Group theoretical approach to nuclear clusterization

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ABSTRACT. The basic concepts and the present status of the semimicroscopic algebraic cluster model are reviewed. Its relation to other approaches is discussed. Possible extensions to heavy nuclei and the connection to cold fission is considered.

RESUMEN. Los conceptos básicos y el estado actual del “modelo semimicroscópico algebraico de cúmulos” son presentados. Su relación con otras aproximaciones es analizada. Posibles extensiones del modelo a núcleos pesados, así como sus implicaciones en la interpretación del proceso de fisión fría son consideradas.

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1. INTRODUCTION

The semimicroscopic algebraic cluster model (SACM) was proposed as an approach to the cluster structure of light nuclei [1, 2]. Special emphasis was put on the unified description of the clusterization in the neighbourhood of the ground state and in the highly excited region of the molecular resonances. The model turned out to have a simple and transparent relation to other models of light nuclei. Particularly interesting is the fact that the common intersection of the three basic nuclear structure models, i.e., the shell, collective and cluster models is provided by the U(3) dynamic symmetry of the SACM. The interrelation of different cluster configurations can also be handled within this approach in a simple manner. This circumstance provides us with a selection rule applicable for fission channels. Recently an attempt has been initiated in order to extend this method to the territory of heavy nuclei, where various forms of radioactive and fission processes attract much attention these days.

2. BASIC CONCEPTS OF THE SACM

This approach [1, 2] is called semimicroscopic, because it combines a microscopically constructed model space with phenomenological interactions. The treatment is algebraic in the sense that both the basis states, and the physical operators are characterised by
the irreducible representations of some Lie algebras, therefore the matrix elements can be determined by group theoretical methods.

In the semimicroscopic algebraic cluster model the internal structure of the clusters is described by the Elliott model of $U^\text{ST}(4) \otimes U_C(3)$ group structure [3], where $U^\text{ST}(4)$ is Wigner's spin-isospin group [4]. The relative motion is treated in terms of the $U_R(4)$ vibron model [5]. There $\sigma$ bosons are added to the $\pi$ bosons, which describe the relative motion. The total number of $\pi$ plus $\sigma$ bosons is kept fixed and plays the role of a cutoff. The group structure of the model for a system of $K$ clusters (having $K - 1$ independent space vector of relative motion) is

$$
U^\text{ST}_C(4) \otimes U_C_1(3) \otimes U^\text{ST}_C(4) \otimes U_C_2(3) \otimes \ldots \otimes U^\text{ST}_C(4) \otimes U_C_K(3)
$$

$$\otimes U_R(4) \otimes U_{R_2}(4) \otimes \ldots \otimes U_{R_{K-1}}(4)
$$

$$
\supseteq U^\text{ST}_C(4) \otimes U_C(3) \otimes U_R(3) \supseteq U^S_2(2) \otimes U(3)
$$

$$
\supseteq U^S_C(2) \otimes O(3) \supseteq U(2) \supseteq O(2).
$$

The basis states of the model are characterised by the labels of irreducible representations of these groups.

A major point in constructing the model space is related to the exclusion of the Pauli-forbidden states and the spurious excitations of the center of mass. One can follow different routes in order to exclude the forbidden states. A simple procedure is to make an intersection between the cluster model basis corresponding to group (1) and that of the (fully antisymmetric) SU(3) shell model of the whole nucleus. This procedure is based on the equivalence of the shell model Hamiltonian, and that of the cluster model. For the simple harmonic oscillator approximation its validity was shown in the early days of cluster studies [6]; for more general (realistic) interactions it is discussed below. The SU(3) basis obtained in this way is free from the Pauli forbidden states (as well as from the spurious excitations), and it describes the cluster configuration, due to its construction.

The physical operators of the model are expressed in terms of the generators of group chain (1) with parameters to be fitted to the experimental data. Depending on the number of clusters, and on the open or closed shell nature of them the group structure can undergo considerable simplification. The simplest case is that of a system consisting of two closed-shell clusters. For such a problem the physical operators are obtained in terms of a single $U_R(4)$ group of the relative motion (simple vibron model).

