Delay time in the $R$ matrix formalism

M. Moshinsky*

Instituto de Física, Universidad Autónoma de México
Apartado postal 20-364, 01000 México, D.F., Mexico.

ABSTRACT. We derive the delay time associated with a single resonance using a schematic model of nuclear reactions based on the $R$ matrix formalism.

RESUMEN. Obtenemos el tiempo de retraso asociado a una sola resonancia usando un modelo esquemático de reacciones nucleares basado en el formalismo de la matriz $R$.

PACS: 03.80.+r

1. INTRODUCTION

For many years [1], the survey of delay time in a nuclear or atomic process has been of paramount interest. Some of the models used to test the ideas consisted of a one dimensional barrier, with or without an intermediate hole, and the analysis of the wave function when initially it was given by a wave packet on the left of the barrier [2].

These models were based on the idea that the essential feature of the problem was the presence of a potential in the Schrödinger equation. But since the seminal work of Wigner [3] we have known that the fundamental concept, at least for neutrons in a nuclear reaction, should be the $R$ matrix. Wigner assumed that as long as the neutron was outside the range of the nuclear forces it behaves as a free particle, but as soon as it touched the surface of a nucleus of $A$ nucleons it became part of a compound state of $A + 1$ nucleons which could be decomposed into an infinite set of eigenstates of the system. This then would give rise to a relation between the wave functions in the different channels of the reaction in which the neutron was involved, with the derivative of this same wave function, both at the surface of the target nucleus. This relation was called an $R$ matrix and for two channels and a single resonance $E_0$ it could be denoted by

$$R(E) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \frac{E_0 - E}{E_0 - E},$$

where $E$ is the energy of the system and $R_{st}, s, t = 1, 2$ being constants of an hermitian matrix [3].

In the view presented in this paper we take the attitude that the interaction should appear in the form of Eq. (1) and starting from it a discussion should be carried out for the effects in the nuclear reaction which should give rise to a delay time in the process.

*Member of El Colegio Nacional.
To carry out our program we shall make use of a schematic but dynamical model [4] of the problem which in its stationary form will give us precisely an $R$ matrix of the form of Eq. (1), which will show how a resonance affects the nature of the process.

2. A SCHEMATIC DYNAMICAL MODEL

Let us consider [4] a space variable $x$ in the interval $-\infty \leq x \leq \infty$ divided into three sections

$$-\infty \leq x < 0; \quad x = 0; \quad 0 < x \leq \infty,$$

and associate respectively within each interval the wave functions

$$\psi_-(x, t); \quad \psi_0(t); \quad \psi_+(x, t),$$

where we note that when $x = 0$, $\psi_0$ can only be a function of $t$.

Now using atomic units $\hbar = m = e = 1$ we assume that $\psi_\pm(x, t)$ satisfy the Schrödinger time dependent equation for free particles in their corresponding intervals

$$i \frac{\partial \psi_\pm(x, t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi_\pm(x, t)}{\partial x^2} \quad (4)$$

and assume, as it is usual, that the probability of finding a particle represented by the above wave functions in any of its intervals is

$$P = \int_{-\infty}^{0} \psi_+(x, t) \psi_-(x, t) \, dx + \psi_0(t) \psi_0(t) + \int_{0}^{\infty} \psi_+(x, t) \psi_+(x, t) \, dx. \quad (5)$$

This probability must remain constant in time and, in fact, equal to 1 if the wave functions are normalized. Thus the derivative of $P$ with respect to time must vanish and using Eq. (4) we obtain

$$i \frac{dP}{dt} = \int_{-\infty}^{0} \left( \frac{1}{2} \frac{\partial^2 \psi_+^*(x, t)}{\partial x^2} \psi_- - \psi_-^* \frac{1}{2} \frac{\partial^2 \psi_-}{\partial x^2} \right) \, dx$$

$$+ \left[ \psi_0^* \left( i \frac{\partial \psi_0}{\partial t} \right) - \left( i \frac{\partial \psi_0}{\partial t} \right)^* \psi_0 \right]$$

