Mass textures and Wolfenstein parameters from breaking the flavour permutational symmetry

A. Mondragón, T. Rivera, and E. Rodríguez-Jáuregui
Instituto de Física, Universidad Nacional Autónoma de México
Apartado postal 20-364, 01000 México, D.F., Mexico
1Deutsches Elektronen-Synchrotron, Theory Group
Notkestrasse 85, D-22603 Hamburg, Germany

Recibido el 14 de marzo de 2001; aceptado el 28 de junio de 2001

We will give an overview of recent progress in the phenomenological study of quark mass matrices, quark flavour mixings and CP-violation with emphasis on the possibility of an underlying discrete, flavour permutational symmetry and its breaking, from which realistic models of mass generation could be built. The quark mixing angles and CP-violating phase, as well as the Wolfenstein parameters are given in terms of four quark mass ratios and only two parameters \( Z^{1/2}, \Phi \) characterizing the symmetry breaking pattern. Excellent agreement with all current experimental data is found.

Keywords: Spontaneous symmetry breaking; unified field theories; hermitian mass matrices; quark mass; CP invariance

Daremos una vista panorámica del progreso reciente en el estudio fenomenológico de las matrices de masas de quarks y mezclas del sabor de los quarks y la violación de CP, con énfasis en la posibilidad de que haya una simetría discreta, permutacional del sabor y su rompimiento a partir de las cuales se puedan construir modelos realistas de la generación de masas. Los ángulos de mezcla de los quarks y la fase que viola CP, así como los parámetros de Wolfenstein se dan en términos de cuatro razones de masas de los quarks y solamente dos parámetros \( Z^{1/2}, \Phi \) que caracterizan el patrón de rompimiento de la simetría. Los resultados se encuentran en excelente acuerdo con todos los datos experimentales mas recientes.

Descriptores: Rompimiento espontaneo de simetría; teorías de campo unificadas; matrices hermitianas; masas de quarks; invariancia de CP

PACS: 12.15.Ff; 11.30.Er; 11.30.Hv; 12.15.Hh

1. Introduction

In this paper we will give a short review of some recent progress in the derivation of quark mass textures and quark mixing matrices from breaking the flavour permutational symmetry. We will also give some new results on the functional dependence of the Wolfenstein parameters on the quark mass ratios and the two flavour symmetry breaking parameters \( Z^{1/2} \) and \( \Phi \). In this way we will be able to relate some features of the assumed flavour symmetry breaking pattern to the properties of the mixing matrix in a very direct, analytical way.

In the standard electroweak theory of particle interactions, quark flavour mixing is governed by the four parameters of the unitary mixing matrix \( V \). In the standard parametrization of the mixing matrix [1], advocated by the particle data group [2], \( V^{PDG} \) is parametrized in terms of three mixing angles \( \theta_{12}, \theta_{13}, \theta_{23} \) and one CP-violating phase \( \delta_{13} \). In the Wolfenstein parametrization [3], the four parameters in \( V^W \) are labelled \( \lambda, A, \rho \) and \( \eta \). These and other phenomenological parametrizations were introduced without taking the possible functional relations between quark masses and flavour mixing parameters into account.

In contrast, the elements of the theoretical quark mixing matrix \( V^{th} \), derived from a simple ansatz on the breaking of the flavour permutational symmetry in two previous papers [4, 5], are explicit functions of the four quark mass ratios \( m_u/m_d, m_c/m_t, m_s/m_b, m_d/m_s \) and only two independent parameters, namely, the CP-violating phase \( \Phi \) and the symmetry breaking parameter \( Z^{1/2} \). The phase equivalence of \( V^{th} \) with \( V^{PDG} \) will allow us to obtain exact explicit, analytical expressions for the Wolfenstein parameters \( \lambda, A, \rho \) and \( \eta \) as functions of the quark mass ratios and the two flavour symmetry breaking parameters \( Z^{1/2} \) and \( \Phi \). Computing the Wolfenstein parameters in the leading order of magnitude in the quark mass ratios, we obtain simple but very accurate, approximate, analytical expressions that relate the properties of the mixing matrix to the characteristic features of the assumed flavour symmetry breaking pattern in a very direct way.

The plan of this paper is as follows. In Sec. 2, the basic concepts and notation are introduced by way of a very brief sketch of the group theoretical derivation of mass matrices with a modified Fritzsch texture. Exact explicit expressions for the elements of \( V^{th} \) in terms of the quark mass ratios and the symmetry breaking parameters are derived in Sec. 3. In Sec. 4, the phase equivalence of \( V^{th} \) and the phenomenological parametrization \( V^{PDG} \) is briefly recalled. Exact explicit expressions for the Wolfenstein parameters as functions of the quark mass ratios and the symmetry breaking parameters are obtained in Sec. 5. Section 6 is devoted to a discussion of the connection between Wolfenstein parameters and the flavour symmetry breaking pattern. We end our paper in Sec. 7 with a summary of results and some conclusions.
2. Mass matrices from the breaking of $S_L(3) \otimes S_R(3)$

A number of authors [4-25] have pointed out that realistic quark mass matrices result from the flavour permutational symmetry $S_L(3) \otimes S_R(3)$ and its spontaneous or explicit breaking. The group $S(3)$ treats three objects symmetrically, while the hierarchical nature of the mass matrices is a consequence of the representation structure $\mathbf{6}$ of $S(3)$, which treats the generations differently. Under exact $S_L(3) \otimes S_R(3)$ permutational symmetry, the mass spectrum for either up or down quark sectors consists of one massive particle in a singlet irreducible representation and a pair of massless particles in a doublet irreducible representation, the corresponding quark mass matrix with the exact $S_L(3) \otimes S_R(3)$ symmetry will be denoted by $M_{3q}$. In order to generate masses for the first and second families, we break the $S_L(3) \otimes S_R(3)$ permutational symmetry adding the terms $M_{1q}$ and $M_{2q}$ to $M_{3q}$. The term $M_{2q}$ breaks the permutational symmetry $S_L(3) \otimes S_R(3)$ down to $S_L(2) \otimes S_R(2)$ and mixes the singlet and doublet representations of $S(3)$. $M_{1q}$ transforms as the mixed symmetry $(\alpha \epsilon \tau)$ in the doublet complex tensorial representation of $S_{\text{diag}}(2) \subset S_L(2) \otimes S_R(2)$. Putting the first family in a complex representation allows us to have a CP violating phase. Then, in a symmetry adapted basis, $M_q$ takes the form

$$M_q = m_{3q} \begin{pmatrix} 0 & A_q e^{-i\phi_q} & 0 \\ A_q e^{i\phi_q} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + m_{3q} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\Delta_q + \delta_q & B_q \\ 0 & B_q & \Delta_q - \delta_q \end{pmatrix} + m_{3q} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (1)$$

The entries in the mass matrix may be readily expressed in terms of the mass ratios $\tilde{m}_{1q} = m_{1q}/m_{3q}$ and $\tilde{m}_{2q} = m_{2q}/m_{3q}$

$$A_q^2 = \tilde{m}_{1q} \tilde{m}_{2q} (1 - \delta_q)^{-1}, \quad \Delta_q = \tilde{m}_{2q} - \tilde{m}_{1q}, \quad B_q = \delta_q \left[ (1 - \tilde{m}_{1q} + \tilde{m}_{2q} - \delta_q) - \tilde{m}_{1q} \tilde{m}_{2q} (1 - \delta_q)^{-1} \right]. \quad (2)$$

