which can be neglected in a first approximation. In light nuclei, however, the spin-orbit interaction has to be taken into account explicitly in order to achieve agreement with experimental data.

Since the spin-orbit force affects the low-energy structure of many light nuclei, it is worthwhile to investigate its effects on the shape of nuclear systems. It is the goal of this contribution to extend the previous study [7] to include a spin-orbit term in addition to the previously considered quadrupole-quadrupole and pairing interactions and to examine the complementarity and competition of these terms in determining the shape of the systems under investigation.

2. THE MODEL AND ITS GEOMETRICAL INTERPRETATION

Realizing the PQM in the framework of the Elliott SU(3) scheme allows for a geometrical interpretation of the many-nucleon states via a relation between the invariants of the SU(3)
group and those of the geometric collective model (GCM). In the Elliott model, which is an algebraic theory based on the three-dimensional harmonic oscillator, basis states are labeled according to their properties under transformations of the unitary group $U(\Omega)$, where $\Omega = (\eta + 1)(\eta + 2)/2$ denotes the degeneracy of the $\eta$-th major oscillator shell under consideration, the special unitary group $SU(3)$, and the angular momentum group $SO(3)$:

$$|\Phi\rangle = |N[f]_\alpha(\lambda\mu)\kappa L, S; JM\rangle .$$

Here $N$ gives the number of particles in the $\eta$-th shell, $[f]$ labels the irreducible representation (irrep) of $U(\Omega)$, $(\lambda\mu)$ refers to the irrep of $SU(3)$, $L$ and $S$ are the orbital and spin angular momenta of the system, respectively, and $J$ is the total angular momentum with projection $M$ along the $z$-axis of the laboratory frame. The quantum numbers $\alpha$ and $\kappa$ are additional labels which are needed to distinguish between multiple occurrences of $(\lambda\mu)$ in a given $[f]$ symmetry and multiple $L$ values in a given $(\lambda\mu)$ irrep, respectively.

A shell-model description of rotational nuclear motion can be given since the $su(3)$ Lie algebra associated with the $SU(3)$ group contracts to $rot(3) = [R^5]_{so(3)}$, the algebra associated with the rotational limit of the geometric collective model. The $su(3)$ algebra is generated by the three components of the orbital angular momentum operator $L = \sum_i L_i$ and the five components of the Elliott (or algebraic) quadrupole operator $Q^a_\mu = \sum_i q^a_\mu = \sqrt{4\pi/5} \sum_i (r_i^2 Y_{2\mu}(\hat{r}_i) + b^2 q^a_{2\mu}(\hat{p}_i))/b^2$ where the sum runs over all particles in the valence shell, $\mu = -2, -1, 0, 1, 2$, and the oscillator length is given by $b = \sqrt{\hbar/m\omega}$; the generators of $rot(3)$ are the components of $L$ and those of the “collective” quadrupole operator, $Q^c_\mu = \sum_i q^c_\mu = \sqrt{16\pi/5} \sum_i r_i^2 Y_{2\mu}(\hat{r}_i)/b^2$. Within a major oscillator shell the matrix elements of $Q^c$ and $Q^a$ are identical, however $Q^c$ couples states belonging to the $\eta$-th shell with those of the $\eta'$-th shell with $\eta' = \eta \pm 2$, whereas the matrix elements of $Q^a$ between states belonging to different shells vanish. It has been demonstrated [11] that the invariants $\text{tr}[(Q^a)^2]$ and $\text{tr}[(Q^c)^3]$, with $\text{tr}[O]$ denoting the trace of the operator $O$, of $ROT(3) = [R^5]_{SO(3)}$, the Lie group associated with $rot(3)$, and those of $SU(3)$, namely $C_2(\lambda\mu)$ and $C_3(\lambda\mu)$, the expectation values of the second and third order Casimir operators, can be linearly related to each other. This, in turn, results in a direct connection between the microscopic quantum numbers $\lambda$ and $\mu$ and the collective shape variables $\beta$ and $\gamma$ [11]:

$$\langle \text{tr}[(Q^a)^2] \rangle = \frac{3}{2}k^2\beta^2 \leftrightarrow C_2(\lambda\mu) = \frac{2}{3}[\lambda^2 + \lambda\mu + \mu^2 + 3(\lambda + \mu)],$$

$$\langle \text{tr}[(Q^c)^3] \rangle = \frac{3}{4}k^3\beta^3\cos 3\gamma \leftrightarrow C_3(\lambda\mu) = \frac{1}{3}(\lambda - \mu)(\lambda + 2\mu + 3)(2\lambda + \mu + 3),$$

where the constant $k = \sqrt{5/9\pi}A\langle r^2 \rangle$ with $A$ being the number of nucleons in the nucleus and $\langle r^2 \rangle$ the mean square radius of the system. The exact relation between $(\beta\gamma)$ and $(\lambda\mu)$ is given by

$$k\beta \cos \gamma = (2\lambda + \mu + 3)/3,$$

$$k\beta \sin \gamma = (\mu + 1)/\sqrt{3},$$

(3)
which implies that each SU(3) irrep \((\lambda \mu)\) corresponds to a unique geometrical shape \((\beta \gamma)\). This correspondence can be illustrated with the help of a grid superposed on the well-known \((\beta \gamma)\)-plane of the geometric collective model as is shown in Fig. 1.\(^1\)

In order to obtain a measure for the “average” deformation of a state one needs to deduce values for the collective model \(\beta\) and \(\gamma\) variables. Since \(\beta\) and \(\gamma\) are simply averages of the microscopic observables, one can use Eqs. (3) to write

\[
k^2 \beta^2 = \frac{2}{3} \langle C_2 \rangle _\nu + 2,
\]

\[
k^3 \beta^3 \cos 3 \gamma = \frac{4}{3} \langle C_3 \rangle _\nu,
\]

where \(\langle O \rangle _\nu \equiv \langle \Psi_\nu | O | \Psi_\nu \rangle\) denotes the expectation value of the operator \(O\) in the \(\nu\)-th eigenstate of the system. To extract information regarding the intrinsic deformation of the \(|\Psi_\nu\rangle\) configurations, the eigenstate \(|\Psi_\nu\rangle\) can be decomposed into SU(3) basis states,

\[
|\Psi_\nu\rangle = \sum_i c_{\nu i} |\Phi_i\rangle,
\]

where the states \(|\Phi_i\rangle\) are as defined in Eq. (1). Since \(C_2\) and \(C_3\) are diagonal in the SU(3) basis it is particularly simple to determine their expectation values in any calculated eigenstate and hence the shape of the nuclear system under investigation.

\(^1\) In the oscillator picture, core configurations couple to the SU(3) irrep \((00)\), and the \((\lambda \mu)\) irreps that can occur are determined by the number of valence protons or neutrons under consideration.
The Hamiltonian of the PQM consists of three parts:

\[ H = H_s + V_Q + V_P, \]

where \( H_s = \sum_j \epsilon_j \hat{n}_j \) includes the harmonic oscillator mean field, as well as all relevant single-particle effects, such as the spin-orbit and orbit-orbit terms. The \( \epsilon_j \) denotes the single-particle energy of the angular-momentum-j orbital and \( \hat{n}_j \) is the number operator counting the fermions in that level, the sum being over all j levels of the shell(s) under consideration. \( V_Q \) denotes the quadrupole-quadrupole interaction with strength \( \chi \), \( V_Q = -\frac{1}{2} \chi Q \cdot Q \), and \( V_P \) is the pairing interaction with strength \( G \). Since the application is restricted to a single major oscillator shell, \( Q \) can be replaced by the Elliott quadrupole operator \( Q = Q^c = Q^a \), and —in the SU(3) basis [Eq. (1)]— the term \( V_Q \) reduces to diagonal form, \( V_Q = -\frac{1}{2} \chi [6C_2 - 3L^2] \) with eigenvalues \( E_Q(\lambda, \mu, L) = -\frac{1}{2} \chi [4\{\lambda^2 + \lambda\mu + \mu^2 + 3\lambda + 3\mu\} - 3L(L+1)] \).

