Coexisting cluster configurations and their symmetries

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The coexistence of cluster configurations of atomic nuclei is discussed in the framework of the semi-microscopic algebraic cluster model. Different two-cluster configurations can be related to each other by multichannel dynamic symmetries, which connect their energy spectra and other observables. It is shown that the multichannel symmetry of binary fragmentations are related to the Talmi-Moshinsky-Smirnov transformations of an underlying three-cluster configuration.

Keywords: Nuclear Clustering; algebraic models

1. Introduction

An atomic nucleus may show evidence for different cluster configurations in different regions of the excitation energy, or even at the same energy. A well-known example is the coexistence of the \(^{12}\text{C}+^{12}\text{C}\) and \(^{20}\text{Ne}+^{4}\text{He}\) configurations in \(^{24}\text{Mg}\). The wave functions of these configurations are not orthogonal to each other, they may have considerable overlap. By applying harmonic oscillator functions this kind of overlaps have been studied extensively by several authors (see e.g. [1] and references therein).

In the semi-microscopic algebraic cluster model (SACM) [2] the clusterization of atomic nuclei is described in a fully algebraic way, i.e. not only the basis states are symmetric, but the interactions (and other physical quantities) as well are represented by operators with well-defined symmetry properties. Therefore, we can study in this framework the possible dynamic symmetries of the hamiltonians of the coexisting cluster configurations, too. The aim of this contribution is to discuss this new symmetry, called multichannel (or multi-configurational) dynamic symmetry (MUSY).

Based on some heuristic arguments a connection was established in previous works [3] between the energy-eigenvalues of different two-cluster-configurations of an atomic nucleus. Here we show that a symmetry which connects two binary fragmentations is related to the Talmi-Moshinsky-Smirnov transformations [4] of an underlying three-cluster configuration.

The structure of this paper is as follows. In Sec. 2 we recall the characteristic features of the dynamic symmetries both for simple and for composite systems. Sec. 3 gives a brief review of the SACM. In Sec. 4 we discuss the multichannel dynamic symmetry. Finally the possible measurable consequences of the MUSY is considered.
In the algebraic models the dynamic symmetries (in the general sense of the previous definition) appear as limiting cases, when the hamiltonian of the system contains the generators of the dynamical group in special combinations, as defined by the invariant operators of a chain of nested subgroups.

Well-known examples of algebraic models are the interacting boson model of the quadrupole nuclear collectivity with $U(6)$ group structure [6], and the vibron model of $U(4)$ group structure [7], which is an algebraic model of the dipole degrees of freedom, e.g. space vector. This latter model is applied in molecular, nuclear, and hadron spectroscopy as well, though in somewhat different manner, corresponding to the different physical nature of the problems [8]. In cluster studies it describes the relative motion of the clusters, therefore it will be presented more in detail in Sec. 3.

2.2. Composite systems

Let us consider a system of two components (1 and 2), each of them described by an algebraic model, and having (at least one) dynamic symmetry

$$G_1 \supset G'_1 \supset G''_1 \supset \ldots; \quad i = 1, 2. \quad (2)$$

If the particle number of the two components are conserved separately, then the algebraic model with group structure

$$G_1 \otimes G_2 \quad (3)$$

usually proves to be a successful approach. The subgroups of Eq. (3) define the relevant dynamic symmetries of the system.

Notable examples are the proton-neutron interacting boson model with $U_p(6) \otimes U_n(6)$ group structure and the interacting boson-fermion model with $U(6) \otimes U_1(m)$ group structure, where $m$ is the number of the single-particle states in the fermion sector [6], or the vibron model for triatomic molecules with $U_1(4) \otimes U_2(4)$ group structure, and vibron-electron model $[U(4) \otimes U_e(m)]$ for the description of the rotational-vibrational motion and electron excitations in a diatomic molecule [9].

The dynamic symmetries of this kind of models are straightforward extensions of those of the simpler systems.

Deeper symmetries, with essentially different nature arise from the embedding of the direct product group $G_1 \otimes G_2$ into a larger group:

$$G_0 \supset G_1 \otimes G_2. \quad (4)$$

Some generators of $G_0$ transform particles of type 1 into particles of type 2, or vice versa, thus giving rise to a much wider symmetry, having stronger restrictions on the physical operators, and providing larger multiplet structure. In this paper we refer to a composite symmetry in this sense. Famous examples are the dynamic supersymmetries, obtained by embedding the direct product group of the boson-fermion models into graded Lie groups [9–11].