If each possible symmetry breaking pattern is now characterized by the ratio $Z_q^2 = B_q/(-\Delta_q + \delta_q)$, the small parameter $\delta_q$ is obtained as the solution of the cubic equation

$$\delta_q \left[ (1 + \tilde{m}_{2q} - \tilde{m}_{1q} - \delta_q)(1 - \delta_q) - \tilde{m}_{1q} \tilde{m}_{2q} \right] - Z_q \left( (\tilde{m}_{2q} + \tilde{m}_{1q} + \delta_q)^2 \right) = 0, \quad (3)$$

which vanishes when $Z_q$ vanishes,

$$\delta_q = Z_q \frac{(\tilde{m}_{2q} - \tilde{m}_{1q})^2}{W_q(Z_q)}, \quad (4)$$

in this expression, $W_q(Z_q)$ is the product of the two roots of Eq. (4) which do not vanish when $Z_q$ vanishes, $W_q(Z_q)$ is a function of the two quark mass ratios $\tilde{m}_{2q}, \tilde{m}_{1q}$, and the symmetry breaking parameter $Z_q$. An exact explicit expression for $W_q$ is given in Eqs. (3.4)-(3.16) of Mondragón and Rodríguez-Jáuregui [4]. A fairly good approximate expression for $W_q(Z_q)$, valid for $0 \leq Z_q < 15$

$$W_q(Z_q) \approx \frac{(1 - \tilde{m}_{1q})(1 + \tilde{m}_{2q}) + Z_q(\tilde{m}_{2q} - \tilde{m}_{1q})}{2(\tilde{m}_{2q} - \tilde{m}_{1q}) + Z[1 + 2(\tilde{m}_{2q} - \tilde{m}_{1q})]} \frac{(\tilde{m}_{2q} - \tilde{m}_{1q})}{(1 - \tilde{m}_{1q})(1 + \tilde{m}_{2q}) + \tilde{Z}_q^2(\tilde{m}_{2q} - \tilde{m}_{1q})} Z_q \left( (\tilde{m}_{2q} - \tilde{m}_{1q}) \right)^2 \left( 2 + (\tilde{m}_{2q} - \tilde{m}_{1q}) \right) \left( 1 + \tilde{m}_{2q} \right) Z_q \left[ 1 + 2(\tilde{m}_{2q} - \tilde{m}_{1q}) \right] \frac{1}{Z_q \left( (\tilde{m}_{2q} - \tilde{m}_{1q}) \right)^2 (1 + \tilde{m}_{2q}) + \tilde{Z}_q^2(\tilde{m}_{2q} - \tilde{m}_{1q}) \left( 2 + (\tilde{m}_{2q} - \tilde{m}_{1q}) \right) \frac{1}{Z_q \left( (\tilde{m}_{2q} - \tilde{m}_{1q}) \right)^2 (1 + \tilde{m}_{2q}) + \tilde{Z}_q^2(\tilde{m}_{2q} - \tilde{m}_{1q}) \left( 2 + (\tilde{m}_{2q} - \tilde{m}_{1q}) \right) \end{pmatrix}. \quad (5)$$

2.1. Symmetry breaking pattern

In the symmetry adapted basis, the matrix $M_{2q}$, in the right hand side of Eq. (1), takes the form

$$M_{2q} = m_{3q} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & Z_q^{1/2} \\ 0 & Z_q^{1/2} & -1 \end{pmatrix}, \quad (7)$$

the parameter $Z_q^{1/2}$ is a measure of the amount of mixing of singlet and doublet irreducible representations of $S_{\text{diag}}(2) \subset S_L(2) \otimes S_R(2)$. The corresponding symmetry breaking term in the Yukawa Lagrangian, $\tilde{q}_i M_{2q} \bar{q}_r$, is a functional of only two fields. Under the permutation of these fields, $\tilde{q}_i M_{2q} \bar{q}_r$ splits into the sum of an antisymmetric term $\tilde{q}_i M_{2q}^{A} \bar{q}_r$, which changes sign, and a symmetric term $\tilde{q}_i M_{2q}^{S} \bar{q}_r$, which remains invariant. The matrix $M_{2q}$ is decomposed as a linear combination of two linearly independent numerical matrices, $M_{2q}^A$ and $M_{2q}^S$, these matrices are of the same form as $M_{2q}$ with mixing parameters $Z_A = -\sqrt{8}$ and $Z_S = 1/\sqrt{8}$, respectively. There is a corresponding decomposition of the mixing parameter $Z_q^{1/2}$,

$$Z_q^{1/2} = N_A Z_A^{1/2} + N_S Z_S^{1/2}, \quad (8)$$

with $1 = N_A + N_S$. In this way a unique linear combination of $Z_A^{1/2}$ and $Z_S^{1/2}$ is associated to the symmetry breaking.
pattern. The pair of numbers \((N_A, N_S)\) is a convenient mathematical label of the symmetry breaking pattern. It will be assumed that the up and down mass matrices are generated following the same symmetry breaking pattern \(Z_u^{1/2} = Z_d^{1/2} = Z^{1/2}\).

### 3. The mixing matrix

The hermitian mass matrix \(M_q\) may be written in terms of a real matrix \(\bar{M}_q\) and a diagonal matrix of phases \(P_q\) as \(P_q \bar{M}_q \bar{P}_q^T\). Then, the mixing matrix \(V^{th}\) is given by

\[
V^{th} = O_q^T P^{u-d} O_d
\]

where \(P^{u-d} = \text{diag}[1, e^{i(\phi_u - \phi_d)}, e^{i(\phi_d - \phi_u)}]\) is the diagonal matrix of the relative phases and \(O_q\) is the orthogonal matrix that diagonalizes \(\bar{M}_q\)

\[
O_q = \begin{pmatrix}
\left(\frac{\bar{m}_{2q_1}f_1}{D_1}\right)^{\frac{1}{2}} & -\left(\frac{\bar{m}_{1q_2}f_2}{D_2}\right)^{\frac{1}{2}} & \left(\frac{\bar{m}_{1q_3}f_3}{D_3}\right)^{\frac{1}{2}} \\
\left(\frac{(1 - \delta_q)f_{1q_1}}{D_1}\right)^{\frac{1}{2}} & \left(\frac{(1 - \delta_q)f_{2q_2}}{D_2}\right)^{\frac{1}{2}} & \left(\frac{(1 - \delta_q)f_{3q_3}}{D_3}\right)^{\frac{1}{2}} \\
\left(\frac{-\bar{m}_{1q_2}f_3}{D_1}\right)^{\frac{1}{2}} & \left(\frac{-\bar{m}_{2q_1}f_3}{D_2}\right)^{\frac{1}{2}} & \left(\frac{f_{1q_2}f_3}{D_3}\right)^{\frac{1}{2}} \\
\end{pmatrix},
\]

where

\[
f_1 = 1 - \bar{m}_1 - \delta_q, \quad f_2 = 1 + \bar{m}_2 - \delta_q, \quad f_3 = \delta_q
\]

\[
D_1 = (1 - \delta_q)(1 - \bar{m}_1)(\bar{m}_{2q_1} + \bar{m}_{1q_1}),
\]

\[
D_2 = (1 - \delta_q)(1 + \bar{m}_42)(\bar{m}_{2q_1} + \bar{m}_{1q_1}),
\]

and

\[
D_3 = (1 - \delta_q)(1 + \bar{m}_{2q_1})(1 - \bar{m}_{1q_1}).
\]

From Eqs. (9) and (10), we derived closed, explicit expressions for all entries in the matrix \(V^{th}\) written in terms of four mass ratios \((\bar{m}_u, \bar{m}_c, \bar{m}_d, \bar{m}_s)\) and two free real parameters \(\Phi = \phi_u - \phi_d\) and \(Z^{1/2}\) [4]. The CP violating phase \(\Phi\) measures the phase mismatch in the \(S_L(2) \otimes S_R(2)\) symmetry breaking patterns in the \(u\) and \(d\)-sectors.