In order to write down expressions for the SU(3) symmetry-breaking one- and two-body terms \( H_s \) and \( V_P \), the formalism of second quantization is employed. The interactions are expanded in terms of single-particle fermion creation and annihilation operators, which are then coupled to SU(3) \( \supset \) SO(3) tensor operators.\(^2\) The SU(3) coupling coefficients as well as reduced matrix elements of the SU(3) unit tensors that occur in the final expressions for \( H_s \) and \( V_P \) can be calculated with the help of existing computer codes [14]; thus the matrix elements of the \( \hat{n}_j \) and pairing operators can be evaluated for any specific SU(3) application.

3. Results

It is well known that the quadrupole-quadrupole interaction plays an important role in driving the nuclear many-body system toward strongly deformed shapes, most of which are experimentally determined to be prolate. In the language of the GCM, most deformed nuclei exhibit an energy minimum near the \( \gamma = 0^\circ \) axis in the \((\beta \gamma)\)-plane. Pairing, on the other hand, has traditionally been (implicitly or explicitly) associated with sphericity, a notion that has been challenged by the results of some recent studies. It was found that seniority-zero states, which are fully-paired eigenstates of the pairing operator, may very well possess an intrinsic quadrupole moment [15] and, furthermore, a detailed investigation of the eigenstates of the pairing operator has demonstrated that pairing favors triaxially deformed many-particle configurations [13, 7].

The latter findings are reproduced in Fig. 2, which gives the \((k\beta, \gamma)\) for ground states of the pure pairing Hamiltonian \((H = V_P)\) for \( N = 2, 4, \ldots, 18 \) particles in the \( fp \)-shell. The results show that in this limit all the systems are —on the average— triaxially deformed; as the number of particles in the \( fp \)-shell increases the deformation becomes stronger and more triaxial, with the maximally asymmetric shape occurring at \( N=10 \) for the half-filled \( \eta=3 \) shell. Adding further nucleons reduces the deformation and a symmetric pattern

\(^2\) The explicit expressions for \( H_s \) and \( V_P \) are given in Refs. [12] and [13, 7], respectively.
emerges as the result of the particle-hole complementarity of the shell model. Similar results have been obtained for the $ds$-shell.

Figure 2 also shows how the deformation changes as a quadrupole-quadrupole interaction is introduced ($H = V_P + V_Q$) and the relative strength $\xi = \chi/G$ of the quadrupole-quadrupole and pairing terms is varied.\(^3\) As expected, the quadrupole term quickly dominates the behavior of the system, even for small values of $\xi$, driving the nuclear many-body system towards prolate shapes if the oscillator shell is less than half filled ($N < \Omega$), oblate shapes if the oscillator shell is more than half filled ($N > \Omega$), and to larger $k\beta$ values at maximum asymmetry for shells which are exactly half filled ($N = \Omega$).

To study the influence of the spin-orbit interaction on these results, the full Hamiltonian [Eq. (6)] was considered for the case of $N = 4$ and $N = 6$ particles in the $\eta = 3$ shell. Since the spin-orbit force has the strongest effect on the largest-$j$ orbital, the energies $\epsilon_{1/2}$, $\epsilon_{3/2}$, and $\epsilon_{5/2}$ of the levels $p_{1/2}, p_{3/2}$, and $p_{5/2}$ were fixed at $0$ and $\epsilon_{7/2}$ of the $f_{7/2}$-orbitals

