The expanding pion liquid and the two-pion correlation function

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We examine the effects that a confining boundary together with hydrodynamical expansion play on two-pion distributions in relativistic heavy-ion collisions. We show that the effects arise from the introduction of further correlations due both to collective motion and the system's finite size. We study the behavior of the correlation function intercept with the system's density. We show that as the density increases, the intercept decreases. The effect is due to the fact that, as the ground state population becomes a significant fraction of the total multiplicity, pions coming from this single quantum state do not partake in the HBT effect, leading to a reduction in the intercept. We also show that, though the effects coming from hydrodynamical expansion lead to a reduction in the apparent source radius with increasing and large average pair momentum \( K \), for small \( K \), the presence of the boundary leads also to a decrease of the apparent source radius with decreasing \( K \). These two competing effects produce a maximum for the effective source radius as a function of \( K \).

**Keywords:** Scattering heavy ion; \( \pi \)-production; \( \pi \)-momentum spectrum; \( \pi \)-correlation function; fireball; temperature

En este trabajo se examinan los efectos introducidos por la presencia conjunta de una frontera y la expansión hidrodinámica colectiva sobre las distribuciones de dos piones en colisiones de iones pesados relativistas. Se muestra que los efectos surgen de la introducción de correlaciones adicionales debidas tanto al movimiento colectivo como al tamaño finito del sistema. Se estudia el comportamiento del parámetro de interceptación como función de la densidad del sistema, mostrándose como es que éste decrece al aumentar la densidad. Este efecto se debe al hecho de que al aumentar la población del estado base para convertirse en una fracción importante de la multiplicidad total, los piones provenientes de este estado cuántico individual no participan del efecto HBT, dando lugar a la reducción del parámetro de interceptación. También se muestra que, pese a que los efectos de expansión hidrodinámica originan una reducción en el radio aparente de la fuente conforme el momento promedio \( K \) del par crece, para valores pequeños de \( K \), la presencia de la frontera origina también que el radio aparente de la fuente disminuya para \( K \) decreciente. La competencia entre estos dos efectos produce un máximo en el radio efectivo de la fuente como función de \( K \).

**Descripciones:** dispersión de iones pesados; producción \( \pi \); espectro de momento \( \pi \); función de correlación \( \pi \); bola de fuego; temperatura

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1. Introduction

In recent years, a very important experimental effort has been geared towards the production of a state of matter where the prevailing extreme conditions of high baryonic densities and/or temperatures, make it possible to produce the so called quark-gluon plasma (QGP), in relativistic heavy ion collisions. The expectation is to detect the fundamental QCD degrees of freedom not necessarily confined within a single nucleon but rather over a larger volume of the order of the dimensions of a nucleus. Strangely enough, while the properties of the QGP have been the subject of intense theoretical study and debate, much less attention has been paid to the properties of hadronic matter at high temperatures and densities, particularly since the most abundantly particles produced in such collisions are pions. An understanding of the properties of pion matter under such conditions is certainly called for, both to be able to discern the signals originating from the different stages of the collision for a clear distinction of a possible QGP formation but also as an interesting subject of study on its own.

Given that the typical number of pions produced one unit around central rapidity in central \( Au + Au \) collisions at energies of order 10 A GeV is \( dN_\pi/dy \sim 300 \) [1], one could estimate the average pion density at freeze out. Under the assumption that the transverse dimensions of the system formed at freeze out are of order of the transverse size of a gold nucleus and that the typical pion formation time is of order 1 fm, the above multiplicity implies that the average pion separation \( d \) at freeze out in the central rapidity region is of order \( d \sim 0.6 \) fm which is less than the average range of the pion strong interaction, \( d_s \sim 1.4 \) fm.