3. Semi-microscopic algebraic cluster model

In the SACM the relative motion of the clusters is described by the vibron model, while their internal structure is treated in terms of the $SU(3)$ shell model. The model space is free from the Pauli-forbidden states (as well as from the spurious motion of the center of mass), i.e. it is constructed in a microscopic way. The physical operators, however, are derived phenomenologically, as expansion in terms of the group generators, and their coefficients are parameters which are fitted to the experimental data.

3.1. Relative motion: vibron model

The vibron model can be viewed as an algebraic description of a rotational vibrational motion in the familiar three-dimensional space. According to the algebraic theory this space is accounted for by the $U(4)$ algebra. (A simple example helps to understand this statement: the symmetry algebra of the harmonic oscillator in this space is $U(3)$, and in order to generate the spectrum one needs an additional degree of freedom.) The spectrum is generated in this treatment by a finite number $(N)$ of bosons, which interact with each other. Due to the physical nature of the problem it is convenient to classify the four bosons into a scalar and a vector bosons, i.e. they carry angular momenta $l = 0$ with positive parity ($\sigma$-bosons) and $l = 1$ with negative parity ($\pi$-bosons). The only non-zero commutation relations of the creation $(\sigma^+, \pi^+)$ and annihilation operators ($\sigma, \pi, \mu = -1, 0, +1$) are:

$$[\sigma, \sigma^+] = 1, \quad [\pi, \pi^+] = \delta_{\mu, \nu}. \quad (5)$$

The particle-number conserving bilinear products of the creation and annihilation operators:

$$B_{m}^{(l)}(l_1, l_2) = \langle b_{l_1}^* \times b_{l_2} \rangle_{m}^{(l)} = \sum_{m_1, m_2} \langle l_1 m_1 l_2 m_2 | m \rangle b_{l_1}^* b_{l_2 m_2}, \quad (6)$$

generate the $U(4)$ algebra. Here $b_{lm}$ stands for $\sigma^+$ or $\pi^+$, while $b_{lm}$ denotes a spherical tensor operator related to the annihilation operator $b_{lm}$, as:

$$b_{lm} = (-1)^{l-m} b_{l-m}. \quad (7)$$

The physical operators are expressed in terms of the elements of the algebra, which are coupled to spherical tensors in Eq. (6), e.g. the hamiltonian has to be an $O(3)$ scalar. The $E1$ and $E2$ transition operators in the lowest order are the dipole and the quadrupole operators, respectively:

$$D_m = B_{m}^{(1)}(1, 0) + B_{m}^{(1)}(0, 1) = [\pi^+ \times \sigma + \sigma^+ \times \pi]^1_m, \quad (8)$$

$$Q_m = \frac{\sqrt{3}}{2} B_{m}^{(2)}(1, 1) = \frac{\sqrt{3}}{2} [\pi^+ \times \pi]^2_m. \quad (9)$$

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Further operators of physical importance are the number of bosons, and the angular momentum: 

\[ n_\pi = B_0^{(0)}(0,0) = |\sigma^+ \times \bar{\sigma}|^{(0)}, \]

\[ n_\pi = \sqrt{3}B_0^{(1)}(1,1) = \sqrt{3}|\pi^+ \times \bar{\pi}|^{(0)}, \]

\[ L_m = \sqrt{2}B_m^{(1)}(1,1) = \sqrt{2}|\pi^+ \times \bar{\pi}|^{(1)}. \]

The model has two dynamical symmetries (limiting cases), providing us with analytical solutions of the eigenvalue problem, and with complete sets of basis states which can be used in the numerical calculations of the general case. The important one, from the viewpoint of cluster studies, is labeled by the algebra-chain:

\[ U(4) \supset U(3) \supset O(3) \]

\[ |N, \ n_\pi, \ L\rangle, \]

where \( n_\pi = N, N - 1, \ldots, 0 \) and \( L = n_\pi, n_\pi - 2, \ldots, 1 \) or 0. (Note, that some operators and their eigenvalues are denoted by the same letter). Because of the bosonic statistics the wave function has to be symmetric, i.e. only the totally symmetric \( U(4) \) representations are relevant. When considering only linear and quadratic operators the energy eigenvalue is

\[ E = \epsilon + \beta L(L + 1) + \gamma n_\pi + \delta n_\pi^2, \]

where \( \epsilon, \beta, \gamma \) and \( \delta \) are parameters to fit. This dynamical symmetry gives a description in terms of an anharmonic oscillator. Neglecting the anharmonic (i.e. the last) term in Eq. (14) and for \( \beta = 0 \) we arrive at the harmonic oscillator limit.