The computation of \(V^{th}\) is straightforward. Here, we will give, in explicit form, only those elements of \(V^{th}\) which will be of use later. From Eqs. (9)–(14), we obtain,

\[
V^{th}_{ud} = \left(\frac{\bar{m}_s(1 - \bar{m}_u - \delta_u)(\bar{m}_d(1 + \bar{m}_s - \delta_d))}{(1 - \delta_u)(1 - \bar{m}_u)(\bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_s)(\bar{m}_c + \bar{m}_d)}\right)^{\frac{1}{2}} \cdot \left(\frac{\bar{m}_u\bar{m}_s}{(1 - \bar{m}_u)(\bar{m}_c + \bar{m}_u)(\bar{m}_d + \bar{m}_s)}\right)^{\frac{1}{2}} + \left(\frac{1 + \bar{m}_c - \delta_u}{1 + \bar{m}_s}\right)^{\frac{1}{2}} \cdot \left(\frac{1 - \bar{m}_u - \delta_u(1 + \bar{m}_s - \delta_d)}{(1 - \delta_u)(1 - \delta_d)(1 + \bar{m}_s)}\right)^{\frac{1}{2}} \cdot e^{i\Phi},
\]

\[
V^{th}_{ub} = \left(\frac{\bar{m}_u(1 - \bar{m}_u - \delta_u)(\bar{m}_d(1 + \bar{m}_s - \delta_d))}{(1 - \delta_u)(1 - \bar{m}_u)(\bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_s)(\bar{m}_d + \bar{m}_s)}\right)^{\frac{1}{2}} + \left(\frac{\bar{m}_u(1 - \bar{m}_u - \delta_u)}{(1 - \delta_u)(1 - \bar{m}_u)(\bar{m}_c + \bar{m}_u)(1 + \bar{m}_s)(1 - \bar{m}_s)}\right)^{\frac{1}{2}} \cdot e^{i\Phi},
\]

\[
V^{th}_{cs} = \left(\frac{\bar{m}_c(1 - \bar{m}_u - \delta_u)(\bar{m}_d(1 + \bar{m}_s - \delta_d))}{(1 - \delta_u)(1 + \bar{m}_c)(\bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_s)(\bar{m}_d + \bar{m}_s)}\right)^{\frac{1}{2}} + \left(\frac{\bar{m}_c(1 - \bar{m}_u - \delta_u)}{(1 - \delta_u)(1 + \bar{m}_c)(\bar{m}_c + \bar{m}_u)(1 + \bar{m}_s)(1 - \bar{m}_s)}\right)^{\frac{1}{2}} \cdot e^{i\Phi},
\]

The best symmetry breaking pattern

In order to find the actual pattern of $S_L(3) \otimes S_R(3)$ symmetry breaking realized in nature, we made a $\chi^2$ fit of the exact expressions for the absolute values of the entries in the mixing matrix $|V_{ij}^{th}|$ and the Jarlskog invariant $J$ to the theoretically determined values of $|V_{ij}^{exp}|$ and $J^{exp}$. We took the values of the running quark masses evaluated at the scale of $m_t$ from H. Fritzsch [27], and Fusaoka and Koide [28], we left the mass ratios $\bar{m}_c, \bar{m}_d$ and $\bar{m}_s$ fixed at their central values $\bar{m}_c = 0.0044$, $\bar{m}_d = 0.0015$ and $\bar{m}_s = 0.034$, but we took the value of $\bar{m}_u = 0.00032$ close to its upper bound. We found that the $S_L(3) \otimes S_R(3)$ flavour symmetry is broken down to $S_L(2) \otimes S_R(2)$ according to a mixed symmetry breaking pattern characterized by $Z^{1/2} = 1/2(Z_S^{1/2} - Z_A^{1/2}) = 81^{32}/32$. A detailed account of the computation may be found in Mondragón and Rodríguez-Jáuregui [4]. Therefore, the theoretical expressions for the entries in the mixing matrix $V^{th}$ are functions of the four mass ratios $(\bar{m}_u, \bar{m}_c, \bar{m}_d, \bar{m}_s)$ with $Z^{1/2} = 81^{32}/32$ and the CP violating phase $\Phi^* = 90^\circ$. The quark mixing matrix $V^{th}$ computed from the theoretical expressions is

$$V^{th} = \begin{pmatrix} \frac{\bar{m}_u (1 + \bar{m}_c - \delta_u)}{(1 - \delta_u)(1 + \bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_d)} & -\frac{\bar{m}_d \bar{m}_s \delta_d}{(1 - \delta_u)(1 + \bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_d)} & \frac{\bar{m}_c (1 + \bar{m}_c - \delta_u)}{(1 - \delta_u)(1 + \bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_d)} \\ \frac{\bar{m}_c (1 + \bar{m}_c - \delta_u)}{(1 - \delta_u)(1 + \bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_d)} & \frac{\bar{m}_d \bar{m}_s \delta_d}{(1 - \delta_u)(1 + \bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_d)} & \frac{\bar{m}_u (1 + \bar{m}_c - \delta_u)}{(1 - \delta_u)(1 + \bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_d)} \end{pmatrix} \exp(\Phi).$$ (18)

3.1. The best symmetry breaking pattern

We look the values of the running quark masses evaluated at the scale of $m_t$ from H. Fritzsch [27], and Fusaoka and Koide [28], we left the mass ratios $\bar{m}_c, \bar{m}_d$ and $\bar{m}_s$ fixed at their central values $\bar{m}_c = 0.0044$, $\bar{m}_d = 0.0015$ and $\bar{m}_s = 0.034$, but we took the value of $\bar{m}_u = 0.00032$ close to its upper bound. We found that the $S_L(3) \otimes S_R(3)$ flavour symmetry is broken down to $S_L(2) \otimes S_R(2)$ according to a mixed symmetry breaking pattern characterized by $Z^{1/2} = 1/2(Z_S^{1/2} - Z_A^{1/2}) = 81^{32}/32$. A detailed account of the computation may be found in Mondragón and Rodríguez-Jáuregui [4]. Therefore, the theoretical expressions for the entries in the mixing matrix $V^{th}$ are functions of the four mass ratios $(\bar{m}_u, \bar{m}_c, \bar{m}_d, \bar{m}_s)$ with $Z^{1/2} = 81^{32}/32$ and the CP violating phase $\Phi^* = 90^\circ$. The quark mixing matrix $V^{th}$ computed from the theoretical expressions is

$$V^{th} = \begin{pmatrix} \frac{\bar{m}_u (1 + \bar{m}_c - \delta_u)}{(1 - \delta_u)(1 + \bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_d)} & -\frac{\bar{m}_d \bar{m}_s \delta_d}{(1 - \delta_u)(1 + \bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_d)} & \frac{\bar{m}_c (1 + \bar{m}_c - \delta_u)}{(1 - \delta_u)(1 + \bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_d)} \\ \frac{\bar{m}_c (1 + \bar{m}_c - \delta_u)}{(1 - \delta_u)(1 + \bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_d)} & \frac{\bar{m}_d \bar{m}_s \delta_d}{(1 - \delta_u)(1 + \bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_d)} & \frac{\bar{m}_u (1 + \bar{m}_c - \delta_u)}{(1 - \delta_u)(1 + \bar{m}_c + \bar{m}_u)(1 - \delta_d)(1 + \bar{m}_d)} \end{pmatrix} \exp(\Phi).$$ (18)

We cannot simply equate $V^{th}$ and $V^{PDG}$ because the arguments of corresponding matrix elements in the two parametrizations are not equal: $\text{arg}(V_{ij}^{th}) \neq \text{arg}(V_{ij}^{PDG})$. This difference is of no physical consequence, it reflects the freedom in choosing the unobservable phases of the quark fields in the mass representation.