Some of the possible consequences of this large pion density produced in relativistic heavy-ion collisions where first studied by Shuryak [2] who coined the term *pion liquid* to refer to the situation where the pion system could not be thought of as existing as a hadron gas but rather, that its properties resembled more those of a liquid of quasipions. In particular, as one of the main characteristics of liquids is the appearance of a surface tension, such state of hadron matter could give rise to a confining boundary that acted as a reflecting surface that could affect the pion distributions just before freeze out. More recently [3], it has been proposed that the equation of state of pion matter could give rise to a phase transition from a gas phase to a more dense phase as the temperature rises close to the temperatures expected to be achieved in relativistic heavy-ion collisions, thus introducing the concept of a *hot pion liquid*. 
Phenomenologically, the development of a surface tension can be incorporated by imposing a sharp boundary for the pion wave functions to evolve just before freeze-out. As a consequence of the finite size of the system during this stage of its evolution, the energy states form a discrete set.

An important difference between statistical systems with and without a boundary is the different density of states at increasing energies, being larger in the case of the former, as illustrated in Fig. 1. The density of states of a finite system approaches that of an unbound one as the size of the system is increased. The above characteristic implies that the transverse inclusive spectrum calculated within a boundary model will exhibit a concave shape at high transverse momentum and could potentially explain the increase of the pion transverse distribution [4] in this kinematical region [5]. The increase of the pion distribution at low transverse momentum could also be explained within the same context by considering the finite chemical potential associated with the mean pion multiplicity per event [6].

More recently, it was realized that an important missing ingredient in the description of the transverse pion spectra within a boundary model was the proper inclusion of hydrodynamical expansion. The phenomenological description in terms of a bound, expanding pion system has been named the expanding pion liquid model and was developed in Ref. 7 and successfully applied to the description of the experimental mid-rapidity, transverse pion spectra in central Au+Au collisions at 11.6A GeV/c [1]. A further natural test ground for the model is the study of multiparticle correlations, in particular two-pion correlations.

A step in this direction has been taken in Ref. 8 where the effects of a pure boundary model (i.e. without collective expansion) have been introduced in the description of the two-pion correlation function. In this work, we incorporate also the effects of hydrodynamical expansion in the calculation of the two-pion correlation function.

As is well known in the study of two-particle correlation functions, the effective size of a system without a boundary decreases monotonically as the average pair momentum is increased [9] when considering the effects of hydrodynamical expansion alone. Physically, this effect is due to the fact that as the average pair momentum increases, the particles in the pair are more likely to be emitted from points close in space. This can also be regarded as the introduction of a further correlation in phase space for the emitted particles that effectively reduces the size of the emission region [10]. On the other hand, as emphasized in Ref. 8, for emission volumes of order of a few average pair wavelengths it is imperative to consider a full quantum density matrix in the description of particle distributions since as the average pair momentum decreases, the correlation region as a function of relative momentum increases thus effectively reducing the apparent size of the system. As we will show in this work, these two competing effects produce a maximum in the effective size of the system at a finite value of the average pair momentum.

2. The expanding pion liquid

Notice that when the system of pions can be considered as confined, its wave functions satisfy a given condition at the boundary. In order to compare the results with the observed particle distributions, the shape of the assumed confining volume could play an important role. It has long been known that the particle momentum distributions are somewhat forward-backward peaked, particularly at energies of the SPS [11], even in the case of central collisions. Nevertheless, for the sake of simplicity and concreteness, here we will assume that the confining volume is spherical. Comparison of the model with data will become better in the central rapidity region where an asymmetry between longitudinal and transverse expansion is less important than in the fragmentation region.

To incorporate the effects of an hydrodynamical flow, we notice that this ordered motion can be represented by a four-velocity field \( u^\mu = \gamma(r) [1, v(r)] \), where \( \gamma \) is the Lorentz-gamma factor and \( v(r) \) is the velocity vector. This field represents a redistribution of momentum in each of the fluid cells, as viewed from a given reference frame (the center of mass in this case), becoming centered about the momentum associated with the velocity of the fluid element. This behavior can be described by the substitution \( p^\mu \rightarrow p^\mu - mu^\mu \), where \( m \) is the pion mass. The term \( mu^\mu \) represents the collective momentum of the given pion fluid element.