For simple two-cluster systems, in which both clusters have closed-shell structure (e.g. \(^{16}\)O+\(^4\)He, or \(^{40}\)Ca+\(^4\)He), the vibron model in its simple form, as presented here, is able to reproduce the excitation spectrum etc. with reasonable agreement [12, 13]. However, in the construction of the model space the internal degrees of freedom of the clusters have to be taken into account in order to appreciate the Pauli exclusion principle, as it is discussed in the next subsection.

### 3.2. Coupled degrees of freedom

The internal structure of the clusters is treated in terms of the Wigner-Elliott shell model, having a symmetry group of \( U^C_T(4) \otimes U_C(3) \), where \( U^C_T(4) \) is the spin-isospin group [14], while \( U_C(3) \) refers to the orbital symmetry [15], and \( C \) stands for cluster. The generators of \( U_C(3) \) are the bilinear products of the oscillator quantum creation and annihilation operators \( a^\dagger, \bar{a} \), with \( l = 1 \) angular momentum, similarly to the \( \pi_1 \) and \( \bar{\pi} \) operators of the vibron model.

\[ U_C(3) \otimes U_R(4) \supset \ U_C(3) \otimes U_C(3) \supset U_C(3) \otimes U_C(3) \supset U(3) \supset SU(3) \supset O(3) \supset O(2) \]

\[ ([n_{1C}, n_{2C}, n_{3C}], [N, 0, 0, 0], [n_\pi, 0, 0], [n_1, n_2, n_3], (\lambda, \mu), K, L, M) \]

Here we have indicated the corresponding labels of the irreducible representations as well.
Let us suppose that the dynamic symmetry of the above group-chain holds (to a reasonable approximation). One can ask the question if there is a relation between the energy spectra of the two configurations. This problem was investigated in the work [3], by applying the following argument. The \( U(3) \) basis states of the different clusterizations are not orthogonal to each other, they may have considerable overlap. This is a consequence of the antisymmetry of the wave function. In some cases the two wave functions can be practically (apart from a normalization factor) identical with each other, being a single-term linear combination of the same shell-model basis state [3]. Therefore, it is natural to require that their energies be the same, too. This requirement establishes a relation between the energy-eigenvalues of the two Hamiltonians, as follows. The energy eigenvalues corresponding to the dynamical symmetry [Eq. (18)] up to linear terms in \( n_x \) can be written as

\[
E = \epsilon + \gamma n_x + \beta L(L + 1) + \theta n_x L(L + 1) + F(\lambda, \mu, L),
\]

where \( F(\lambda, \mu, L) \) is a function of the indicated quantum numbers. The \( n_x \) quantum number depends on the fragmentation: \( n_{x_k}, k = 1, 2 \), while the \( \lambda, \mu \) and \( L \) quantum numbers do not. \( E \) and its parameters also carry the channel index \( k \). If, however, several states are common in the two configurations, then their energies are the same, according to our requirement. Let the relation of the relative motion quantum numbers be \( n_{x_1} = n_{x_2} + n_{x_0} \). Then

\[
\gamma_1 = \gamma_2 = \gamma, \quad \epsilon_1 = \epsilon_2 + \gamma n_{x_0},
\]

\[
\beta_1 = \beta_2 = \beta, \quad \theta_1 = \theta_2 + \theta n_{x_0},
\]

\[
F_1(\lambda, \mu, L) = F_2(\lambda, \mu, L)
\]

have to hold. They guarantee the identical energies for the common states of the two fragmentations, and give a unique relation between the cluster-cluster interactions of two different configurations.

4. Multichannel dynamic symmetry

We consider two different clusterizations of an atomic nucleus, both of them consisting of two clusters

\[
c : C_1 + C_2, \quad d : D_1 + D_2.
\]

Their relation can be established by considering some transformations of a three-cluster configuration, as follows.

Let us suppose that the relation of the mass-numbers of the clusters are

\[
A_{C_1} \geq A_{C_2}, \quad A_{D_1} \geq A_{D_2}, \quad A_{D_1} \geq A_{C_1}.
\]

what can be done without any loss of generality. (In the example of the \(^{28}\)Si nucleus, mentioned before, \( C_1 : ^{16}\)O, \( C_2 : ^{12}\)C, \( D_1 : ^{24}\)Mg, and \( D_2 : ^{4}\)He.) Let us consider the three-cluster configuration

\[
(C_1) + (CD) + (D_2), \quad (CD) = (C_2 - D_2) = (D_1 - C_1).
\]