In the mass basis, the quark charged currents take the form

$$J^\mu = \frac{g}{\sqrt{2}} q_i \gamma^\mu V_{ij}^{th} q^i_j.$$

A redefinition of the phases of the quark fields which leaves $J^\mu$ invariant, will change the argument of $V_{ij}^{th}$ but leave the moduli $|V_{ij}^{th}|$ invariant,

$$V_{ij}^{th} \to V_{ij}^{th} = e^{-i\chi_i^3} V_{ij}^{th} e^{i\chi_j^3}.$$

The phases $\chi_i^3$ and $\chi_j^3$ occurring in Eq. (24) will be determined from the requirement that corresponding entries in $V^{th}$ and $V^{PDG}$ be equal,

$$|V_{ij}^{th}| e^{i[w_i^3 - (\chi_i^3 - \chi_j^3)]} = |V_{ij}^{PDG}| e^{iw_i^3_{PDG}},$$

where $w_i^3_{PDG}$ are the Wolfenstein parameters from breaking the flavour permutational symmetry.
in this expression \( w_{ij}^{\text{th}} \) and \( w_{ij}^{\text{PDG}} \) are the arguments of \( V_{ij}^{\text{th}} \) and \( V_{ij}^{\text{PDG}} \) respectively. Since the moduli \( |V_{ij}^{\text{th}}| \) and \( |V_{ij}^{\text{PDG}}| \) are equal, the arguments of the entries in the two parametrizations are related by the set of nine equations

\[
\chi_i^u - \chi_j^d = w_{ij}^{\text{th}} - w_{ij}^{\text{PDG}}. \tag{26}
\]

The set of Eqs. (26) relates the differences of the unobservable quark field phases to the differences of the arguments of corresponding entries in \( V^{\text{th}} \) and \( V^{\text{PDG}} \). Using an elimination procedure for all possible combinations \((\chi_i^u - \chi_j^d) \) we derive a set of nine equations, only four of which are linearly independent. Since, in \( V^{\text{PDG}} \) there are five entries with non-vanishing arguments, namely, \( w_{i3}^{\text{PDG}} = -\delta_{13}, w_{21}^{\text{PDG}}, w_{23}^{\text{PDG}}, w_{31}^{\text{PDG}} \) and \( w_{32}^{\text{PDG}} \), we require still one more equation relating the arguments of the entries of the two-parametrizations. This is obtained from the phase relations between the determinants of the two matrices, \( V^{\text{th}} \) and \( V^{\text{PDG}} \). From Eqs. (24) and (25), it follows that

\[
\det V^{\text{th}} = \det [X_u^t V^{\text{PDG}} X_d], \tag{27}
\]

in this expression \( X_u \) and \( X_d \) are the diagonal unitary matrices of phases occurring in Eq. (24). The quark field phases themselves are determined only up to a common additive constant. Since the quark field phases are unobservable, without loss of generality, we may fix one of them, and solve for the others. In this way, if we set \( \chi_i^d = 0 \), we get the diagonal matrices of phases required to compute the phase transformed \( V^{\text{th}} \)

\[
X_u = \text{diag}[e^{iw_{11}^{\text{th}}}, e^{i(-w_{12}^{\text{th}}-w_{13}^{\text{th}}+2\Phi^*)}, e^{i(-w_{23}^{\text{th}}-w_{13}^{\text{th}}+2\Phi^*)}] \tag{28}
\]

and

\[
X_d = \text{diag}[1, e^{i(w_{12}^{\text{th}}-w_{13}^{\text{th}})}, e^{i(w_{13}^{\text{th}}-w_{23}^{\text{th}}-2\Phi^*)}]. \tag{29}
\]

Hence, with the help of Eqs. (28) and (29), we verify that

\[
X_u^t V^{\text{th}} X_d = V^{\text{PDG}}, \tag{30}
\]

is satisfied as an identity, provided that \( |V_{ij}^{\text{th}}| = |V_{ij}^{\text{PDG}}| \). Then, the CP-violating phase occurring in \( V^{\text{PDG}} \) is

\[
\delta_{13} = w_{11}^{\text{th}} + w_{13}^{\text{th}} + w_{13}^{\text{th}} + w_{23}^{\text{th}} + w_{32}^{\text{th}} - 2\Phi^*. \tag{31}
\]

From Eq. (30) and the equality of the moduli of corresponding entries in \( V^{\text{PDG}} \) and \( V^{\text{th}} \), the three mixing angles \( \theta_{13}, \theta_{23}, \theta_{12} \) and the CP-violating phase \( \delta_{13} \) which appear in the phenomenological parametrization \( V^{\text{PDG}} \) as free, independent parameters are expressed as functions of four quark mass ratios and only two flavour symmetry breaking parameters \( Z^{*1/2} \) and \( \Phi^* \) [5]. The numerical values of the mixing angles computed from our expressions are

\[
\sin \theta_{12}^* = 0.22, \quad \sin \theta_{23}^* = 0.040, \quad \sin \theta_{13}^* = 0.0034, \quad \sin \delta_{13}^* = 0.966. \tag{32}
\]

These values coincide almost exactly with the central values of the same mixing parameters determined from a fit to the experimental data [5]. The predicted value of the CP violating phase is

\[
\delta_{13}^* = 72.3^\circ. \tag{33}
\]

For the three inner angles \( \alpha, \beta \) and \( \gamma \) of the unitarity triangle, we get, \( \alpha = 83^\circ, \beta = 22^\circ \), and \( \gamma = 75^\circ \) in good agreement with current data on CP violation in the \( K^0 \to \bar{K}^0 \) mixing system [2,35] and oscillations in the \( B_s^0 \to \bar{B}_s^0 \) system [5,30,31,34].

5. The Wolfenstein parameters

The quark mixing matrix \( V \) was parametrized by Wolfenstein [3] in terms of the four parameters \( \lambda, A, \rho \) and \( \eta \). As originally proposed, it is an approximate parametrization in which the hierarchy of mixings is exhibited writing each element of the mixing matrix as an expansion in powers of the small parameter \( \lambda = |V_{ud}| \sim 0.22 \). The Wolfenstein parametrization has several nice features. In particular, in conjunction with the unitarity triangle, it offers a very transparent geometrical picture of the structure of the mixing matrix and, as will be shown below, it allows one to relate some features of the assumed flavour symmetry breaking pattern to the properties of the mixing matrix in a very direct, analytical way. However, one drawback of the Wolfenstein parametrization, as originally defined, is that the mixing matrix is not exactly unitary.

In order to give an exactly unitary representation of the mixing matrix which maintains the nice features of the Wolfenstein parametrization, we will follow Buras et al. [29], and define the parameters \( \lambda, A, \rho \) and \( \eta \) in terms of the mixing angles and the CP-violating phase of the standard parametrization \( V^{\text{PDG}} \) through the relations

\[
\lambda = \sin \theta_{13} \cos \theta_{13}, \tag{34}
\]

\[
A \lambda^2 = \sin \theta_{23} \cos \theta_{13}, \tag{35}
\]

and

\[
A \lambda^3 (\rho - i \eta) = \sin \theta_{12} e^{-i \delta_{13}}. \tag{36}
\]

From these defining equations, it follows that,

\[
\rho = \frac{\sin \theta_{12}}{\sin \theta_{12} \cos \theta_{12} \sin \theta_{23} \cos \delta_{13}} \tag{37}
\]

and

\[
\eta = \frac{\sin \theta_{12}}{\sin \theta_{12} \cos \theta_{12} \sin \theta_{23} \cos \delta_{13} \sin \delta_{13}} \tag{38}
\]

We observe that Eqs. (34)–(38) define a change of variables.

