An algebraic model for three-cluster giant molecules

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After an introduction to the algebraic U(7) model for three bodies, we present a relation of a geometrical description of three-cluster molecules to the algebraic U(7) model. Stiffness parameters of oscillations between each of two clusters are calculated and translated to the model parameter values of the algebraic model. The model is applied to the trinuclear system $^{132}$Sn+$\alpha$+$^{116}$Pd which occurs in the ternary cold fission of $^{252}$Cf.

Keywords: Nuclear clusters; algebraic model

Después de una introducción del modelo algebraico de U(7) para tres cuerpos, presentamos la descripción de un modelo geométrico de una molécula nuclear de tres cúmulos con el modelo U(7). Se calculan los parámetros de rigidez entre pares de cúmulos y se les traduce a los valores del modelo algebraico. El modelo es aplicado al sistema $^{132}$Sn + $\alpha$ + $^{116}$Pd.

Descriptores: Cúmulos nucleares; modelo algebraico

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1. Introduction

The recent announced discovery of $^{10}$Be accompanied cold fission in the spontaneous fission of $^{252}$Cf [1] is leading to the astonishing result that long living trinuclear molecules may exist. The experiment was triggered to the electromagnetic transition $2^+_1 \rightarrow 0^+^+_1$ of $^{10}$Be. The transitions of the accompanying clusters where measured in coincidence with the one of $^{10}$Be and the level spectrum was reconstructed, leading to an identification of the system $^{96}$Sr + $^{10}$Be + $^{140}$Ba. As a surprise, no Doppler broadening of the lines was observed as one expected when the nuclei decay in flight. This finding can be explained, assuming that the system emits the $\gamma$ rays at rest. Taking into account the life time of the $2^+_1$ level in Be, which is 0.125 ps, the system has to live at least as long as this. Independently, if this experiment is confirmed, there are some arguments that nuclear molecules of this type might exist:

a) The cold fission of $^{252}$Cf into three clusters is observed with an $\alpha$-particle as the lightest nucleus [2].

b) If an $\alpha$ particle can be emitted, there is no reason to believe that larger clusters cannot be emitted too. As an example serves the observation of heavy cluster radioactivity [3], though larger clusters are produced with a much smaller probability.

c) $^{10}$Be consists of a core of two $\alpha$-particles with two loosely bound neutrons [4] (and references therein), the latter being ideal to provide binding.

d) In theoretical calculations of molecular potentials, minima are reproduced [5] (and references therein) for the system observed.

Most of these arguments are, however, circumstantial and cannot prove the existence of these types of molecules. More experiments have to be done on these lines.

In a series of articles [5] a geometrical model for three-cluster molecules was published. It is based on an earlier model for two-cluster molecules [6]. In order to obtain an analytical solution, several strong assumptions had to be made:

i) The system is in a linear configuration.

ii) The inclination angles of the nuclear symmetry axis to the axis which defines the linear orientation should be very small.

iii) The light cluster has to be sandwiched between the two heavy ones.

The intention of this contribution is to present an algebraic model, inspired by previous work on baryon structure [7], which might overcome these restrictions. The advantage of this model is the easiness in how one can describe complicated systems, which would require complex procedures in the geometrical model. Also it permits to discuss possible dynamical symmetry limits, allowing for an analytical description of the spectrum. This latter part will not be discussed here. The main idea of the algebraic model was already presented in [8]. However, the problem remained, how to obtain the model parameters starting from known stiffness parameters, describing the pairwise interaction between the
clusters. On one side we will have the picture of three masses connected via a spring (see Fig. 1) and on the other side there is the algebraic model. The mapping can not be one to one but it should reproduce at low energies similar frequencies of the vibrational modes. For the rotational part we will assume a simple expression. The model does not contain yet the vibrational contributions of the individual clusters, i.e., no $\beta$ and $\gamma$ vibrations. In other words, the clusters are treated without structure yet. Without any inner structure, a geometrical model can still be carried out, as shown in this paper. However, the introduction of structure to the clusters will be straightforward within the algebraic model and without the problems involved in the geometrical treatment. As one possibility, one can introduce the IBA-1 hamiltonian [9] for the individual clusters, using a dynamical symmetry limit for simplicity. As it turned out in Ref. [5], the dominant contribution to the energy of a linear configuration comes from the relative motion of the nuclei and not from the deformation of the clusters. The deformation plays a role in the vibrational modes of the individual clusters.