An important limit emerges when the Hamiltonian can be expressed in terms of the Casimir invariants of the subgroup-chain. This is called dynamic symmetry, and when it holds, an analytic solution of the energy eigenvalue problem is available. For the simple case of two closed-shell clusters it is characterised by the group-chain (with the corresponding labels of irreducible representations)

$$
U_R(4) \supseteq U_R(3) \supseteq O_R(3),
$$

$$
| N, n_\pi, L \rangle, \quad n_\pi = N, N - 1, \ldots, 0; \quad L = n_\pi, n_\pi - 2, \ldots, 1 \text{ or } 0.
$$

(2)
Here $L$ is the angular momentum of a state, $n_{\pi}$ is the number of oscillator quanta in the relative motion, and $N$ is its upper limit representing a cutoff in the model space. When considering only linear and quadratic operators in the Hamiltonian it reads

$$H = \epsilon + \gamma_R C^{(1)}_{U_R(3)} + \delta_R C^{(2)}_{U_R(3)} + \beta C^{(2)}_{O_R(3)},$$

(3)

where $C$ denotes a Casimir operator of the degree given as an upper index. The corresponding eigenvalue is

$$E = \epsilon + \gamma_R n_{\pi} + \delta_R n_{\pi}^2 + \beta L(L + 1),$$

(4)

with $\gamma_R' = \gamma_R + 3\delta_R$. This simple energy formula is able to reproduce the spectrum of the well-established $^{16}$O + $\alpha$ cluster bands of $^{20}$Ne [7].

When there are more than two clusters, or they have open shell structure, then other degrees of freedom couple to the relative motion. U(3) dynamically symmetric Hamiltonians can always be constructed in a simple way. For example, when one of the clusters has non-closed-shell structure with isospin zero and spin non-zero, a simple Hamiltonian with U(3) dynamic symmetry can be written as

$$H = \epsilon + \gamma_R n_{\pi} + \delta_R C^{(2)}_{SU_R(3)} + \delta C^{(2)}_{SU_C(3)} + \delta C^{(2)}_{SU_S(3)} + \theta K^2 + \beta L^2 + \xi L \cdot S.$$  

(5)

Here the subscript $C$ refers to cluster, $S$ is the spin of the core in its L-S coupled shell model description, and $K$ distinguishes the different bands having the same SU(3) quantum numbers.

3. **APPLICATION TO SINGLE CONFIGURATION**

A consistent description of the energy spectra of the $A = 16 - 20$ nuclei is given in Ref. 8 in terms of core-plus-alpha-particle configurations. More specifically the U(3) dynamic symmetry approach is applied to the $^{12}$C + $\alpha$, $^{14}$C + $\alpha$, $^{15}$N + $\alpha$ and $^{16}$O + $\alpha$ systems, with the standard form of the Hamiltonian of Eq. (5). The parameters have a smooth $A$-dependence, based on which an interpolation could be made for those of the $^{13}$C + $\alpha$ system. ($\xi$ had the same value as in the $^{15}$N + $\alpha$ system, and $\theta = 0$ was taken.) The predicted $^{13}$C + $\alpha$ spectrum shows remarkable similarity to that of the $^{17}$O, observed experimentally.

For the $E2$ and $E1$ transitions of the $^{18}$O nucleus a detailed analysis was carried out in Ref. 9 in terms of core-plus-alpha-particle configuration.

The low- and high-lying states of $^{38}$Ar have been described in terms of core-plus-alpha-particle model [10] too, including new experimental data as well.

The $^{12}$C + $^{12}$C and $^{28}$Si + $^{28}$Si resonance spectra can also be described together with the low-lying states of the $^{24}$Mg and $^{56}$Ni nuclei, respectively [11, 12].

4. **RELATIONS TO OTHER MODELS**

The relation of the SACM to other approaches can be investigated most easily for the simplest cluster configuration, i.e., for two structureless clusters.
In the SU(3) shell model the most important part of the Hamiltonian contains a single-particle harmonic oscillator term, and effective two-nucleon interactions of quadrupole-quadrupole nature

\[ H = H_{\text{HO}} + \chi Q \cdot Q. \]  

(6)

Recalling that the (algebraic) \( Q \) operator (which acts only within a major shell) is related to the second-order Casimir operator of SU(3) \( C_{SU(3)}^{(2)} \) by

\[ C_{SU(3)}^{(2)} = 2Q \cdot Q + \frac{3}{4}L \cdot L, \]

(7)

and taking into account the simple \( H_{\text{HO}} = C_{U(3)}^{(1)} \) equation, one realizes that the algebraic Hamiltonian of Eq. (3) contains exactly the terms of Eq. (6). The dynamically symmetric Hamiltonian of the SACM (3) is more general than that of the shell model (7), because (i) it contains an anharmonic term \( n_n^2 \), and (ii) the relative strength of the Casimir operators of the U(3) and O(3) groups is not fixed.