When we make the change of variables in \( V^{\text{PDG}} \), we obtain a new expression for this same matrix written in terms of \( \lambda, A, \rho \) and \( \eta \), which we will call \( V^W \). Although written in terms of different sets of parameters, when Eqs. (34)–(38) are satisfied, all corresponding entries in \( V^W \) and \( V^{\text{PDG}} \) are equal in modulus and phase and \( V^W \) satisfies unitarity exact-
The matrix elements of $V^W$ are

$$V^W_{ud} = \sqrt{1 - \lambda^2 - A^2\lambda^6(\rho^2 + \eta^2)},$$

$$V^W_{us} = \lambda,$$

$$V^W_{ub} = A\lambda^3(\rho - i\eta),$$

$$V^W_{cd} = -\lambda - \frac{A^2\lambda^5(\rho + i\eta)\sqrt{1 - A^2\lambda^6(\rho^2 + \eta^2)} - \lambda^2}{1 - A^2\lambda^6(\rho^2 + \eta^2)},$$

$$V^W_{cs} = \frac{\sqrt{1 - \lambda^2 - A^2\lambda^6(\rho^2 + \eta^2)(1 - A^2\lambda^4 - A^2\lambda^6(\rho^2 + \eta^2)) - A^2\lambda^6(\rho + i\eta)}}{1 - A^2\lambda^6(\rho^2 + \eta^2)},$$

$$V^W_{cb} = A\lambda^2,$$

$$V^W_{ed} = A\lambda^3(1 - \rho - i\eta)\sqrt{1 - A^2\lambda^4 - A^2\lambda^6(\rho^2 + \eta^2)^2} - 1 - A^2\lambda^6(\rho^2 + \eta^2),$$

$$V^W_{es} = -A\lambda^2\sqrt{1 - \lambda^2 - A^2\lambda^6(\rho^2 + \eta^2)} - 1 - A^2\lambda^6(\rho^2 + \eta^2),$$

$$V^W_{eb} = \sqrt{1 - A^2}\lambda^2 - A^2\lambda^6(\rho^2 + \eta^2).$$

We recover the original Wolfenstein expression for the mixing matrix if we expand all entries in powers of $\lambda$ and keep terms up to $\lambda^3$,

$$V^W = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \lambda & A\lambda^2(\rho - i\eta) \\
-\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\
A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix} + O(\lambda^4).$$

We may now turn to the question of relating the Wolfenstein parametrization to the theoretical mixing matrix $V^t$ derived from breaking the flavour permutational symmetry. In a previous paper [5] we found that the theoretical mixing matrix, $V^t$, and the standard parametrization $V^{PDG}$ advocated by the particle data group [1, 2] give an equally good representation of the experimentally determined moduli, $|V_{ij}^{\exp}|$, of the mixing matrix. More precisely, when the best set of values of the free parameters of each parametrization is used, the moduli of corresponding entries in $V^t$ and $V^{PDG}$ are numerically equal and give an equally good representation of the experimentally determined moduli of the mixing matrix $|V_{ij}^{\exp}|$. Hence, we are justified in writing

$$|V^t_{ij}| = |V^{PDG}_{ij}|,$$

even though $V^t$ has only two free, real, and independent adjustable parameters, while the number of adjustable parameters in $V^{PDG}$ is four. From this Equation and (22) we obtain the following relations,

$$\sin \theta_{12} = \frac{|V^t_{us}|}{\sqrt{1 - |V^t_{us}|^2}},$$

$$\sin \theta_{13} = \frac{|V^t_{ub}|}{|V^t_{ub}|},$$

$$\sin \theta_{23} = \frac{|V^t_{cb}|}{\sqrt{1 - |V^t_{ub}|^2}}.$$
\[
\eta = \tan \left( w_{u_d}^t + w_{u_s}^t - w_{u_b}^t + w_{c_b}^t - 2\Phi^* \right).
\]

Finally, exact, explicit expressions for the Wolfenstein parameters as functions of the four quark mass ratios and the two flavour symmetry breaking parameters are obtained if we substitute in Eqs. (55)-(58) the expressions for \( V_{us}^t, V_{cb}^t, V_{ub}^t \), and \( w_{ij}^t \) derived in our previous calculations [4,5].

When we substitute the expression (15) for \( |V_{us}| \) in Eq. (55), we find for \( \lambda \) the following expression

\[
\lambda = D_{us}^{1/2} \left( A_{us} + B_{us} + 2A_{us}^{1/2}B_{us}^{1/2} \cos \Phi \right)^{1/2},
\]

where

\[
A_{us} = \bar{m}_u \bar{m}_d \bar{m}_c \bar{m}_s (1 - \bar{m}_u - \bar{d}_u) (1 + \bar{m}_s - \bar{d}_d),
\]

\[
B_{us} = \bar{m}_u \bar{m}_s (1 - \bar{d}_u) (1 - \bar{d}_d) (1 + \bar{m}_s - \bar{d}_d) + \bar{d}_d \bar{m}_s (1 + \bar{m}_s - \bar{d}_u) (1 - \bar{m}_u - \bar{d}_d),
\]

and

\[
D_{us} = (1 - \bar{d}_u) (1 + \bar{m}_s - \bar{d}_d) (1 + \bar{m}_s - \bar{d}_d) (1 + \bar{m}_s - \bar{d}_d).
\]

The dependence of \( \lambda \) on \( \Phi \) is explicitly exhibited in Eq. (59). If we recall that the best value of \( \Phi \) is \( \Phi^* = \pi/2 \), Eq. (59) simplifies to

\[
\lambda^* = \frac{(A_{us}^* + B_{us}^*)^{1/2}}{D_{us}^{1/2}}.
\]

The notation \( A_{us}^*, B_{us}^* \) and \( D_{us}^* \) means that \( \delta_u^* \) and \( \delta_d^* \) are evaluated at the best values of \( \Phi \) and \( Z \), \( Z^{1/2} = \sqrt{81/32} \), and \( \Phi^* = \pi/2 \).

From Eqs. (59)-(63), we see that \( \lambda \) is also weakly dependent on \( Z \) through the dependence on \( Z \) of the small quantities \( \delta_u \) and \( \delta_d \).

An expression for the parameter \( A \) as function of the four quark mass ratios and the two symmetry breaking parameters is obtained in a similar way. From Eq. (56) and Eqs. (15) and (18) we get

\[
A = \frac{D_{us} \left[ A_{cb} + B_{cb} + 2A_{cb}^{1/2}B_{cb}^{1/2} \cos \Phi \right]^{1/2}}{D_{cb}^{1/2} A_{us} + B_{us} + 2A_{us}^{1/2}B_{us}^{1/2} \cos \Phi},
\]

where \( A_{us}, B_{us} \) and \( D_{us} \) are given in Eqs. (60)-(62) and

\[
A_{cb} = \left[ \bar{m}_c \bar{m}_b \bar{m}_c (1 - \bar{m}_c - \bar{d}_u) (\bar{m}_b - \bar{d}_d)^2 \right] Z,
\]

\[
B_{cb} = \bar{m}_c \left( (1 + \bar{m}_c - \bar{d}_u) (1 - \bar{d}_d) (1 + \bar{m}_d - \bar{d}_d) (1 + \bar{m}_s - \bar{d}_d) (\bar{m}_c - \bar{m}_d)^2 \right) W_d(Z),
\]

and

\[
D_{cb} = (\bar{m}_c + \bar{m}_b) (1 - \bar{d}_u) (1 - \bar{d}_d) (1 + \bar{m}_c) (1 + \bar{m}_s) (1 - \bar{m}_d).
\]

From expressions (65) and (66), we see that the terms \( A_{cb} \) and \( B_{cb} \) are proportional to \( Z \). This proportionality has its origin in the factors \( \delta_u \) and \( \delta_d \), occurring in the right hand side of Eqs. (65) and (66), which according to Eq. (5), are proportional to \( Z \).

The dependence of \( A \) on \( \Phi \) is explicitly exhibited in Eq. (64). When the parameters \( \Phi \) and \( Z \) are set equal to the best values \( \Phi^* = \pi/2 \) and \( Z^{1/2} = \sqrt{81/32} \), the expression (64) simplifies to give

\[
A^* = \frac{D_{us} \left[ A_{cb}^* + B_{cb}^* \right]^{1/2}}{D_{cb}^{1/2} A_{us}^* + B_{us}^*}.
\]

where the asterisk means that \( \delta_u \) and \( \delta_d \) are evaluated at the best value of \( Z \), \( Z^{1/2} = \sqrt{81/32} \).