\(^3\) The inverted triangles correspond to increments of 0.1 in $\xi$, starting with $\xi = 0.0$ and ending with $\xi = 1.0$. For realistic nuclei, the parameter $\xi$ may span a wide range of values, for example, for $fp$-shell nuclei values between 0.04-0.6 occur.
level was varied from 0 to \(-\hbar \omega\). The results for the \(N = 6\) case are displayed in Fig. 3. Shown are the \((k\beta, \gamma)\) values for a separation \(\Delta \epsilon_{7/2} = 0\), \(\hbar \omega/10\), and \(\hbar \omega/3\) (top), \(\hbar \omega/2\) (middle) and \(\hbar \omega\) (bottom) of the \(f_{7/2}\) orbital from its \((p_{1/2}, p_{3/2}, f_{5/2})\) partners. The curve for \(\Delta \epsilon_{7/2} = 0\) has been repeated in all three portions of the graph in order to provide some visual guidance. The leftmost circles represent a vanishing quadrupole-quadrupole interaction (i.e., \(\chi = 0\)), and from left to right the ratio \(\xi \equiv \chi/G\) increases in increments of 0.05, with the rightmost point corresponding to \(\xi = 2.0\).\(^4\)

It is observed that for both vanishing and finite spin-orbit interaction strengths, the quadrupole-quadrupole interaction drives the system towards prolate shapes. However, while for a weak spin-orbit force \((\Delta \epsilon_{7/2} = 0 \rightarrow \hbar \omega/10)\) the quadrupole term quickly dominates the behavior of the system, for \(\Delta \epsilon_{7/2} = \hbar \omega/3 \rightarrow \hbar \omega\) one finds that the spin-orbit term counteracts this effect; the quadrupole-quadrupole interaction has to reach a certain minimum strength \((\chi_0)\) in order to significantly affect the shape of the nucleus. However, once this strength \(\chi_0\), which depends strongly on \(\Delta \epsilon_{7/2}\) \((\chi_0 \approx 0.035 \hbar \omega\) for \(\Delta \epsilon_{7/2} = \hbar \omega/3\), 0.055 \(\hbar \omega\) for \(\Delta \epsilon_{7/2} = \hbar \omega/2\), and 0.115 \(\hbar \omega\) for \(\Delta \epsilon_{7/2} = \hbar \omega\)\) is reached, the \(Q \cdot Q\) term quickly drives the system towards its maximum (prolate) elongation. For a weak spin-orbit force this maximum is easily reached (see the top part of the figure), while a large \(\Delta \epsilon_{7/2}\) requires a strong \(Q \cdot Q\) term to force the system into its maximally prolate (or oblate) deformed state. It is also noteworthy to observe that increasing the spin-orbit interaction decreases the \(\beta\)-deformation, while the expectation value for \(\gamma\) is less affected. That is, the spin-orbit term has a strong effect on the elongation of a nucleus but it does not change its triaxiality significantly, while the primary effect of the quadrupole-quadrupole interaction is to drive a given system away from triaxiality, towards prolate \((N < \Omega)\) or oblate \((N > \Omega)\) shapes.

In order for \(k\beta\) and \(\gamma\) to yield a meaningful measure for the deformation of a nuclear system, their expectation values as well as the widths of their distributions need to be known. Figure 4 illustrates how for different values of \(\Delta \epsilon_{7/2}\) the widths of the distributions change as a function of the strength \(\chi\) of the quadrupole-quadrupole interaction. For \(\Delta \epsilon_{7/2} = 0\) (vanishing spin-orbit force) and \(\Delta \epsilon_{7/2} = \hbar \omega/10\) (weak spin-orbit force) and small \(\chi\)-values \(\Delta \beta/\beta \approx 20\%-25\%\) and \(\Delta \gamma \approx 15^\circ\), that is, the \(\gamma\)-distribution has a considerable width, allowing for a wide spread in \(\gamma\). The softness of the deformation implies that even though pairing favors triaxial nuclear shapes, the quadrupole-quadrupole interaction is able to easily drive the system away from the triaxial pairing minimum towards prolate or oblate shapes. As \(Q \cdot Q\) gets stronger and begins to dominate the Hamiltonian, the shape of the nuclear system becomes sharper, in particular in the \(k\beta\)-variable. As shown in Fig. 4c and d, with increasing spin-orbit interaction strength \((\Delta \epsilon_{7/2} = \hbar \omega/2 \rightarrow \hbar \omega)\) the distributions in both \(k\beta\) and \(\gamma\) become more spread out. The quadrupole-quadrupole interaction still pushes the system towards larger \(k\beta\) and smaller \(\gamma\), but it is no longer able to clearly define the shape of the nucleus.