The eigenfunctions of the confined, expanding system of pions are thus obtained as the solutions to the equation

\[
\left\{ -\left(\frac{i}{\partial t} - m\gamma(r) \right)^2 + \left[ -i\nabla - m\gamma(r)v(r) \right]^2 + m^2 \right\} \psi(r, t) = 0 \tag{1}
\]

and in the approximation, we expand \( \gamma(r) \) to the lowest non-

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**Figure 1.** Number of states for a spherically symmetric system with a sharp boundary at \( R = 6 \text{ fm} \) (---) compared to a system without boundary (- - -). Notice how the presence of the boundary makes the number of states grow faster at large energies.
trivial order, these are given by (see Ref. 7 for details)

\[\psi_{nlt'}(\mathbf{r},t) = \frac{A_{nl}}{\sqrt{2E_{nl}}} e^{-iE_{nl}t+i\mathbf{m}\beta^2/(2R)} Y_{l'}^{m'}(\mathbf{r}) \times e^{-\alpha_{nl}^2 r^2/2R} F_l\left(\frac{l+3/2}{2} - \frac{E_{nl}^2}{4\alpha_{nl}^2}, l + \frac{3}{2}; \frac{\alpha_{nl}^2 r^2}{2}\right), \tag{2}\]

The parameter \(0 < \beta < 1\) represents the surface fireball hydrodynamical velocity, \(F_l\) is a confluent hypergeometric function and \(Y_{l'}^{m'}\) is a spherical harmonic. The quantities \(A_{nl}\) are the normalization constants and are found from the condition

\[\int d^3r \psi_{nlt'}^*(\mathbf{r},t) \frac{\partial}{\partial t} \psi_{nlt'}(\mathbf{r},t) = 1. \tag{3}\]

The parameters \(\alpha_{nl}\) and \(\varepsilon_{nl}\) are related to the energy eigenvalues \(E_{nl}\) by

\[\alpha_{nl}^2 = m(E_{nl} - m)^2 / R^2, \quad \varepsilon_{nl} = E_{nl}(E_{nl} - 2m), \tag{4}\]

with \(E_{nl}\) given as the solutions to

\[1 F_l\left(\frac{l+3/2}{2} - \frac{\varepsilon_{nl}^2}{4\alpha_{nl}^2}, l + \frac{3}{2}; \frac{\alpha_{nl}^2 R^2}{2}\right) = 0. \tag{5}\]

Equation (2), along with the energy eigenvalues and definitions in Eqs. (4) and (5), constitute the set of (properly normalized) eigenfunctions in terms of which the various multi-particle distributions can be expressed. The system’s finite size and the strength of the collective expansion are given in terms of the parameters \(R\) and \(\beta\), respectively. In order to further proceed, it is necessary to specify the kind of ensemble that describes the statistical properties of the pion system.

### 3. Two-pion correlation function

In order to describe the situation where equilibrium has been attained (which we assume here), the proper statistical distribution for the ensemble is thermal. For the purposes of this section, we closely follow Ref. 8 to which we refer the reader for details. Let \(\lambda\) represent the set of quantum numbers \(\{nlm\}\). The corresponding occupation number \(N_\lambda\) for a given state is written as

\[N_\lambda = \exp\left(\frac{E_\lambda - \mu}{T} - 1\right), \tag{6}\]

where \(T\) is the system’s temperature and \(\mu\) the chemical potential, related to the average total multiplicity \(N\) by

\[N = \sum_\lambda \exp\left(\frac{E_\lambda - \mu}{T} - 1\right). \tag{7}\]

Let \(\psi_\lambda(p)\) represent the Fourier transformed wave function for the state with quantum numbers \(\lambda\), namely

\[\psi_\lambda(p) = \int d^3r e^{-ipr} \psi_\lambda(r). \tag{8}\]