In what follows we will briefly present the U(7) model and the hamiltonian. In Sec. 3 the mapping of the geometrical picture to the algebraic model is given and the approximations discussed. In Sec. 4 we apply it to the cold ternary splitting $^{122}$Sn+$^{40}$Ca+$^{114}$Pd where calculations of the nuclear molecular potential indicates a possible minimum. In Sec. 5 conclusions are drawn.

The work presented here describes the current status of the research topic, which has still to be completed in near future.

2. The U(7) model

In Ref. 7 an algebraic model for three particles with identical mass was proposed, in the context of the orbital excitations of quarks. In [10] the model was extended to three particles with, in general, different masses, intended to be applied to atomic molecules. The U(7) model, whose name will become more transparent further below, can be applied to any kind of three particle systems, as the three-cluster molecule [1, 5], discussed in the introduction.

The number of internal degrees of freedom are six and for each relative coordinate we can introduce boson creation and annihilation operators, carrying negative parity. The basic concept of the U(7) model is to introduce a cutoff through the addition of a scalar boson of positive parity. With this the spherical components of the creation operators are given by

$$p_{\rho,m}^+ \quad p_{\lambda,m}^+ \quad s_1^+, \quad m = -1, 0, 1. \tag{1}$$

The total number of bosons $N = n_\rho + n_\lambda + n_s$ is conserved, which implies that the total number of $p$-bosons is restricted between zero and $N$.

Taking all possible double bilinear products of a creation with an annihilation operator, we obtain the algebra U(7) with its 49 generators. A convenient form of the generators in terms of tensors with definite angular momentum is given in Eq. (5) of Ref. 10. The classification and the structure of possible dynamical subgroups is given in Ref. 7. The most convenient basis, with respect to which the model Hamilton operator will be diagonalized, is the one given by Eq. (4.2) of Ref. 7. The basis states are given by

$$|N, (n_\rho, L_\rho), (n_\lambda, L_\lambda); LM_L\rangle, \tag{2}$$

with $n_\rho$ and $n_\lambda$ the number operator of the $\rho$- and $\lambda$- oscillation quanta, respectively. The $L_\rho$ and $L_\lambda$ are the angular momenta of the $\rho$ and $\lambda$ part, $L$ is the total angular momentum and $M_L$ its projection.

As the model hamiltonian we use the one given by Eq. (12) of Ref. 10 plus a rotational energy contribution. We will resume it here for completeness:

$$H = \alpha_0 L^2 + A P_1^1 P_1 + C P_2^1 P_2 + C P_3^1 P_3$$

$$+ D(P_1^1 P_2 + P_2^1 P_1) + E(P_1^3 P_3 + P_3^1 P_1)$$

$$+ F(P_2^1 P_3 + P_3^1 P_2). \tag{3}$$

The operators $P_1$ are defined via

$$P_1^1 = p_{\rho}^+ \cdot p_{\rho}^+ + p_{\lambda}^+ \cdot p_{\lambda}^+ - R_0 s_1^+ s_1^+, \quad P_2 = \sin^2 \beta_0 p_{\rho}^+ \cdot p_{\rho}^+ - \cos^2 \beta_0 p_{\lambda}^+ \cdot p_{\lambda}^+,$$

$$P_3^1 = \sin (2\beta_0) p_{\rho}^+ \cdot p_{\rho}^+ \cdot p_{\lambda}^+ \cdot p_{\lambda}^+ - \cos \gamma_0 (\sin^2 \beta_0 p_{\rho}^+ \cdot p_{\rho}^+ + \cos^2 \beta_0 p_{\lambda}^+ \cdot p_{\lambda}^+). \tag{4}$$