$$+ \int_{0}^{\infty} \left( \frac{1}{2} \frac{\partial^2 \psi_+^*(x, t)}{\partial x^2} \psi_+ - \psi_+^* \frac{1}{2} \frac{\partial^2 \psi_+}{\partial x^2} \right) \, dx$$

$$= \left[ \frac{1}{2} \left( \frac{\partial \psi_+^*}{\partial x} \right)_0 \right] \psi_-(-0) - \psi_-^*(-0) \left[ \frac{1}{2} \left( \frac{\partial \psi_-}{\partial x} \right)_0 \right]$$

$$+ \psi_0^* \left( i \frac{\partial \psi_0}{\partial t} \right) - \left( i \frac{\partial \psi_0}{\partial t} \right)^* \psi_0$$

$$+ \left[ -\frac{1}{2} \left( \frac{\partial \psi_+^*}{\partial x} \right)_0 \right] \psi_+(-0) - \psi_+^*(-0) \left[ -\frac{1}{2} \left( \frac{\partial \psi_+}{\partial x} \right)_0 \right] = 0. \quad (6)$$
where we carried out the integrations explicitly as the integrands are exact differentials and assume that the wave functions and their derivatives go to 0 at $x = \pm \infty$ so only their values at $x = 0$ remain.

Following the analysis of Ref. 4 we want now to obtain linear relations between the terms involving the $\psi$ and $\psi^*$ separately that make the right hand side of Eq. (6) vanish. For purpose we define

$$z_1 \equiv -\frac{1}{2} \left( \frac{\partial \psi_-}{\partial x} \right)_0, \quad z_2 \equiv \psi_0, \quad z_3 \equiv \frac{1}{2} \left( \frac{\partial \psi_+}{\partial x} \right)_0,$$

$$z_4 \equiv \psi_-(0), \quad z_5 \equiv -i \left( \frac{\partial \psi_0}{\partial t} \right)_0, \quad z_6 \equiv \psi_+(0),$$

so that the expression (6) becomes

$$\sum_{i=1}^{3} (z_i^* z_{3+i} - z_{3+i}^* z_i) = 0. \quad (7)$$

This bilinear form is of a familiar type [5] and it vanishes when we have the linear relations

$$z_{3+i} = \sum_{j=1}^{3} C_{ij} z_j, \quad (9)$$

implying

$$z_{3+i}^* = \sum_{j=1}^{3} C_{ij}^* z_j^*, \quad (10)$$

as substituting in Eq. (8) it becomes identically 0 if the matrix $\|C_{ij}\|$ is hermitian.

Thus we get the boundary conditions

$$\psi_-(0) = C_{11} \left[ -\frac{1}{2} \left( \frac{\partial \psi_-}{\partial x} \right)_0 + \frac{1}{2} \left( \frac{\partial \psi_+}{\partial x} \right)_0 \right] + C_{12} \psi_0 + C_{13} \left[ -\frac{1}{2} \left( \frac{\partial \psi_+}{\partial x} \right)_0 \right],$$

$$\left( -i \frac{\partial \psi_0}{\partial t} \right)_0 = C_{21} \left[ -\frac{1}{2} \left( \frac{\partial \psi_-}{\partial x} \right)_0 + \frac{1}{2} \left( \frac{\partial \psi_+}{\partial x} \right)_0 \right] + C_{22} \psi_0 + C_{23} \left[ -\frac{1}{2} \left( \frac{\partial \psi_+}{\partial x} \right)_0 \right],$$

$$\psi_+(0) = C_{31} \left[ -\frac{1}{2} \left( \frac{\partial \psi_-}{\partial x} \right)_0 + \frac{1}{2} \left( \frac{\partial \psi_+}{\partial x} \right)_0 \right] + C_{32} \psi_0 + C_{33} \left[ -\frac{1}{2} \left( \frac{\partial \psi_+}{\partial x} \right)_0 \right]. \quad (11)$$

that supplement the equations (4).