In a similar way, we find that \( \rho^2 + \eta^2 \) is obtained from Eqs. (57) and (15), and Eqs. (16) and (18),

\[
\rho^2 + \eta^2 = \frac{D_{cb}D_{us}}{D_{ub}} \left[ A_{us} + B_{us} + 2A_{us}^{1/2}B_{us}^{1/2} \cos \Phi \right] \left[ A_{cb} + B_{cb} + 2A_{cb}^{1/2}B_{cb}^{1/2} \cos \Phi \right],
\]
where
\[ A_{ub} = \tilde{m}_d \tilde{m}_s \tilde{m}_c (1 - \tilde{m}_u - \delta_u) \left( \frac{\tilde{m}_s - \tilde{m}_d}{W_d(Z)} \right)^2 Z, \]
\[ B_{ub} = \left[ -\tilde{m}_u \left( 1 + \tilde{m}_c - \delta_u \right) \frac{(\tilde{m}_s - \tilde{m}_d)^2}{W_d(Z)} + \tilde{m}_u (1 - \delta_u) (1 - \delta_d) \right] Z, \]
and
\[ D_{ub} = (1 - \delta_u) (1 - \delta_d) (\tilde{m}_c + \tilde{m}_s) (1 - \delta_d). \]

The dependence of \( \rho^2 + \eta^2 \) on \( \cos \Phi \) is explicitly exhibited in Eq. (69). When \( \Phi \) is set equal to its best value \( \Phi^* = \pi/2 \) we obtain the somewhat simpler expression
\[ \rho^*^2 + \eta^*^2 = \frac{D_{ub} A_{ub} + B_{ub}}{D_{ub}^* A_{ub}^* + B_{ub}^*}, \]
where, as before, the asterisk means that \( \delta_u \) and \( \delta_c \) are evaluated at the best value of \( Z \). \( Z_{1/2} = \sqrt{81/32} \).

The parameter \( \rho^2 + \eta^2 \) depends only weakly on \( Z \), because the \( Z \) factor in the numerator of Eq. (69), coming from \( A_{ub} \) and \( B_{ub} \), cancels with the \( Z \) factor in the denominator of the same expression, coming from \( A_{cb} \) and \( B_{cb} \). The remaining dependence on \( Z \) comes from the factors \( (1 - \delta_d) \), but \( \delta_d(Z) \) is much smaller than one for all values of \( Z \).

Finally, the ratio \( \eta/\rho \) is obtained from Eqs. (58) and (53),
\[ \frac{\eta}{\rho} = \tan (w_{ub} + w_{us} - w_{ub} + w_{cb} + w_{ct} - 2\Phi). \]

The arguments \( w_{ij} \) may be computed from the expressions
\[ \tan w_{ij} = \frac{B_{ij} \sin \Phi}{A_{ij}^2 + B_{ij}^2 \cos \Phi}, \]
where \( A_{ij} \) and \( B_{ij} \) are given in Eqs. (60)-(62), (65)-(67), and (70)-(72).

When the symmetry breaking parameter \( Z_{1/2} \) and the CP-violating phase \( \Phi \) are set equal to their best values \( Z_{1/2} = \sqrt{81/32} \) and \( \Phi^* = \pi/2 \), Eq. (45) gives
\[ \tan w_{ij} = \left( \frac{B_{ij}^*}{A_{ij}^*} \right)^{1/2}. \]

In explicit form [5]
\[ w_{ud}^* = \tan^{-1}\left[ \sqrt{\frac{\tilde{m}_u \tilde{m}_d}{\tilde{m}_c \tilde{m}_s}} \left( \sqrt{(1 - \delta_u) (1 - \delta_d)} + \sqrt{\delta_u^* \delta_d^* \left( 1 + \tilde{m}_c - \delta_u^* \right) \left( 1 + \tilde{m}_d - \delta_d^* \right)} \right) \right], \]
\[ w_{us}^* = \pi - \tan^{-1}\left[ \sqrt{\frac{\tilde{m}_u \tilde{m}_s}{\tilde{m}_c \tilde{m}_d}} \left( \sqrt{(1 - \delta_u^*) (1 - \delta_d^*)} + \sqrt{\delta_u^* \delta_d^* \left( 1 + \tilde{m}_c - \delta_u^* \right) \left( 1 + \tilde{m}_d - \delta_d^* \right)} \right) \right], \]
\[ w_{ub}^* = \tan^{-1}\left[ \sqrt{\frac{\tilde{m}_u}{\tilde{m}_c \tilde{m}_d \tilde{m}_s}} \left( \sqrt{(1 - \delta_u) (1 - \delta_d^*)} - \sqrt{\delta_u^* \left( 1 + \tilde{m}_c - \delta_u^* \right) \left( 1 + \tilde{m}_d - \delta_d^* \right) \left( 1 + \tilde{m}_s - \delta_d^* \right)} \right) \right], \]
\[ w_{cb}^* = \pi - \tan^{-1}\left[ \sqrt{\frac{\tilde{m}_c}{\tilde{m}_u \tilde{m}_d \tilde{m}_s}} \left( \sqrt{(1 - \delta_u^*) (1 - \delta_d^*)} - \sqrt{\delta_u^* \left( 1 + \tilde{m}_u - \delta_u^* \right) \left( 1 + \tilde{m}_d - \delta_d^* \right) \left( 1 + \tilde{m}_s - \delta_d^* \right)} \right) \right], \]
and
\[ w_{tb}^* = \tan^{-1}\left[ \sqrt{\frac{1}{\tilde{m}_u \tilde{m}_c \tilde{m}_d \tilde{m}_s}} \left( \sqrt{(1 - \delta_u^*) (1 - \delta_d^*)} + \sqrt{\left( 1 + \tilde{m}_c - \delta_u^* \right) \left( 1 + \tilde{m}_d - \delta_d^* \right) \left( 1 + \tilde{m}_s - \delta_d^* \right) \left( 1 + \tilde{m}_d - \delta_d^* \right)} \right) \right]. \]

From Eq. (75) we see that when \( \Phi \) vanishes, all arguments \( w_{ij} \) vanish and the right hand side of Eq. (74) also vanish. Therefore, when all entries in \( V^{1P} \) are real, \( \eta \) vanishes and all entries in \( V^W \) are real. In fact, from Eqs. (74) and (75) it may be readily verified that \( \tan (w_{ud} + w_{us} - w_{ub} + w_{cb} + w_{ct} - 2\Phi) = \sin \Phi \).

Hence, for each fixed value of \( Z_{1/2} \), CP violation is maximal when \( \sin \Phi \) attains its maximum value, that is, \( \Phi^* = \pi/2 \). It is in this sense, that we may say that the best value of \( \Phi \), \( \Phi^* = \pi/2 \), extracted from a \( \chi^2 \) fit to the experimental data [4, 5] corresponds to maximum CP violation, see also Fritzsch [27].

6. Wolfenstein parameters and flavour symmetry breaking

The connection between the assumed flavour symmetry breaking pattern and the properties of the quark mixing matrix are clearly exhibited in the functional dependence of the Wolfenstein parameters on the symmetry breaking parameters \( Z \) and \( \Phi \). In order to simplify the discussion, we will compute \( \lambda \), \( A \), \( \rho \) and \( \eta \) in the leading order of magnitude in the quark mass ratios.