where the parameter $R_0 = \sqrt{\rho_0 + \lambda_0 \cdot \lambda_0}$ describes the extension of the system. The $\beta_0$ is defined by the relative size of $\rho$ and $\lambda$, i.e., $\lambda_0 = R_0 \cos \beta_0$ and $\rho_0 = R_0 \sin \beta_0$, and $\gamma_0$ gives the angle between the two vectors. The index zero indicates the equilibrium position of the system around which the oscillations take place. The parameters $R_0$, $\beta_0$, and $\gamma_0$ can be related to the definition of a coherent state, which at these values describes the approximate eigenstate of the system [10]. The coherent state is defined via

$$|NR_0, \beta_0 \gamma_0 \rangle = \frac{1}{\sqrt{N!}} (\hat{a}_1^\dagger)^N |0\rangle, \tag{5}$$

where

![Figure 1](https://example.com/figure1.png)
is called the *condensate boson*. For more details, see Ref. 10 and references therein.

In order to describe the motion around the equilibrium position, fluctuation bosons are introduced in Ref. 10, which are orthogonal to \( b_c \). These are \( b_c \) describing the breathing mode, \( b_c \), the butterfly mode, and \( b_c \) is the mode where the angle \( \gamma \) between the vectors \( \rho \) and \( \lambda \) is changing (shearing mode).

A Bogoliubov treatment is applied, where the \( b_c \) and \( b_c \) are substituted by their expectation value \( \sqrt{N} \) and only leading terms in \( N \) are taken into account. The hamiltonian obtained has the form

\[
H_B = \sum_{\alpha_1, \alpha_2} \epsilon_{\alpha_1, \alpha_2} b_{\alpha_1} b_{\alpha_2}
\]

with \( \alpha_1, \alpha_2 = u, v, w \). The frequencies are given by [10]

\[
\begin{align*}
\epsilon_u &= 4 ANR_0^2, \\
\epsilon_v &= BN_{R_0^2} \sin^2 \frac{2\beta_0}{1 + R_0^2}, \\
\epsilon_w &= C N_{R_0^2} \sin^2 \frac{(2\beta_0) \sin^2 \frac{\gamma_0}{1 + R_0^2}}, \\
\epsilon_{uv} &= 2DN_{R_0^2} \sin \frac{2\beta_0}{\sqrt{1 + R_0^2}}, \\
\epsilon_{uu} &= 2EN_{R_0^2} \sin \frac{\gamma_0}{\sqrt{1 + R_0^2}}, \\
\epsilon_{uvw} &= F N_{R_0^2} \sin^2 \frac{(2\beta_0) \sin \frac{\gamma_0}{1 + R_0^2}}.
\end{align*}
\]

These are the estimates of the lowest frequencies, which we will need later in order to relate them to those calculated in a nuclear interaction model.

### 3. Stiffness Parameters

In order to determine the parameters of the U(7) model, in a first step the stiffness of oscillations between each two of the three clusters is determined. For illustration, see Fig. 1 where the interaction between two clusters is presented via a spring. Afterwards we will go through several steps until we can compare the energies at low values to those of the hamiltonian of Eq. (7). Because the hamiltonian, in a geometrical picture, is not identical to the one of the U(7) model, we will require that the lowest frequencies are similar. The geometrical potential, we start with, is itself an approximation to the one calculated between the nuclei, i.e. a mapping which reproduces the lowest frequencies is more than sufficient. The mapping is necessary because in the algebraic model there is no direct relation to the stiffnesses which are calculated in a nuclear interaction model.

The stiffnesses are calculated using a double folding calculation. The details are given in [11] and here we only mention that the potential is obtained by evaluating the multidimensional integral

\[
V_{ij} = \int dr_i \int dr_j \rho_i(r_i)\rho_j(r_j)\psi(r_j - r_i + r_i),
\]

where \( \rho_i \) indicate the mass density distribution of the \( k \) cluster and \( \psi \) the two-body interaction between two nucleons \( i \) and \( j \). For that a M3Y interaction is chosen [12]. To the above potential a repulsive part is added as proposed in Ref. 13. The strength of the compression term is chosen in such a way that for total overlap of a pair of clusters a double normal density of the nuclear matter is obtained. This repulsive component which arises in the interaction potential is proportional to the volume overlap of the two densities and contributes to the integral of Eq. (9) only for \( |r_i - r_j| \leq R \leq |r_i + r_j| \).