This behavior can be understood if one considers the symmetry-breaking effects of the spin-orbit force, as is done in Table I. Shown is the eigenstate decomposition for a system

\(^4\) The pairing interaction strength \(G\) was fixed at 0.01 \(\hbar \omega\), a value which corresponds to about 0.1 MeV in the \(fp\)-shell and leads to a pairing gap of 1 MeV \(\equiv 0.1 \hbar \omega\); the quadrupole-quadrupole interaction strength was varied from 0 to 0.02 \(\hbar \omega \equiv 0.2\) MeV.
FIGURE 3. A plot of \((k\beta, \gamma)\) for calculated eigenstates for 6 particles in the \(fp\)-shell. The strength of the pairing interaction is held constant at \(G = 0.01 \hbar \omega\), while the quadrupole-quadrupole force increases in increments of 0.005 \(\hbar \omega\) from \(\chi = 0\) to \(\chi = 0.02 \hbar \omega\). Five curves are shown, corresponding to different strengths of the spin-orbit interaction. Specifically, the graphs for \(\Delta \epsilon_{7/2} = 0\), \(\hbar \omega / 10\), and \(\hbar \omega / 3\) are plotted in the top portion of the figure with \(\chi\) increasing from the left end of each curve \((\chi = 0)\) to the right end \((\chi = 0.02 \hbar \omega)\). The central part of the figure displays the graph for \(\Delta \epsilon_{7/2} = \hbar \omega / 2\) and repeats the \(\Delta \epsilon_{7/2} = 0\) case (solid line) in order to guide the eye. The bottom portion of the figure shows the graph for \(\Delta \epsilon_{7/2} = \hbar \omega\) and, again, gives the \(\Delta \epsilon_{7/2} = 0\) reference curve (solid line).
FIGURE 4. Expectation values for $k\beta$ and $\gamma$, along with their uncertainty ranges ($k\beta \pm \Delta k\beta, \gamma \pm \Delta \gamma$) for 6 particles in the $fp$-shell, as a function of the quadrupole-quadrupole interaction strength $\chi$. Each plot gives the expectation values (solid lines) of $k\beta$ (indicated by circles) and of $\gamma$ (indicated by triangles), as well as the associated uncertainty ranges (dashed lines). Four plots are shown, corresponding to different strengths of the spin-orbit interaction: a) $\Delta \epsilon_{7/2} = 0$, b) $\Delta \epsilon_{7/2} = \hbar \omega / 10$, c) $\Delta \epsilon_{7/2} = \hbar \omega / 2$, and d) $\Delta \epsilon_{7/2} = \hbar \omega$. 
of $N = 4$ particles in the $fp$-shell,\footnote{The $N = 4, J = 0$ system has been selected since it allows for a listing of all occurring basis states.} with the parameters in the Hamiltonian (6) chosen to be $G = 0.01 \hbar \omega$, $\chi = 0.005 \hbar \omega$ and $0.015 \hbar \omega$, and $\Delta \epsilon_{7/2}$ is varied from 0 to $\hbar \omega$. For a vanishing spin-orbit interaction (columns 2 and 6) the ground state of the system is a pure $S = 0$ state, with the $(\lambda \mu) = (8, 2)$ component being the dominant one in the expansion. The small amount of mixing between different SU(3) irreps with the same spin $S$ is due to the pairing interaction, which is a spin scalar but breaks the SU(3) symmetry. Already for a small spin-orbit force ($\Delta \epsilon_{7/2} \approx \hbar \omega/10$) the symmetry-breaking effects of this interaction become clearly visible, and for a strong spin-orbit force ($\Delta \epsilon_{7/2} \rightarrow \hbar \omega$) the SU(3) structure of the system is completely destroyed. The symmetry-preserving influence of the quadrupole-quadrupole interaction, which is a SU(3) scalar, can be observed by comparing columns 2-5 with columns 6-9. The $Q \cdot Q$ term is (as expected) found to counteract the symmetry-breaking effects of both the two-body pairing and the one-body spin-orbit forces. Hence the results confirm that for a nuclear system where the quadrupole-quadrupole interaction plays a dominant role (as signified experimentally by rotational spectra), the SU(3) basis becomes very valuable since it lends itself to a straightforward truncation. For systems with a strong spin-orbit force it has been shown that the pseudo-SU(3) scheme yields good solutions to the nuclear many-body problem [16]. In both cases, however, if there are strong pairing correlations present, mixing between different irreps becomes very important and thus complicates the search for a suitable truncation scheme.