The parameter \( \lambda \), computed from Eqs. (59)-(62) in the leading order of magnitude is

\[
\lambda \approx \left[ \frac{m_u + m_d}{m_c} + 2 \frac{m_u m_d}{m_c m_s} \left( 1 + \frac{m_u}{m_c} \right) \left( 1 + \frac{m_d}{m_s} \right) \right]^{1/2} \cos \Phi .
\]  

In this approximation, \( \lambda \) is a function of the ratios of the light quark masses \( m_u/m_c, m_d/m_s \) and \( \cos \Phi \) but, it is independent of \( Z \). Since the coefficient of \( \cos \Phi \) in the right hand side of Eq. (53) is smaller than \( m_u/m_c + m_d/m_s \), \( \lambda \) is only weakly sensitive to changes in the CP-violating phase \( \Phi \). When \( \Phi \) is set equal to its best value \( \Phi^* = \pi/2 \), we obtain the corresponding best value of \( \lambda \),

\[
\lambda^* \approx 0.221 ,
\]  

while the exact numerical value of \( \lambda^* \), computed from Eqs. (56)-(59) with \( Z^{1/2} = \sqrt{81/32} \) and \( \Phi^* = \pi/2 \), is

\[
\lambda^* = 0.222 \pm 0.003 .
\]  

From Eqs. (85) and (86), it is apparent that the difference between the exact value of \( \lambda^* \) and the value of \( \lambda^* \) computed in the leading order of magnitude is not significant. Since the dependence of \( \lambda \) on \( Z \) and the heavy quark masses is wholly contained in this very small difference, it is evident that \( \lambda \) is almost completely insensitive to changes in the values of the symmetry breaking parameter \( Z \) and the values of the heavy quark masses \( m_b \) and \( m_t \).

The Wolfenstein parameter \( A \), computed from Eqs. (64)-(67) in the leading order of magnitude of the quark mass ratios is

\[
A \approx Z^{1/2} \left[ \frac{\bar{m}_u - \bar{m}_d}{m_d + m_u} \right] \left( \frac{\bar{m}_c - \bar{m}_u}{m_c + m_u} \right) \left( \frac{m_d}{m_c + m_s} \right) \left( \frac{m_u}{m_c + m_s} \right) \cos \Phi .
\]  

From this expression, the meaning of the Wolfenstein parameter \( A \) in terms of the flavour symmetry breaking pattern becomes evident. Since \( A \) is proportional to the symmetry breaking parameter \( Z^{1/2} \), a vanishing value of \( A \) would imply a vanishing value of \( Z^{1/2} \), which means that the singlet and doublet representations of \( S_L(3) \otimes S_R(3) \) would not be in a pure singlet representation while the light quarks in each sector, \( u, c \), and \( d, s \) would be in a pure doublet representation.

In that case, the mixing matrix \( V \) and the quark mass matrices \( M_{ij} \) would be real and block diagonal, as can be seen from Eqs. (1), (7) and (9)-(11),

\[
V^W = \begin{pmatrix}
\sqrt{1 - \lambda^2} & \lambda & 0 \\
-\lambda & \sqrt{1 - \lambda^2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  

and

\[
M_{ij}^{H} = m_{34} \begin{pmatrix}
0 & \bar{m}_{2q} \bar{m}_{1q} & 0 \\
\bar{m}_{2q} \bar{m}_{1q} & \bar{m}_{1q} - \bar{m}_{2q} & 0 \\
0 & 0 & 1
\end{pmatrix}_{ij}.
\]  

In this case, there would not be any CP-violating term in \( V^W \).

Coming back to Eq. (87), we see that, when \( Z^{1/2} \) does not vanish, \( A \) is only weakly dependent on \( \cos \Phi \). The best value of \( A \) is obtained setting \( Z^{1/2} = \sqrt{81/32} \) and \( \Phi^* = \pi/2 \), in this way, we get

\[
A^* \approx Z^{1/2} \left[ \frac{\bar{m}_u - \bar{m}_d}{m_d + m_u} \right] \left( \frac{\bar{m}_c - \bar{m}_u}{m_c + m_u} \right) \left( \frac{m_d}{m_c + m_s} \right) \left( \frac{m_u}{m_c + m_s} \right) \cos \Phi .
\]  

The numerical value of \( A^* \) computed in the leading order of magnitude from Eq. (90) is

\[
A^* \approx 0.87 ,
\]  

while the exact value of \( A^* \) computed from Eqs. (64)-(67) is

\[
A^* = 0.82 \pm 0.041 .
\]  

The difference between the leading order of magnitude and the exact values of \( A^* \) is less than 6%. The allowed range of values of \( A \) determined from experimental data as given by Y. Nir [32], is

\[
A^* = 0.0826 \pm 0.041 .
\]  

In terms of the flavour symmetry breaking pattern, the numerical value of \( A^* \), extracted from a fit to the experimental data, gives a direct measure of the amount of mixing of singlet and doublet representations of the flavour permutational group \( S_L(3) \otimes S_R(3) \).

The parameters \( \rho^2 \) and \( \eta^2 \) may be computed from expression (69) which may be rewritten as
\[ \rho^2 + \eta^2 = (\rho^* \eta^*) \left( 1 + \frac{2 \sqrt{A_{ub} B_{ub}} \cos \Phi}{A_{ub} + B_{ub}} \right), \]

\[ \left( 1 + \frac{2 \sqrt{A_{us} B_{us}} \cos \Phi}{A_{us} + B_{us}} \right) \left( 1 + \frac{2 \sqrt{A_{cb} B_{cb}} \cos \Phi}{A_{cb} + B_{cb}} \right), \]

where \( \rho^* \eta^* \) is the value of \( \rho^2 + \eta^2 \) obtained when the CP-violating phase \( \Phi \) is set equal to its best value \( \Phi^* = \pi/2 \).

Computing Eq. (94) in the leading order of magnitude in the quark mass ratios, we obtain

\[ \rho^2 + \eta^2 \approx (\rho^* \eta^*) \left( 1 + \frac{1}{2 \frac{m_d}{m_u}} \cos \Phi \right), \]

and

\[ \rho^* \eta^* \approx \frac{m_u m_s}{m_c m_d} \left( 1 + \frac{m_d}{m_u} \right) \left( 1 + \frac{m_u}{m_c} \right). \]

From Eqs. (95) and (96), we see that, in this approximation \( \rho^2 + \eta^2 \) is independent of the symmetry breaking parameter \( Z^{1/2} \). In this same approximation, \( \rho^2 + \eta^2 \) is a function of \( \cos \Phi \). The numerical value of the \( \cos \Phi \) dependent factor in the right hand side of Eq. (95) is

\[ \frac{1 + 0.157 \cos \Phi}{1 - 0.0013 \cos \Phi} \]

This expression shows that \( \rho^2 + \eta^2 \) is sensitive to small changes in \( \Phi \). When \( \Phi \) is set equal to its best value \( \Phi^* = \pi/2 \), this factor reduces to 1.

The numerical value of \( \rho^* \eta^* \) computed from Eq. (96) is

\[ \rho^* \eta^* \approx 0.1647, \]

while the exact value, computed from Eq. (73) is

\[ \rho^* \eta^* = 0.1641. \]

The difference between the exact value of \( \rho^* \eta^* \) and the value computed in the leading order of magnitude is not significant. From Eq. (96) we see that \( \rho^* \eta^* \) computed in the leading order of magnitude in the quark mass ratios, is a function of \( m_u/m_c \) and \( m_d/m_u \) but it is independent of the masses of the heavy quarks \( m_t \) and \( m_b \).