Finally the potential is expanded in multipoles. The resulting potential can exhibit at some point a minimum. A quadratic expansion around this minimum is made, giving the stiffness \( C_{ij} \) of the potential between cluster \( i \) and \( j \).

The relevant degrees of freedom of the three particle system, excluding the center of mass motion, can be cast into Jacobi coordinates [10]

\[
\begin{align*}
\rho &= r_2 - r_1, \\
\lambda &= \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2},
\end{align*}
\]

where this definition deviates by some factors to the one given in [10]. It is not relevant for the further steps.

Doing the same for the coordinates at the equilibrium position, we can define \( \delta \rho = \rho - \rho_e \) and \( \delta \lambda = \lambda - \lambda_e \). This allows us to relate the spatial differences between two clusters to the difference vectors \( \delta \rho \) and \( \delta \lambda \). Further we have

\[
\begin{align*}
r_2 - r_1 &= \rho, \\
r_3 - r_1 &= \frac{m_2}{m_1 + m_2} \rho - \lambda, \\
r_2 - r_3 &= \frac{m_1}{m_1 + m_2} \rho + \lambda,
\end{align*}
\]

where \( m_k \) refer to the masses involved.

The classical hamiltonian, from which we start, and where the center of mass motion is excluded already, is given by

\[
H_{cl} = \frac{\mu_{12}}{2} \delta \rho^2 + \frac{\mu_{123}}{2} \delta \lambda^2 + \frac{C_\rho}{2} \delta \rho^2 + \frac{C_\lambda}{2} \delta \lambda^2 + C_\rho \lambda \delta \rho \cdot \delta \lambda,
\]

where the interaction between two clusters is given via a double folding potential.
with

\[ C_{\rho} = C_{21} + \frac{C_{23}m_1^2 + C_{31}m_2^2}{(m_1 + m_2)^2}, \]
\[ C_{\lambda} = C_{23} + C_{31}, \]
\[ C_{\rho\lambda} = \frac{C_{23}m_1 - C_{31}m_2}{m_1 + m_2}, \]

(13)

and \( \mu_{12} \) is the reduced mass between cluster 1 and 2, while \( \mu_{123} \) is the reduced mass between cluster 3 and the combined mass \( (m_1 + m_2) \) at the position of the center of mass of the first two clusters.

When the vector \( \lambda \) is defined along the molecular \( x \)-axis and the vector \( \rho \) in the molecular \( xy \)-plane (note that the molecular \( z \)-axis is defined perpendicular to the plane of the molecule) the potential acquires the form

\[ V_{cl} = \frac{C_{\rho}}{2} \delta \rho^2 + \frac{C_{\rho}}{2} \rho_0^2 \gamma^2 + C_{\rho\lambda} \delta \lambda^2 \]
\[ + C_{\rho\lambda} \left( \cos \gamma_0 \delta \rho \lambda - \rho_0 \sin \gamma_0 \delta \gamma \lambda \right), \]  

(14)

where \( \gamma_0 \) and \( \rho_0 \) are the equilibrium values of \( \gamma \) and \( \rho \) at the potential minimum and an expansion up to quadratic terms was made. We are interested in the pure vibrational part only. Excluding the rotational part and possible mixings the kinetic energy is given by

\[ \hat{T}_{\text{vib}} = \frac{\mu_{12}}{2} \delta \rho^2 + \frac{\mu_{12} \rho_0^2 \gamma^2}{2} \delta \gamma^2. \]  

(15)

The \( \rho \) and \( \lambda \) give the length of the corresponding vectors.