This interaction is, of course, much more general than that of the harmonic oscillator, for which the connection between the shell model and the cluster model was established [6]. Nevertheless, it fulfills the important conditions, needed for interrelating the two models: (i) the energy spectra are identical in the two cases, and (ii) the eigenstates of this Hamiltonian are basis states for both models. Via the shell model the cluster states are also connected to the collective model states [3]. (For example, they have well-defined quadrupole deformation [13].) Therefore, we can say that the common intersection of the three basic structure models of light nuclei is provided by the U(3) dynamic symmetry.

In the fully microscopic description of cluster configurations specific effective two-nucleon forces are applied. Our phenomenologic algebraic Hamiltonians can be compared to them in a simple way. The set of SU(3) basis states is the same in the two approaches, therefore one can calculate the matrix elements of different interactions between the same basis states, and equate them. This has been done for the \( ^{16}\text{O} + \alpha \) system in Ref. 14. It turns out that the general features of the effective two-nucleon forces are similar to those of the phenomenologic one that gives the best fit to the experimental data, however, the corresponding \( \epsilon, \gamma_R, \delta_R, \beta \) values [of Eq. (4)] are somewhat different.

For more general cluster configurations the relations to other models can be worked out in a similar fashion, however, the formalism becomes more involved.

A further interesting question is how the algebraic interaction of Eq. (3) is related to potentials. The relation of the present algebraic description to the geometric picture is discussed in Ref. 15. Here we give a short overview. The main idea is to define the potential as the expectation value of the algebraic Hamiltonian with respect to a trial state

\[ V(\alpha) = \langle \alpha | H | \alpha \rangle. \]

(8)

The trial state has a similar form as that used in the physics of atomic molecules [16, 17]. The difference appears through the condition that a minimal number of \( \pi \) bosons is required. For a trial state we use

\[ |\alpha\rangle = N_{M,n_\alpha}(\alpha \cdot \pi^\dagger)^{n_\alpha} [\sigma^\dagger + (\alpha \cdot \pi^\dagger)]^M |0\rangle, \]

(9)
where $N_{M,n_o}$ is the normalization factor, $N = n_o + M$ and $|0\rangle$ is the boson vacuum. The boson vacuum is not structureless but contains information about the SU(3) structure of the clusters.

The calculation is straightforward and is given in detail in Ref. 15. The parameters $\alpha_{1m}$ have to be related to the relative distance. This is done by defining

$$\langle \alpha | r_m | \alpha \rangle = r_m,$$

where the operator $r_m$ had to be modified due to the introduction of the $\sigma$ bosons [17]. Also a part had to be added [15] in order to reproduce the average position of the wave function of the three dimensional harmonic oscillator with zero $\sigma$ and minimal number of $\pi$ bosons. The variable at the right can be interpreted as the relative vector as long as the variation of the expectation value is not too large. For small values the interpretation as a distance parameter has to be questioned.

Using the above formulas and Ref. 15 it is easy to obtain a potential as a function in the relative distance variable $r$ and the deformation variables of the clusters. The expressions are getting simple when we consider the limit of large total number of $\pi$ plus $\sigma$ bosons. As an example the expectation value of the number operator for $\pi$ bosons is just

$$\langle n_\pi \rangle \approx N(\alpha \cdot \alpha) \approx \frac{m\omega}{2\hbar} (r - r_o)^2.$$  

Note that it corresponds to a shifted oscillator, with the minimum at

$$r_o = \sqrt{\frac{\hbar}{m\omega}} n_o,$$

where $\sqrt{\frac{\hbar}{m\omega}}$ is the oscillator length defined by the united system and $n_o$ is the minimal number of $\pi$ bosons required by the Wildermuth condition. It shows that the minimal number of $\pi$ bosons determines where the minimum of the molecular potential is located.