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\[
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\]

what can be done without any loss of generality. (In the example: \(^{16}\)O+\(^{8}\)Be+\(^{4}\)He). Let us choose the following two sets of Jacobi coordinates (see Fig. 1)

\[
\begin{align*}
t_c &= r_{D_2} - r_{CD}, \\
s_c &= r_{C_1} - \frac{M_{D_2} r_{D_2} + M_{CD} r_{CD}}{M_{D_2} + M_{CD}}, \\
t_d &= r_{C_1} - r_{CD}, \\
s_d &= r_{D_2} - \frac{M_{C_1} r_{C_1} + M_{CD} r_{CD}}{M_{C_1} + M_{CD}},
\end{align*}
\]

where \( M \) is the mass and \( r \) is the space vector of the corresponding coordinate-set. Then, obviously, the clusterization \( C_1 + C_2 \) corresponds to the coordinate-set \( c \) with some restriction on \( t_c \), while clusterization \( D_1 + D_2 \) corresponds to the coordinate-set \( d \) with some restriction on \( t_d \).

The transformation from the clusterization \( C_1 + C_2 \) to that of \( D_1 + D_2 \) amounts up to a transformation between the two sets of Jacobi coordinates: \( t_c, s_c \) and \( t_d, s_d \). This kind of transformation has been studied extensively, in order to relate the harmonic oscillator wave functions of the two configurations [16,4], and it is known as the Talmi-Moshinsky-Smirnov transformation. For the special case discussed here it reads (in terms of the oscillator quantum creation vector operators) [4]

\[
\begin{pmatrix}
\pi^1(s_d) \\
\pi^1(t_d)
\end{pmatrix} =
\begin{pmatrix}
\cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\
\sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{pmatrix}
\begin{pmatrix}
\pi^1(s_c) \\
\pi^1(t_c)
\end{pmatrix},
\]

where

\[
\cos \left( \frac{\theta}{2} \right) = -\left[ \left( \frac{M_{D_2} M_{C_1}}{(M_{D_2} + M_{CD})(M_{C_1} + M_{CD})} \right) \right]^{1/2}.
\]

These transformations are known to have a \( U_q(2) \) group structure [16], where \( q \) refers to the quasispin group, which acts in the particle index space [17]. Since the weight operator of the quasispin group is given by the difference of the number operators

\[
S_0 = \frac{1}{2}(n_1 - n_2),
\]

the relation of the energy-eigenvalues for the two different two-cluster configurations, as shown by Eqs. (19) and (20),
seems to be justified. Work is in progress in order to fully explore the group-structure of the MUSY, and construct its physical operators.

5. Applications

The multichannel symmetry establishes a strict correlation between the observables of different clusterizations. In this sense it is a very restrictive symmetry, however, because of the same feature it has a very strong predictive power as well, e.g. the hamiltonian of one cluster configuration may completely determine the energy spectrum of another cluster configuration, without any free parameter (and without any ambiguity in the model space, due to its microscopic construction).

As for the experimental observation of cluster spectra of different configurations, they are limited by several physical and technical circumstances (like penetrability, level density, energy resolution etc.). Therefore, detailed comparison between the experimental data and model calculations can be expected only in the low-lying (ground-state) region, while in the highly excited energy domain, one can compare mainly the gross features (like e.g. level density of a given spin-parity), not so much the states one-by-one.

After the intuitive introduction of the multichannel symmetry, as discussed in Sec. 3.2, some applications of this kind have been carried out [3, 18]. In Ref. 3 e.g. the \(^{24}\text{Mg}+^{4}\text{He}\), and \(^{16}\text{O}+^{12}\text{C}\) spectra were described with the same hamiltonian in a wide energy range. Furthermore, the density of \(\alpha\)-particle (scattering or capture) resonances were predicted by this hamiltonian, without any parameter fitted in the relevant energy region, and the result turned out to be in good agreement with the experimental observation.

6. Summary and conclusions

In this contribution we have discussed a new symmetry, called multichannel dynamic symmetry (MUSY), which connects different fragmentation channels (or clusterizations) of an atomic nucleus.

The multichannel symmetries of binary fragmentations are shown to be related to the Talmi-Moshinsky-Smirnov transformations of an underlying three-cluster configuration.

The multichannel symmetry is very restrictive (in the sense of requiring very strong correlations between the different cluster configurations), and due to the same reason it seems to have a strong predictive power.

The first tests of the new symmetry seem to be promising, however, the available applications of the multichannel symmetry is extremely limited; much work remains to be done in order to check if this strongly restrictive symmetry is realised in nuclear spectra.

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