With the normalization adopted in Eq. (2), the one-pion momentum distribution can be written as

\[P_1(p) \equiv \frac{d^3 N}{d^3 p} = \frac{1}{(2\pi)^3} \sum_\lambda 2E_\lambda \rho_\lambda(p) \psi_\lambda(p). \tag{9}\]

Similarly, and under the assumption of a complete factorization of the two-particle density matrix, the two pion momentum distribution can be written as

\[P_2(p_1, p_2) \equiv \frac{d^3 N}{d^3 p_1 d^3 p_2} = P_1(p_1)P_1(p_2) \]

\[+ \frac{1}{(2\pi)^3} \sum_\lambda \left[2E_\lambda \rho_\lambda(p_1) \psi_\lambda(p_1) \psi_\lambda(p_2)\right]^2, \tag{10}\]

from where the two-pion correlation function \(C_2\) can be written, in terms of \(P_1\) and \(P_2\), as

\[C_2(p_1, p_2) \equiv \frac{P_2(p_1, p_2)}{P_1(p_1)P_1(p_2)} = 1 \]

\[+ \frac{\sum_\lambda E_\lambda \rho_\lambda(p_1) \psi_\lambda(p_2)}{\sum_\lambda E_\lambda \rho_\lambda(p_1) \psi_\lambda(p_2)} + \frac{\sum_\lambda E_\lambda \rho_\lambda(p_1) \psi_\lambda(p_2)}{\sum_\lambda E_\lambda \rho_\lambda(p_1) \psi_\lambda(p_2)} \tag{11}\]

Notice that as a consequence of the factorization assumption, the correlation function is such that \(C_2(p, p) = 2\). This is usually referred to as the completely chaotic pion production scenario [12], which is the situation expected to occur in a heavy ion collision, given the considerable rescattering experienced by pions in the production region. In contrast, if the particles were produced completely coherently, they would occupy a pure quantum state and the two-pion momentum distribution would be simply the product of two single-pion momentum distributions, leading to the absence of the HBT effect.

### 4. The effective radius

Armed with the eigenfunctions describing the confined and expanding pion system, Eq. (2) and with the explicit expression for the two-pion correlation function in Eq. (11), it is possible to perform an analysis to describe the behavior of \(C_2\) as a function of the several variables and parameters involved. For the spherically symmetric problem described here, the correlation function depends on the angle between the pion pair momenta. For the sake of simplicity, let us consider the case in which both momenta \(p_1\) and \(p_2\) are parallel. In this case, the summation in the numerator of the second term in Eq. (11) can be simplified, with the aid of the addition theorem for the spherical harmonics

\[\sum_{m'=-l}^l Y_{l}^{m'}(\hat{p}_1)Y_{l}^{m'*}(\hat{p}_2) = \frac{2l + 1}{4\pi}. \tag{12}\]
creases as \( \lambda \) increases. However, when the ground state population becomes a significant fraction of the total multiplicity, e.g. for \( \mu = 315 \text{ MeV}, N_g/N = 0.37 \) and \( \lambda \sim 0.62 \) (- - -). When almost all of the particles occupy the ground state, e.g. for \( \mu = 317.5 \text{ MeV}, N_g/N = 0.77 \), the correlation function becomes flatter.

Figure 2. a) \( C_2(q) \) for a fixed value of \( K = 500 \text{ MeV} \) and for \( R = 6 \text{ fm} \) (---) and \( R = 10 \text{ fm} \) ( - - -). The width of \( C_2 \) decreases as \( R \) is increased, b) \( C_2(q) \) for a fixed value of \( R = 6 \text{ fm} \) and for \( K = 300 \text{ MeV} \) (---), \( K = 500 \text{ MeV} \) ( - - -) and \( K = 700 \text{ MeV} \) (-----). For the chosen values of \( K \) the width of \( C_2 \) increases as \( K \) is increased, and c) \( C_2(q) \) for a fixed value of \( R = 6 \text{ fm} \) and for \( K = 50 \text{ MeV} \) (---), \( K = 100 \text{ MeV} \) ( - - -) and \( K = 200 \text{ MeV} \) (-----). For the chosen values of \( K \) the width of \( C_2 \) decreases as \( K \) is increased. In all cases, the temperature, surface expansion velocity and chemical potential have been held fixed to \( T = 120 \text{ MeV}, \beta = 0.5c \) and \( \mu = 0 \) respectively.