The procedure was applied to different systems. The simplest one is $^{16}\text{O} + \alpha \rightarrow ^{20}\text{Ne}$ which consists of two spherical clusters. The minimal number of $\pi$ bosons is 8, which gives for $r_o$ about 5 fm. This value agrees very well with a simplistic assumption of a box distribution for the mass and that the two clusters touch each other. Also the estimated energy of the first excited $1^{-}$ state results in 7 MeV compared to the first observed experimental value of 5.8 MeV. In conclusion we obtain fairly consistent results.

The other example we discussed is the $^{12}\text{C} + \alpha \rightarrow ^{16}\text{O}$ case. Here one cluster is highly deformed and oblate. Our procedure allows to obtain a potential which does not only depend on the relative distance of the two clusters but also on the orientation of the clusters with respect to each other. In this case we have the dependence of the potential on the angle, giving the orientation of the symmetry axis of the $^{12}\text{C}$ cluster with respect to the molecular $z$-axis, defined as being along the axis connecting the two clusters. Again the qualitative agreement what one would expect for the position of the molecular minimum and for the first excited $1^{-}$ state is good (see Ref. 15). In the last example we can also discuss the energy dependence of the potential as the orientation of the $^{12}\text{C}$ cluster is changed. As a favorable orientation we obtain the one where the $\alpha$ particle approaches along the symmetry axis of the $^{12}\text{C}$ cluster.
5. Multiple Configurations

An atomic nucleus can be composed of various cluster configurations. The cluster configurations are directly related to reaction channels, therefore, they play an essential role in understanding phenomena like radioactivity, fission and fusion.

The treatment of various cluster configurations in the SACM is based on a composite symmetry called multichannel (or multiconfigurational) dynamic symmetry [18]. By saying that this symmetry has a composite nature, we mean the following. U(3) dynamic symmetries can hold for different cluster configurations, as discussed above (i.e., in a single channel manner). Having U(3) dynamic symmetries in different cluster configurations does not mean that the (intercluster and coupling) interactions in the different channels are the same, or are simply related to each other. If, however, the transformations that take one cluster configuration to another (i.e., the transformations between the two sets of the Jacobi coordinates) leave the interactions invariant, then we say that a multichannel U(3) dynamic symmetry holds. Obviously, this means that the interactions are highly symmetric, i.e., they are very special ones. The harmonic oscillator interactions are known to have this feature [19], but they are too simple ones, being far from realistic. It turns out, however, that interactions corresponding to a U(3) dynamic symmetry have also this nature [18]. They are largely anharmonic, and proved to be realistic for several systems, as illustrated below. These interactions were determined based on the constraints the multichannel symmetry imposed on the energy-eigenvalue equations of different cluster configurations [18].

Having the same Hamiltonian for different configurations, one gets very close relations between the corresponding basis states as well (just like in case of the connection between the shell and cluster models). Let us consider for example the case of states which have U(3) quantum numbers with a single multiplicity, both in the shell model description of the whole nucleus and in different cluster model bases. Due to the fact that the Hamiltonians have exactly the same eigenvalue for the different configurations, the relation of the U(3) bases of these two cluster descriptions is very simple. Each basis state of a cluster configuration is a linear combination of those basis states of the shell model, which belong to the same energy. Since the basis states of different U(3) labels are orthogonal, these wavefunctions are identical with each other. Another way of expressing the same result is the following: the general effect of the antisymmetrization is that seemingly different cluster configurations can overlap to a large extent. The consequences are interesting from different respects. For special nuclear states we obtain a selection rule for cold fission and fusion, while for the energy spectra we end up with correlations between the distributions of different cluster configurations and between the interactions of different reaction channels.

In Ref. 20 the link between the superdeformed and cluster states of alpha-like nuclei ($N = Z =$ even) was explored. It turns out, that in most cases several different clusterizations have the same wavefunction and in addition, the same cluster structure appears at different excitation energies of the same nucleus, depending on the relative orientations of the deformed clusters.

In Ref. 21 the allowed and forbidden binary fission modes of ground-state-like configurations in $sd$ shell nuclei were determined.
For the studies of complex spectra the consequence of the multichannel dynamic symmetry is that it gives a strong correlation between the distributions of different cluster configurations at low and high energies [18]. The high-lying cluster states are usually populated as resonances in heavy-ion reactions. Their analysis in terms of multichannel dynamical symmetry involves much more constraints, than in terms of the phenomenological models (algebraic or geometric), therefore we have much less ambiguities in this description than usual. For example, for some cases parameter-free predictions can be given for the energy spectrum in an interval determined by the experimental circumstances.

