Introduction

Problems in different areas can be modelled by a one dimensional conservation equation of the form

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = F(\rho),$$

where the source term $F(\rho)$ represents net growth and saturation processes. We shall assume that $F(\rho)$ has at least two equilibrium points, at $u = 1$ and $u = 0$, corresponding to a stable and an unstable equilibrium point. The flux $\vec{j}$ is given by Fick’s law

$$\vec{j} = -D(\rho) \vec{\nabla} \rho,$$

where the diffusion coefficient $D(\rho)$ may depend on the density or in simple cases be taken as a constant. In one dimensional this leads to the equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left( D(\rho) \frac{\partial \rho}{\partial x} \right) + F(\rho).$$

It has been established, for a wide class of such equations, that sufficiently localized initial conditions evolve asymptotically into a traveling monotonic wavefront $q(z = x - ct)$ joining two equilibrium states. This asymptotic travelling wave is the solution of the ordinary differential equation

$$(D(q)q_z)_z + cq_z + F(q) = 0,$$

with

$$q(-\infty) = 1, \quad q > 0, \quad q' < 0 \quad \text{in} \quad (\infty, \omega), \quad q(\omega) = 0.$$

In previous work we have shown that a variational principle can be formulated for the speed.

There is another class of problems which in a simplified setting reduce to a modified reaction diffusion equation, the propagation of a chemotactic species. Assume that $\rho$ represents the density of a species which diffuses, and, is chemotactic to a certain chemical of density $s$. In addition to the diffusion due to the gradient of the species itself, an additional flux proportional to the gradient in the chemical is present. This flux is directed towards the regions of higher chemical concentration, it is therefore proportional to $\vec{\nabla} s$. Moreover the flux will be proportional to the density of the species $\rho$.

The conservation Eq. (1) including the two fluxes is then

$$\rho_t + \left[ -D \rho_x + \rho \chi s_x \right]_x = f(\rho),$$

where $\chi$ represents the chemotactic sensitivity. In the simplest setting, the density of the chemical $s$ diminishes only due to its consumption by the chemotactic species. In this case

$$s_t = -k \rho,$$

where $G$ is an increasing function of $\rho$. It is readily seen that travelling wave solutions can be reduced to a single reaction diffusion equation. For the simplest case of constant diffusion rates this is

$$\rho_t = D \rho_{xx} - \frac{\chi k}{c} (\rho^2)_x + f(\rho).$$
Suitably scaled, travelling wave solutions of this equation reduce to the solution of the ordinary differential equation

\[ q_{zz} + (c - \mu q)q_z + f(q) = 0, \]

with

\[ \lim_{z \to -\infty} q = 1, \quad \lim_{z \to \infty} q = 0, \quad q_z < 0. \]

We show that the speed of the travelling fronts can be obtained from a variational principle, and that chemotaxis will have a significant effect on the speed of the travelling front above a certain critical value \( \mu_c \). The main result is

\[ c_* = \sup_g \left[ \frac{2 \int_0^1 \sqrt{f(-g')} du}{\int_0^1 g(u) du} + \mu \int_0^1 u g(u) du \right], \tag{1} \]

where \( g(u) \) is a positive monotonic decreasing function.

From here it will follow that

\[ 2\sqrt{f'(0)} \leq c_* \leq 2 \sup_u (1 + f'(u) + \mu u). \tag{2} \]

Several extensions are possible, density dependent diffusion coefficient, as well as a different dependence of the chemical depletion on the density of the species can be treated in a similar fashion. For a rigorous proof and further details we Ref. 4.

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