The classical Hamiltonian is quantized using the Pauli-Podolsky procedure [18]. The expression is given by

\[ T_{\text{vib}} = -\frac{\hbar^2}{2 \mu_{12}} \frac{\partial^2}{\partial \rho^2} - \frac{\hbar^2}{2 \mu_{12}} \frac{\partial^2}{\partial \lambda^2} \]
\[ - \frac{\hbar^2}{2} \left( \frac{1}{\mu_{12} \rho_0^2} + \frac{1}{\mu_{123} \lambda_0^2} \right) \frac{\partial^2}{\partial \gamma^2}. \]  

(16)

In order to estimate the vibrational energies we set, for a brief moment, \( C_{\rho\lambda} = 0 \). This is the case where the coupling terms of the potential between \( \lambda \), \( \rho \), and \( \gamma \) vanish. The vibrational part of the Hamiltonian can be solved analytically with the frequencies

\[ \varepsilon_\rho = \hbar \sqrt{\frac{C_{\rho}}{\mu_{12}}}, \]
\[ \varepsilon_\lambda = \hbar \sqrt{\frac{C_{\lambda}}{\mu_{123}}}, \]
\[ \varepsilon_\gamma = \hbar \sqrt{C_{\rho\lambda} \left( \frac{1}{\mu_{12} \rho_0^2} + \frac{1}{\mu_{123} \lambda_0^2} \right)}. \]  

(17)

They refer to the fluctuations in \( \rho \), \( \lambda \) and \( \gamma \) respectively.

For the rotational part we assume a simple form \( \alpha L^2 \).

More terms can be introduced, like \( L^2_\rho \) or \( L^2_\lambda \) (see Ref. 10). As for the value \( \alpha \) we take \( 1/2 \Theta_0 \), with \( \Theta_0 = (\mu_{12} \rho_0^2 + \mu_{123} \lambda_0^2) \) which is the largest moment of inertia possible. For the rotational kinetic energy we finally have

\[ T_{\text{rot}} = \frac{\hbar^2}{2} \left( \frac{\mu_{12} \rho_0^2 + \mu_{123} \lambda_0^2}{2} \right) L^2, \]  

(18)

which is still too simple because in general mixing terms like \( L_k \), \( L_{k_0} \) occur. These terms have to be simulated by some combination of operators of the U(7) model, which has not been done yet.

A problem is still related to the interaction part. If we substitute the fluctuation coordinates \( \delta \rho \), \( \delta \lambda \) and \( \delta \gamma \) in terms of their boson creation and annihilation operators, terms of the type \( p^\dagger p^\dagger \) appear, not conserving the number of bosons. One possible proposal is to substitute it by

\[ \delta \rho \delta \lambda \rightarrow \frac{1}{2} (\delta \rho \delta \lambda + \pi_{\rho \pi_{\rho}}), \]
\[ \delta \gamma \delta \lambda \rightarrow \frac{1}{2} (\delta \gamma \delta \lambda + \pi_{\gamma \pi_{\gamma}}), \]  

(19)

where \( \pi_{\rho}, \pi_{\gamma} \) and \( \pi_{\pi_{\gamma}} \) are the conjugate momenta to the corresponding coordinates. Here we really change the original Hamiltonian. The change is justified by three arguments:

1) The frequencies in Eq. (17) are of the order of several MeV, which implies that the mixing between the oscillations, given by the terms \( p^\dagger p^\dagger \), is probably small \((2\hbar \omega \alpha \text{ excitations})\).

2) The parameter in front of this interaction turns out to be small for the system discussed.

3) The Hamiltonian, we started with, is itself an approximation and changing it to the more advantageous form of the U(7) model might not change much. Nevertheless, in other situations the procedure might not be valid.

As the next step we relate the fluctuations in \( \rho \), \( \lambda \) and \( \gamma \) to the breathing (u), the butterfly (v) and the shearing mode (w). Inspired by [10] we take

\[ p^\dagger_{\rho} = \sin \beta \rho b^\dagger_\rho - \cos \beta \rho b^\dagger_\rho, \]
\[ p^\dagger_{\lambda} = \cos \beta \rho b^\dagger_\rho + \sin \beta \rho b^\dagger_\rho, \]
\[ p^\dagger_{\gamma} = b^\dagger_\omega. \]  

(20)

With this we find as the final vibrational Hamiltonian

\[ H_{\text{vib}} = \varepsilon_u b^\dagger_u b_u + \varepsilon_v b^\dagger_v b_v + \varepsilon_w b^\dagger_w b_w \]
\[ + \varepsilon_{uv}(b^\dagger_u b_v + b^\dagger_v b_u) + \varepsilon_{uw}(b^\dagger_u b_w + b^\dagger_w b_u) \]
\[ + \varepsilon_{vw}(b^\dagger_v b_w + b^\dagger_w b_v). \]  

(21)
The double folding potential [Eq. (9)] provides us the value which we chose to be the equilibrium position of our molecule. We obtained for the equilibrium positions a triangular molecule.