The $^{24}\text{Mg} + ^{4}\text{He}$ and $^{12}\text{C} + ^{16}\text{O}$ configurations of the $^{28}\text{Si}$ nucleus have been studied in Ref. 18. The relation of different clusterisations of the $^{24}\text{Mg}$ nucleus is investigated in Ref. 22.

6. Heavy nuclei

The SACM has been proposed for the description of light nuclei, because this approach is largely based on the SU(3) i.e., on an L-S coupled shell model. This coupling scheme is not valid for heavy nuclei; the SU(3) symmetry breaks down, mainly due to the spin-orbit interactions. Recent shell model studies, however, revealed some possibilities to overcome this difficulty, and restore an SU(3) symmetry also for heavy nuclei. Based on these modified shell model schemes, we have considered the extensions of the algebraic cluster model to some problems of heavy nuclei as well.

In the pseudo SU(3) scheme [23] a part of the shell model space is equipped with an SU(3) symmetry, which is obtained by omitting the intruder (i.e., opposite parity single particle) states from each major shell. The nucleons sitting in the intruder states have to be handled separately (typically in a seniority scheme). Usually it is supposed that the nucleons of the normal-parity subspace determine the dynamics, while intruders follow this dynamics due to an adiabatic coupling.

Based on the pseudo SU(3) extension of the SACM two problems have been addressed so far. (i) In Ref. 24 we have investigated the question, whether the $^{210}\text{Pb} + ^{14}\text{C}$ cluster configuration is present in the ground-state wave function of the $^{224}\text{Ra}$ nucleus. The answer turns out to be affirmative, and in addition the low-lying bands could be described in terms of this cluster configuration. (ii) The work in Ref. 25 has dealt with the effects of the microscopic structure on the cold binary fission channels of the $^{252}\text{Cf}$ nucleus, some of which have recently been observed [26].

The other possibility of making use of the SU(3) symmetry is still related to the real SU(3) scheme. Although it is not valid for heavy nuclei, as it is for the light ones, it does not necessarily mean that it is useless. What it really means is that we can not restrict ourselves to the leading (i.e., to a single) representation of the orbital part of the wavefunction, and spin $S = 0$. For strong spin-orbit interactions shell model studies indicate [27] that the states fall into bands with extremely small interband quadrupole matrix elements. Moreover, the states obtained by diagonalising the Hamiltonian are remarkably close to the quasi-representations obtained by strong coupling of spin $S = 1$ with the SU(3) irrep of the orbital part. These investigations seem to suggest that the
leading term approximation can be considerably improved by taking into account also the $S = 1$ contribution, and still the problem remains tractable.

Based on this approximation recently we have investigated the effect of the structural selection rule for the Mo emission of the ground state of the $^{252}\text{Cf}$ in spontaneous fission [28]. The mass-number dependence of the reciprocal forbiddenness [25] for the Mo decay turns out to have a single-bumped structure, fairly similar to that of the yield-curve found experimentally.

7. **Conclusions**

In this contribution we have presented a brief review on the Semicroscopic Algebraic Cluster Model with some of its applications. In this approach the model space is constructed microscopically, and the interactions we apply are phenomenological ones. Nevertheless, their relation to effective two-nucleon-forces can be established.

The model seems to be capable of reproducing the bulk properties of a large amount of experimental data for light nuclei in a wide energy range. Furthermore, one can make predictions (i) based on the mass number dependence of its operators, and (ii) based on the multichannel symmetry of its interactions. The latter one seems to be especially valuable, since (to our best knowledge) no other model is known to relate the distributions of different cluster configuration of a nucleus.

The applications of the model have been carried out so far within the U(3) dynamic symmetry approach. This symmetry plays an important role also in connecting the cluster picture to that of the shell model and collective model as well.

The extension of the model to heavy nuclei has been initiated, but it is far from being complete. It is related to several interesting questions of the present day nuclear studies, like e. g., exotic clusterization, cold fission, and the validity of real or pseudo SU(3) symmetry.

8. **Acknowledgments**

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**References**