In Fig. 5 the angular momentum dependence of the above findings is investigated for four particles in the $fp$-shell. The curves in the top portion of the plot show the results for $J = 0$ and $\Delta \epsilon_{7/2} = 0$, $\hbar \omega/10$, and $\hbar \omega$. The system is found to behave similarly to the six-particle system described above: An increase in the strength of the quadrupole-quadrupole interaction leads to larger values of $k/3$ while $\gamma$ decreases. The latter effect is not as prominent as it is for the $N = 6$ system, which is due to the fact that the maximally prolate deformed Pauli-allowed irrep is $(\lambda \mu) = (8, 2)$, whereas for six particles it is $(\lambda \mu) = (12, 0)$. In both cases, the introduction of a spin-orbit force ($\Delta \epsilon_{7/2} = 0 \rightarrow \hbar \omega$) results in reduced expectation values for $k/3$ while $\gamma$ is not significantly affected. The curves in the middle and bottom portions of Fig. 5 show for the $N = 4$ example that the situation for $J = 2$ is very similar to the case of $J = 0$, and the results for $J = 4$ differ slightly. The general trend for $J = 4$ agrees with the findings for $J = 0$ and $J = 2$, although the asymptotic behavior of the curves seems to indicate a new development. For $\Delta \epsilon_{7/2} = \hbar \omega/10$ and $\hbar \omega$ and $\chi \rightarrow \infty$, the curves approach the point $(k/3, \gamma) = (7.08, 8.18^\circ)$, while the limit point for $\Delta \epsilon_{7/2} = 0$ lies at $(k/3, \gamma) = (7.21, 13.9^\circ)$. In the limit $\chi \rightarrow \infty$ three basis states of the $N = 4$ system become degenerate, namely $|N[f]\alpha(\lambda \mu)\kappa L, S; J M\rangle = |4[22]1(8, 2)14, 0; 4\rangle, |4[22]2(8, 2)14, 0; 4\rangle$, and $|4[211]1(9, 0)13, 1; 4\rangle$. Since neither the quadrupole-quadrupole interaction nor pairing mixes different U(3) symmetries ($[f]$ irreps), the lowest eigenstate for $\Delta \epsilon_{7/2} = 0$ and $\chi \rightarrow \infty$ is a pure $(\lambda \mu) = (8, 2)$ state, which is associated with $(k/3, \gamma) = (7.21, 13.9^\circ)$. The spin-orbit force, on the other hand, mixes different spins (and thus $[f]$ irreps), hence for $\Delta \epsilon_{7/2} = \hbar \omega/10$ and $\hbar \omega$ an increase in $\chi$ leads to a combination of the above basis states which corresponds to $(k/3, \gamma) = (7.08, 8.18^\circ)$. Thus, aside from peculiar effects which can
Table I. Eigenstate decomposition for a system of $N = 4$ particles in the fp-shell. The contributions $|c_{\lambda \mu}|^2$ (summed over the multiplicity index $\alpha$) are shown for different basis states $|\Phi_i\rangle$, labelled by their $(\lambda \mu)$ and spin $S$ content, to the ground state of the system for $G = 0.01 \hbar \omega$ and various values of $\chi$ and $\Delta \epsilon_{7/2}$.