This is the most general interaction we can get, but to obtain the $R$ matrix (1) we can assume that there is no direct self-interaction between the channels themselves or between the $-$ and $+$ channels.
Furthermore, if all $C_{ij}$ are equated to 0 except $C_{22}$ we are left with the equation

$$-i\partial \psi_0 / \partial t = C_{22} \psi_0$$

which implies that $C_{22}$ is the negative of the energy of the compound state so we can designate it as $C_{22} = -E_0 \equiv -\frac{1}{2} k_0^2$. We then see that for the stationary state where all $\psi_{\pm}(x, t), \psi_0(t)$ depend on time through $e^{-iEt}$ we obtain finally

$$\begin{bmatrix} \psi_-(0) \\ \psi_+(0) \end{bmatrix} = \begin{bmatrix} C_{12} C_{21} & C_{12} C_{23} \\ C_{32} C_{21} & C_{32} C_{23} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \left( \frac{\partial \psi_-}{\partial x} \right)_0 \\ \frac{1}{2} \left( \frac{\partial \psi_+}{\partial x} \right)_0 \end{bmatrix}.$$  \hspace{1cm} (12)

Thus we get through (4) and (11), with the restrictions (12), a dynamical set of equations, i.e., time dependent ones, that in the stationary limit gives us exactly Wigner's relation (1) with the slight change of a $-$ sign in $(\partial \psi_- / \partial x)_0$ due to the fact that the interval there is $-\infty \leq x < 0$ instead of $0 < x \leq \infty$.

3. Solution of the Problem

In Eq. (2) we note from the hermitian character of $\|C_{ij}\|$ that $C_{12} = C_{12}^*, C_{13} = C_{31}^*$, and for simplicity we shall assume all of this coefficients as real and equal i.e.,

$$C_{11} = C_{33} = C_{13} = C_{31} = a.$$  \hspace{1cm} (13)

We have then to solve the equation

$$\begin{bmatrix} \psi_-(0) \\ \psi_+(0) \end{bmatrix} = \frac{2a^2}{k_0^2 - k^2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \left( \frac{\partial \psi_-}{\partial x} \right)_0 \\ \frac{1}{2} \left( \frac{\partial \psi_+}{\partial x} \right)_0 \end{bmatrix}.$$  \hspace{1cm} (14)

Assuming that initially we have a plane wave of unit amplitude coming from the left we see that the solution of (14) should have the form

$$\begin{align*}
\psi_-(x, k) &= e^{ikx} + A e^{-ikx} \quad -\infty \leq x < 0 \\
\psi_+(x, k) &= B e^{ikx} \quad \quad \quad 0 < x \leq \infty.
\end{align*}$$  \hspace{1cm} (15)

Substituting in (14) we obtain the relations

$$\begin{bmatrix} 1 + A \\ B \end{bmatrix} = \frac{2a^2}{k_0^2 - k^2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} k (1 - A) \\ \frac{1}{2} k B \end{bmatrix},$$  \hspace{1cm} (16)

Which lead to two linear equations for the coefficients $A$, and $B$ from which we get

$$B = \frac{i2a^2 k}{k^2 - k_0^2 + i2ka^2}.$$  \hspace{1cm} (17)
4. DELAY TIME IN THE $R$ MATRIX FORMALISM

Besides introducing the concept of $R$ matrix [3], Wigner also discussed the concept of delay time [5] showing that it could be related to the derivative with respect to the energy of the phase shift $\delta$ of the outgoing wave $\psi_+(x,t)$.

As the coefficient of the latter is the $B$ of (17) we require to write it as

$$B = |B|e^{i\delta}. \quad (18)$$

From (17) we immediately see that

$$\delta = \arctan \frac{k^2 - k_0^2}{2ka^2}, \quad (19)$$

and, in our units where $E = \frac{1}{2}k^2$, the Wigner delay time [5], which we shall denote by $\tau$, is given by

$$\tau = \frac{d\delta}{dE} = \frac{1}{k} \frac{d\delta}{dk}. \quad (20)$$

Replacing (19) in (20) we finally obtain

$$\tau = \frac{2a^2}{k} \frac{(k^2 + k_0^2)}{(k^2 - k_0^2)^2 + 4k_0^2a^4}, \quad (21)$$

which has the Lorentzian form one expects for the case of a single resonance, whose maximum value is taken at $k = k_0$ giving rise to

$$\tau = \frac{1}{k_0a^2}. \quad (22)$$

We have thus derived the delay time for a single resonance in the $R$ matrix formalism.

REFERENCES