These values have to be compared to the ones given in Eq. (8).

### 4. Application to $^{132}$Sn+$\alpha+$116Pd

The double folding potential [Eq. (9)] provides us the values of the stiffness coefficients in the quadratic approximation as has been discussed elsewhere [5]. Since the present model copes with spherical clusters only the monopolar term is retained in the expansion of the potential. For the $\alpha$ particle we use a gaussian-like density with a width parameter $\beta = 0.69$ fm$^{-1}$. For the strength of the compression term $V_{\text{comp},0}$ we used for exemplification 2 choices:

i) Like in previous papers [5] we soughted to reproduce the nuclear compression modulus $K$ in accordance to the nuclear EOS (nuclear equation of state) [14] by varying $V_{\text{comp},0}$ separately for all the three pairs (see first line of Table I where all the values are in MeV).

ii) We used also a unique value of $V_{\text{comp},0} = 300$ MeV as have been done very recently in a calculation of $\alpha$-like nuclear molecules life-times [15](see second rows of Table I). This choice obviously leads to sensitively larger nuclear compressibilities, which according to experiments on giant monopole resonance should range between 180 and 240 MeV [16].

Along with the stiffness parameters we obtained also the location of the absolute minima of the total ternary potential [17] which we chose to be the equilibrium position of our triangular molecule.

In Fig. 2 the situation is illustrated, where at the position 1 there is the $^{132}$Sn, at 2 the $^{116}$Pd and at 3 the $\alpha$ particle. We obtained for the equilibrium positions $\rho_0$ and $\lambda_0$ the values 11 fm and 5.14 fm respectively. For the angle $\gamma_0$ we obtain $\approx 84^\circ$.

Taking as $N = 10$ and using Eqs. (8), (13) and (22), we obtain the parameters $A$ to $F$ and $a_\perp$ shown in Table II.

### Table I. Relevant parameters of the theory, including the values for the stiffness of the potential parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
<th>Value 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{\text{comp},12}$</td>
<td>187.5</td>
<td>228.7</td>
<td>225.5</td>
<td>214.9</td>
</tr>
<tr>
<td>$V_{\text{comp},23}$</td>
<td>300.0</td>
<td>300.0</td>
<td>300.0</td>
<td>383.0</td>
</tr>
<tr>
<td>$V_{\text{comp},13}$</td>
<td>327.0</td>
<td>324.0</td>
<td>324.0</td>
<td>324.0</td>
</tr>
<tr>
<td>$R_{12\text{min}}$</td>
<td>11.0</td>
<td>7.37</td>
<td>7.65</td>
<td>5.14</td>
</tr>
<tr>
<td>$R_{23\text{min}}$</td>
<td>11.1</td>
<td>7.53</td>
<td>7.82</td>
<td>5.34</td>
</tr>
<tr>
<td>$R_{13\text{min}}$</td>
<td>163.8</td>
<td>25.43</td>
<td>12.63</td>
<td>218.8</td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>97.3</td>
<td>17.77</td>
<td>10.29</td>
<td>84.0</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>228.7</td>
<td>7.37</td>
<td>7.65</td>
<td>173.8</td>
</tr>
<tr>
<td>$\gamma_0$</td>
<td>97.3</td>
<td>17.77</td>
<td>10.29</td>
<td>84.0</td>
</tr>
</tbody>
</table>