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<th>$\chi = 0.015 \hbar \omega$</th>
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<td>0.00</td>
</tr>
<tr>
<td>$(0 \ 3)$</td>
<td>1</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
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</tr>
<tr>
<td>$(5 \ 2)$</td>
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</tr>
<tr>
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</tr>
<tr>
<td>$(0 \ 6)$</td>
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</tr>
<tr>
<td>$(2 \ 2)$</td>
<td>2</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

be ascribed to the mixing of degenerate states, the trends observed for the $J = 0$ case survive higher angular momenta. Furthermore, if one considers the widths of the $k/3$ and $\gamma$ distributions for $N = 4$ (not shown here), it turns out that for $J = 0, 2,$ and $4$ the spread in both the $k/3$ and $\gamma$ variables displays a behavior similar to that discussed for the $N = 6, J = 0$ case.

4. Summary

The pairing-plus-quadrupole model has been investigated in the framework of the SU(3) shell model for the case of $N = 4$ and 6 identical particles in a single major harmonic-oscillator shell. The restricted Hilbert space allowed for a schematic study of the competing
FIGURE 5. A plot of \((k\beta, \gamma)\) for calculated eigenstates for 4 particles in the \(fp\)-shell with angular momentum \(J = 0\) (top), \(J = 2\) (center), and \(J = 4\) (bottom). The strength of the pairing interaction is held constant at \(G = 0.01 \hbar \omega\), while the quadrupole-quadrupole force increases in increments of 0.01 \(\hbar \omega\) from the left end of each curve (\(\chi = 0\)) to the right end (\(\chi = 0.02 \hbar \omega\)). Three curves are shown for each angular momentum value, corresponding to different strengths of the spin-orbit interaction (\(\Delta \epsilon_{7/2} = 0\) - solid lines, \(\hbar \omega/10\) - dashed lines, and \(\hbar \omega\) - dotted lines).
pairing-plus-quadrupole, pairing, and spin-orbit interactions in a truncation-free environment.

While the quadrupole-quadrupole interaction preserves and enhances the SU(3) symmetry of nuclear systems, both pairing and the spin-orbit force break SU(3). Pairing is a scalar in the spin space, it only mixes different \((\lambda\mu)\) irreps of SU(3) but not different \([\alpha]\) irreps of U(Ω). A description of those triaxially deformed many-particle configurations which pairing favors requires a large number of SU(3) basis states; hence the \(k\beta\) and \(\gamma\) deformations induced by pairing are rather soft, with \(\Delta\beta/\beta \approx 20\%-25\%\) and \(\gamma\) on the order of 10°. It is the softness of this deformation that allows the quadrupole-quadrupole interaction to easily drive a given nuclear system away from a triaxial pairing minimum towards prolate or oblate shapes. The quadrupole term is diagonal in the SU(3) scheme, therefore, as \(Q \cdot Q\) begins to dominate the Hamiltonian, it not only drives the system towards larger \(k\beta\) and smaller (larger) \(\gamma\) for shells which are less (more) than half filled, but it also sharpens the shape of the nucleus.

The spin-orbit interaction breaks both the SU(3) symmetry and (by mixing different spins) the U(3) symmetry of nuclear systems. A weak spin-orbit force pushes the system toward smaller \(k\beta\)-values while not affecting the \(\gamma\)-deformation significantly, whereas a strong spin-orbit term succeeds in softening the \(k\beta\) and \(\gamma\) deformations to the point where a shape can no longer be clearly defined. These trends were shown to be fairly angular momentum independent. Although the symmetry-breaking effects of the spin-orbit interaction are strong, they do not preclude the use of the SU(3) scheme, since for systems with a dominant spin-orbit force the pseudo-spin concept can be applied which then leads to a pseudo-realization of SU(3). The advantage of this realization is that the symmetry-breaking spin-orbit interaction in the new representation is weak enough to yield good pseudo-SU(3) quantum numbers. The challenge that remains is to find an analogous concept that will allow for a proper treatment of the pairing interaction in heavy nuclei.

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References