Figure 2 shows the behavior of \( C_2(q) \) as a function of \( q \), the magnitude of the pair momentum difference \( q = p_2 - p_1 \). Figure 2a shows \( C_2(q) \) for a fixed value of the magnitude \( K = 500 \text{ MeV} \) of the average pair momentum \( K = (K_g + K_t)/2 \) for two values of the system radius \( R = 6, 10 \text{ fm} \). Notice that for a fixed \( K \) the width of the correlation function decreases as \( R \) is increased. Figure 2b shows \( C_2(q) \) for a fixed value of \( R = 6 \text{ fm} \) and three values of \( K = 300, 500, 700 \text{ MeV} \). For the chosen values of \( K \), the width of the correlation function increases as \( K \) is increased. Figure 2c shows \( C_2(q) \) for a fixed value of \( R = 6 \text{ fm} \) and three values of \( K = 50, 100, 200 \text{ MeV} \). Notice that in this case, the width of the correlation function decreases as \( K \) is increased. In all these Figures, the temperature and surface expansion velocity have been held fixed to \( T = 120 \text{ MeV} \) and \( \beta = 0.5c \). These values for \( T \) and \( \beta \) are chosen in accordance to the analysis in Ref. 13 where a correlation between the transverse flow velocity and the freeze out temperature is found in such a way that higher temperatures imply lower expansion velocities and vice versa. Since, at least for AGS energies, not too high temperatures are reached during the collision, the above value for \( T \) implies that for \( \beta \), thus, the free parameters for the model can be taken either as \( R \) and \( T \) or \( R \) and \( \beta \).

It is also worth mentioning that in both of the above Figures, the value of the chemical potential \( \mu \) appearing in Eq. (11) has been fixed to \( \mu = 0 \). The behavior of \( C_2(q) \) for different values of the chemical potential (i.e. different average multiplicities) is shown in Fig. 3. Notice that varying the chemical potential up to values below the onset of Bose-Einstein condensation (BEC) [6] does not introduce changes in the shape of the correlation function. This can be understood by noticing that even when we increase the system’s density and thus the value of \( \mu \), we are not introducing any further correlation among the bound but otherwise non-interacting set of particles. This situation changes when, for a given temperature and system’s size, \( \mu \) is beyond the value to allow for the ground state to accommo-
date a significant fraction of the particle population [6]. This is also shown in Fig. 3. In this case, the ground state population has to be treated separately from the pions coming from the excited states, since the former originates from a pure quantum state, as opposed to the assumption leading to Eq. (10). The weight assigned to the pions coming from the ground state is equal to the ratio of the ground state population $N_g$ to the total average multiplicity $N$. Correspondingly, the correlation function $C_2(q)$ becomes flatter and the intercept with the vertical axis occurs for values smaller than 2, that is, $\lambda = 1 - N_g/N$. Thus, as the multiplicity increases, this behavior signals that for the given temperature and volume, pions are predominantly emitted from the ground state. This is in agreement with the analysis in Ref. 14.