### Table II. Parameter values of the U(7) hamiltonian.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
<th>Value 4</th>
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</thead>
<tbody>
<tr>
<td>$A$ (MeV)</td>
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<td>1.4507</td>
<td>0.018401</td>
<td>—</td>
</tr>
<tr>
<td>$B$ (MeV)</td>
<td>71.57 MeV</td>
<td>71.57 MeV</td>
<td>71.57 MeV</td>
<td>—</td>
</tr>
<tr>
<td>$C$ (MeV)</td>
<td>17.20 MeV</td>
<td>17.20 MeV</td>
<td>17.20 MeV</td>
<td>—</td>
</tr>
<tr>
<td>$D$ (MeV)</td>
<td>228.7 MeV</td>
<td>228.7 MeV</td>
<td>228.7 MeV</td>
<td>—</td>
</tr>
</tbody>
</table>

### Figure 2. The geometry of the system $^{132}$Sn + $\alpha+$116Pd.

Setting the $C_{\rho\lambda}$ equal to 0 does change $A$, $B$, $C$, and $D$ only at the last digits shown, however $E$ and $F$ are 0. This situation is the same as in the geometric picture when we extracted the eigenfrequencies $\varepsilon_\rho$, $\varepsilon_\lambda$, and $\varepsilon_\perp$ for $C_{\rho\lambda} = 0$. The values are $\varepsilon_\rho = 8.386$ MeV, $\varepsilon_\lambda = 17.20$ MeV and $\varepsilon_\perp = 71.57$ MeV. The latter vibrational state is too high in order to be believed to exist. After the mapping, this division into $\varepsilon_\rho$ and $\varepsilon_\perp$, which are nearly separated, gets lost and there is a strong mixing between the breathing (u) and the butterfly (v) mode, which manifests itself in a large $\varepsilon_\perp$. In order to compare the numerical evaluation to the estimates of the frequencies as given in Eq. (17) and using the parameter values as given in Table I, we have to put $C_{\rho\lambda} = 0$ in order that the mixing between the $\rho$ and $\lambda$-modes does not alter the position of the vibrational states. We obtain as the lowest frequency for the (u) and (v) mode the values 9.1851 MeV and 17.7067 MeV, while for the (v) mode we have $\approx 73$ MeV. These values are sufficiently close to the ones given above.

When the mixing due to $C_{\rho\lambda}$ is taken into account, the spectrum changes by shifting the vibrational modes to lower energies. In Fig. 3 we present the spectrum obtained. According to this, the ground state band is severely squeezed as for the case of the linear molecule [5]. However, for the excited vibrational states there is no clear band structure apparent. This is the same finding as in Ref. 8.
5. Conclusions

In this contribution we presented a mapping from the geometrical picture of a three-cluster molecule, which is not aligned but in a triaxial configuration, due to the existence of an absolute minima in the three-cluster potential off the molecular axis. The stiffness parameters of the interaction potential between each of two clusters are mapped to the parameters of the hamiltonian of the U(7) model. The mapping is approximate and is done in such a way that the lowest vibrational frequencies are the same in both pictures.

As a result we applied the model to the system $^{132}$Sn+$\alpha$+$^{116}$Pd. The ground state band is severely compressed with a distance of two subsequent states of the order of keV. The vibrational modes are at several MeV, which indicates a strong separation between rotational and vibrational modes as in the atomic molecules. Though, the rotational states in the vibrational bands are farther apart as in the ground state band, the distance between them is of the order of 100 keV's, as can be seen in Fig. 3.

Due to an increasing Coriolis force in this triangular system, due to $1/N$ corrections, there is no apparent band structure visible for the vibrational bands. One way to see a direct evidence of a nuclear molecule is to look for transitions of vibrational states to states in the ground state band. The transition line, which has a value of several MeV, should split into several other lines with only a few keV apart.

In future, we will add the dynamics of nuclear vibrations to the model. As one possibility, one can use for each cluster an IBA-I hamiltonian [9]. This permits dynamical symmetries in order to approximate the nuclear structure of each cluster, as was done similarly in the geometric description of a linear configuration [5], where only spherical or prolateally deformed nuclei could be described. In case of an IBA-I hamiltonian, many more possibilities are allowed. The work is still in progress and the current presentation describes preliminary results.

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