Finally, a simple but fairly accurate expression for the parameter \( \eta/\rho \) as function of the quark mass ratios and the two symmetry breaking parameters \( \Phi \) and \( Z^{1/2} \) may be obtained from Eq. (58), which may be written as

\[ \frac{\eta}{\rho} \approx \tan \left[ \frac{w_{us} + w_{cb} - w_{ub} - w_{cs}}{w_{ud} + w_{cs} + w_{tb} - 2 \Phi^*} \right]. \]
According to Eqs. (28)–(31), the term in parenthesis in the right hand side of Eq. (100) is equal to the phase of the element $V_{e3}^{PDG}$ of the standard parametrization,

$$
V_{e3}^{PDG} = w_{ud}^{th} + w_{cb}^{th} + w_{tb}^{th} - 2\Phi^*.
$$

(101)

Taking the numerical values of the quantities in the right hand side of Eq. (70) from our previous computation [5], see also Eq. (19), we find

$$
w_{e3}^{PDG} \leq 0.01^\circ,
$$

(102)

which shows that $w_{e3}^{PDG}$ is a very small number. Therefore, in an excellent approximation, we may write

$$
\frac{\eta}{\rho} = \tan \left( (w_{us}^{th} + w_{cb}^{th} - w_{ub}^{th} - w_{cs}^{th}) \right),
$$

(103)

which may be written as

$$
\frac{\eta}{\rho} = \frac{\tan w_{ij}^{th} \sin \Phi}{1 + \tan w_{ij}^{th} \cos \Phi},
$$

(104)

where $w_{ij}^{th}$ is the value of $\arg[V_{ij}^{th}]$ obtained when the CP-violating phase $\Phi$ of the theoretical mixing matrix is set equal to its best value $\Phi^* = \pi/2$. Hence, all terms in the numerator of the right hand side of Eq. (104) are proportional to $\sin \Phi$. Therefore, the vanishing of $\Phi$, i.e. the absence of a phase mismatch in the symmetry breaking patterns of the $u$ and $d$-type quark mass matrices implies the vanishing of $\eta$, that is, the absence of a CP-violating term in $V^{W}$.

Now, we may compute $\eta^*/\rho^*$ from Eqs. (74)–(76) and the explicit expressions for $w_{ij}^{th}$ given in Eqs. (76)–(81). Computing in the leading order of magnitude in the quark mass ratios, we get

$$
\frac{\eta^*}{\rho^*} \approx \sqrt{\frac{m_d m_c}{m_u m_u} \left( \frac{1}{m_c - m_u} - \frac{1}{m_d - m_d} \right) \left( \frac{1}{m_s - m_u} \right) \left( \frac{1}{m_s - m_d} \right)}.
$$

(105)

The numerical value of $\tan^{-1} (\eta^*/\rho^*)$ computed from Eq. (105) is

$$
\delta_{13}^\star = \tan^{-1} \left( \frac{\eta^*}{\rho^*} \right) \approx 69^\circ,
$$

(106)

while the exact numerical value of $\tan^{-1} (\eta^*/\rho^*)$, computed from Eqs. (74) and (76)–(78) with $Z^{1/2} = \sqrt{81/32}$ and $\Phi^* = \pi/2$, is

$$
\delta_{13}^\star = \tan^{-1} \left( \frac{\eta^*}{\rho^*} \right) = 72.3^\circ.
$$

(107)

From expressions (105) and (106), the ratio $\eta/\rho$ is not very sensitive to changes in the numerical value of the symmetry breaking parameter $Z^{1/2}$, but it is very sensitive to variations in the parameter $\Phi$. As explained above, the vanishing of $\Phi$ implies the vanishing of $\eta$. In other words, when there is no phase mismatch in the symmetry breaking patterns of the $u$ and $d$-type quark mass matrices, $\eta$ vanishes. The value $\sin \Phi^* = 1$, extracted from a $\chi^2$ fit to the experimental data, indicates that the phase mismatch of the symmetry breaking patterns of the $u$- and $d$-type mass matrices is maximum. The ratio $\eta^*/\rho^*$ computed in the leading order of magnitude in the quark mass ratios is independent of the symmetry breaking parameter $Z^{1/2}$. As a function of the quark masses, $\eta^*/\rho^*$ is very sensitive to the quark mass ratios $m_u/m_s$ and $m_u/m_c$, while the dependence of $\eta^*/\rho^*$ on the other quark mass ratios is very weak.

7. Conclusions

In this paper, we have given a short overview of some recent progress in the derivation of quark mass matrices and quark mixing matrices from breaking the flavour permutation symmetry. In particular, we derived exact analytical expressions relating the Wolfenstein parameters to the four quark mass ratios $m_u/m_s$, $m_c/m_t$, $m_d/m_b$, $m_s/m_b$ and the two symmetry breaking parameters $Z^{1/2}$ and $\Phi$. The strong hierarchy in the masses of the quark families, $m_{2q} \gg m_{1q} \gg m_{1q'}$ makes it possible to derive simple but very accurate approximate expressions for the Wolfenstein parameters that relate the properties of the mixing matrix to the characteristics of the assumed flavour symmetry breaking pattern. Computing in the leading order of magnitude in the quark mass ratios, we found the following results.

i) The parameter $\lambda$, as function of the quark masses, depends only on the two light quark mass ratios $m_u/m_s$ and $m_d/m_b$, and it is almost completely insensitive to the heavy quark masses $m_s$ and $m_t$. As function of the symmetry breaking parameters, $\lambda$ is sensitive to changes in the CP-violating phase $\Phi$ but it is almost...
completely insensitive to changes in the value of the symmetry breaking parameter $Z^{1/2}$. The phase $\Phi$ is a measure of the phase mismatch in the $S_L(2) \otimes S_R(2)$ symmetry breaking patterns of the $u$- and $d$-sectors. Therefore, the numerical value of $\lambda$ gives a clear, if not very precise, indication about the amount of mismatching which is responsible for the violation of CP. The most recent determination of $\lambda = 0.225 \pm 0.0019$ [35] is consistent with a best value of $\Phi^* = 90^\circ$ corresponding to maximum CP-violation.

ii) The parameter $A$ is proportional to the symmetry breaking parameter $Z^{1/2}$ but it is only weakly sensitive to changes in $\Phi$. A non-vanishing value of $A$ means that singlet and doublet representations of $S_L(3) \otimes S_R(3)$ are mixed, with the heaviest quark in each sector, $t$ and $b$, in the singlet representation and the two light quarks in each sector, $u, c$ and $d, s$, in a doublet representation of the flavour permutational group. As function of the quark masses, $A$ is a function of the four quark mass ratios, its magnitude is very sensitive to $m_d/m_s$ and $m_u/m_c$, that is, it is sensitive to the strong hierarchy in quark masses.

iii) Computed in the leading order of magnitude in the quark mass ratios, the parameter $\rho^2 + \eta^2$ is independent of the symmetry breaking parameter $Z^{1/2}$ but it is sensitive to changes in the phase mismatch parameter $\sin \Phi$. As a function of the quark masses, $\rho^2 + \eta^2$ is a function of the light quark mass ratios $m_u/m_c$ and $m_d/m_s$ only.

iv) The parameter $\eta/\rho$ is proportional to the phase mismatch parameter $\sin \Phi$, but it is independent of the symmetry breaking parameter $Z^{1/2}$. Therefore, $\eta/\rho$ gives a direct measure of the phase mismatch in the symmetry breaking patterns of $u$- and $d$-type quark mass matrices. The best value $\sin \Phi^* = 1$, extracted from a $\chi^2$ fit to the experimentally determined values of the moduli of the entries in the mixing matrix, $|V_{ij}^{\text{exp}}|$, the Jarlskog invariant $J^{\text{exp}}$ and the inner angles of the unitary triangle of the $K^0 - \bar{K}^0$ system, indicates that the phase mismatch of the symmetry breaking patterns of $u$- and $d$-type quark mass matrices is maximum.

8. Acknowledgements

This work was partially supported by DGAPA-UNAM under contract No. PAPIIT-IN125298 and by CONACYT (México) under contract 32238E.

---


34. M. Neubert, “Exploring the Weak Phase $\gamma$ in $B^{\pm} \to \pi K$ Decays,” hep-ph/9904321.

35. M. Ciuchini et al., LAL 00-77; ROME-1307/00; RM3-TH/00-16; hep-ph/0012308