Another property of the bound and expanding system of pions that can be extracted from the correlation function $C_2(q)$ is the behavior of the system's effective radius $R_{\text{eff}}$ as a function of $K$. The relevant quantity to pay attention to is the ratio $\eta = T/\gamma(R)m\beta$ of the energy scale associated with random motion, i.e. $T$, to the energy scale associated with ordered motion, i.e. $\gamma(R)m\beta$. For $K$ small compared to $\eta T$ —that is, when the average pair momentum is mostly due to random motion— $R_{\text{eff}}$ is an increasing function of $K$. This can be understood by noticing [8] that increasing $K$ corresponds to increasing both of the momenta in the pion pair. Correspondingly, the quantum states that contribute to the momentum distributions $P_1$ and $P_2$ are those with increasingly larger quantum numbers. But, according to Eq. (6), these states are suppressed by their statistical weight and therefore, only those other states with smaller quantum numbers —and consequently with a larger spread in coordinate space— can contribute significantly to the correlation function, which in turn drops faster as a function of $q$ with increasing $K$, leading to an increase in size of the apparent region of particle emission. However, for $K > \eta T$, the collective motion dominates over the thermal motion in $K$ and the relevant physical effect that dictates the behavior of $R_{\text{eff}}$ is the correlation between the spatial region of emission of pions and the pair momentum introduced by the collective expansion, in such a way that faster pions are more likely to be emitted from points close in space [9] leading to a reduction in size of the apparent region of particle emission.

\begin{equation}
C_2(P_1, P_2) = 1 + \frac{\int d^3x \ G\left(\frac{P_1 + P_2}{2}\right) e^{-i(P_1 - P_2) \cdot x} \ [\int d^3x \ G(x, p_1) \ [\int d^3x \ G(x, p_2)]}}{[\int d^3x \ G(x, p_1) \ [\int d^3x \ G(x, p_2)]}.
\end{equation}

For the bound and non-expanding system, the eigenfunctions are given in terms of Bessel functions of the first kind [6] (see also Ref. 8). The corresponding expression for $R_{\text{eff}}$ is obtained from that of $C_2(q)$ by also fitting gaussians of the form given by Eq. (13). Notice that the curve representing the effective radius as a function of $K$ for a bound but nonexpanding system grows with $K$, in agreement with the analysis of Ref. 8. In contrast, the curve representing the effective radius for an unbound but expanding system decreases monotonically as $K$ is increased, also in agreement with the analysis of Ref. 9.

5. Conclusions

We have studied the effects that a confining boundary together with hydrodynamical expansion at freeze out, play on the two-pion correlation function, in the context of relativis-
tic heavy-ion collisions. We have argued that the confining boundary could be produced as a consequence of the high pion density that can be achieved at freeze out in central collisions.

We have shown that for a given system's volume and temperature, varying the multiplicity, and therefore the chemical potential, does not introduce any changes in the correlation function when \( \mu \) is below the values for BEC. Nevertheless, the intercept of the function \( C_2(q) \) occurs for values less than 2 when the chemical potential is beyond the value to allow for BEC. A similar behavior can be expected for a given pion density if the freeze out temperature is below the critical temperature for BEC. However, this is a less likely scenario in this kind of collisions.

We have found the behavior of \( C_2(q) \) when varying either \( R \) or \( \gamma \) keeping the other variable fixed. Since the importance of correlation analyses rests basically on the information that it can provide about the physical size of the system produced during the collision, a major result of the present work is the functional dependence of the effective system radius \( R_{\text{eff}} \) with the magnitude of the average pair momentum \( K \). We have shown that the interplay of the energy scales associated with collective and random motion, \( \gamma(R) m \beta \) and \( T \), respectively, produce a maximum for \( R_{\text{eff}} \) at a value \( K \sim \eta T \), where \( \eta = T/\gamma(R) m \beta \). The physical origins of this behavior are the combined effects of the confining boundary and hydrodynamical expansion. In the regime where \( K \) is basically due to random motion, the boundary effects are the most important and \( R_{\text{eff}} \) grows as a function of \( K \). However, in the regime where \( K \) is basically due to collective expansion, the effective size of the system is dictated by the correlation between the points of emission and the pair momentum and \( R_{\text{eff}} \) decreases as a function of \( K \). This behavior signals that the pion system feels the presence of the boundary at freeze out and can be used to experimentally determine such effects in relativistic heavy-ion collisions.

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