The Foldy-Wouthuysen transformation from a functional integral approach

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The Foldy-Wouthuysen transformation is obtained from the functional integral, Lagrangian, approach. As a toy model we used the QCD Lagrangian. This leads to a derivation of the effective Lagrangian for non-relativistic quantum chromodynamics and the heavy quarks effective field theory.

Keywords: Qcd, heavy quarks.

Se obtiene la transformación de Foldy-Wouthuysen usando el punto de vista de la integral funcional. Como modelo de juguete se usa el lagrangiano de QCD. Esto conduce a una derivación del lagrangiano efectivo para cromodinámica cuántica no relativista y para la teoría de campos efectiva de quarks pesados.

Descriptores: Qcd, quarks pesados.

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Textbooks discuss the Foldy-Wouthuysen transformation from the Hamiltonian point of view, as it was originally formulated [1]. A natural question to ask about is how this transformation looks like when the functional integral formalism is used. Here we present a derivation of such result using the QCD lagrangian.

In order to discuss our results let us consider the QCD Lagrangian

\[ \mathcal{L}_{QCD} = \mathcal{L}_G + \mathcal{L}_L + \mathcal{L}_H, \]

where \( \mathcal{L}_G \) is the lagrangian for the gluons fields and \( \mathcal{L}_{L,H} \) is the fermionic part for the light and heavy quarks respectively, i.e.,

\[ \mathcal{L}_{L,H} = \bar{\psi}_{L,H} \left[ i \mathcal{D} - m_{L,H} \right] \psi_{L,H}, \]

where \( \mathcal{D} = \partial + igA \).

In order to define the heavy quark mass one must neglect the hard gluons contributions while the soft gluons one, by definition, are contained in the heavy quark mass. This assumption is valid when the heavy quark mass is much heavier than the scale of QCD and, under these conditions, the heavy quarks can be considered as non-relativistic particles. As a consequence of this, one focus the attention to the heavy modes sector of the partition function

\[ Z_H[A] = \int D\bar{\psi}_H D\psi_H e^{iS_H}. \]

Heavy quarks interact with the light modes through the gluon field and this is weak if measured at the scale of the heavy fermion mass (we will omit the subscript H from now on). In this case the original bispinor \( \psi \) can be written in terms of a slowly varying bispinor \( \phi \) as

\[ \psi(x) = e^{-imt} \phi(x) \]

(Notice that this reparametrizations of the fields can be seen as changing the origin from where the energy \( E \) is measured, i.e., as defining \( E_0 = E - m \), where \( \phi \), in the leading approximation, carries no information about the heavy quark mass. The only contribution coming from the mass of the heavy quarks appears in the corrections in powers of \( 1/m \).

Using (4) one find that the heavy quark Lagrangian is

\[ \mathcal{L} = \bar{\phi}(i\mathcal{D} - m(1 - \gamma_0))\phi. \]

This can be written explicitly in terms of the large \( \varphi \) and small \( \chi \) components of \( \phi \) as

\[ \mathcal{L} = \varphi^\dagger iD_0 \varphi + \chi^\dagger [iD_0 + 2m] \chi + \varphi^\dagger i\sigma \cdot D\chi + \chi^\dagger i\sigma \cdot D\varphi, \]

where we have used the standard Dirac’s representation for gamma matrices. Thus, the partition function for the heavy quarks in terms of \( \varphi \), and \( \chi \) is

\[ Z_H[A] = \int D\varphi^\dagger D\varphi D\chi^\dagger D\chi e^{iS_H}. \]
In the case of the originally formulated Foldy-Wouthuysen transformation, we have the following situation: one wants to decouple the small and large spinor components at the level of the Dirac equation. That is, one wants the decoupling to occur after using the equations of motion.

From the functional integral point of view, the decoupling is made first and is straightforward: one just needs to perform the change of variables (with unit Jacobian)

\[
\varphi' = \varphi, \\
\varphi'^\dagger = \varphi^\dagger, \\
\chi' = \chi + \left[ i D_0 + 2m \right]^{-1} \sigma \cdot D \varphi, \\
\chi'^\dagger = \chi^\dagger + \varphi^\dagger i \sigma \cdot D \left[ i D_0 + 2m \right]^{-1}
\]  

in (3). Then, the new lagrangian reads (omitting the primes)

\[
\mathcal{L} = \varphi^\dagger \left[ i D_0 + \frac{1}{2m} D^2 + \frac{g}{2m} \sigma \cdot B \right] \varphi, \\
+ \chi^\dagger \left[ i D_0 + 2m \right] \chi.
\]  

This lagrangian describes the (non-local) dynamics of relativistic heavy quarks in terms of two components spinors.

We observe that, as in the case of the original Foldy-Wouthuysen transformation, the non-relativistic limit of (9) can be pursued in a systematic way; it is straightforward because one expands the operator \((i D_0 + 2m)^{-1}\) in powers of \(1/m\), i.e.

\[
\left( i D_0 + 2m \right)^{-1} = \frac{1}{2m} \left( 1 - \frac{i}{2m} D_0 + \frac{i^2}{4m^2} (D_0)^2 - \ldots \right).
\]  

Then, the effective lagrangian for the heavy quarks coming from (9) becomes

\[
\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \ldots
\]  

The first term in (11), after using \(\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma^k\), reads

\[
\mathcal{L}^{(0)} = \varphi^\dagger \left[ i D_0 + \frac{1}{2m} D^2 + \frac{g}{2m} \sigma \cdot B \right] \varphi,
\]  

which is just the lagrangian for a non-relativistic quark in a (chromo)magnetic field and includes the Pauli term.

Even if expanded in this way, the resulting Lagrangian is not written in the standard form, i.e., \(\mathcal{L} = p \dot{q} - H\), from where one can infer directly the way the Hamiltonian looks like. To that end must use the equations of motion for both \(\varphi^\dagger\) and \(\varphi\) when calculating those contributions higher that \(\mathcal{L}^{(0)}\).

The higher order corrections are more laborious to find but the calculation is straightforward. Thus, the order \(1/m^2\) correction is

\[
\mathcal{L}^{(1)} = \frac{g}{8m^2} \varphi^\dagger \left[ \left[ D, E \right] + i \sigma \cdot \left( D \times E - E \times D \right) \right] \varphi,
\]  

where the terms in the RHS are the well known non-abelian Darwin and spin-orbit ones, and in the commutator \([D, E]\) an inner product is understood.

The next higher order terms \((1/m^3)\) can be computed following the same procedure, thus one gets that \(\mathcal{L}^{(2)}\) is

\[
\mathcal{L}^{(2)} = \frac{1}{8m^3} \varphi^\dagger \left[ D^4 + g \left( D^2, \sigma \cdot B \right) + g^2 \left( B^2 - E^2 \right) \right. \\
\left. + i g^2 \sigma \cdot \left( B \times B - E \times E \right) \right] \varphi,
\]  

where the second line is a genuine QCD contribution.

All these terms correspond to those already found in the literature (see e.g. [2] and references therein).

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