Generalized chiral membrane dynamics

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We develop the dynamics of the chiral superconducting membranes (with null current) in an alternative geometrical approach. Besides of this, we show the equivalence of the resulting description with the one known Dirac-Nambu-Goto (DNG) case. Integrability for chiral string model is obtained using a proposed light-cone gauge. In a similar way, domain walls are integrated by means of a simple ansatz.

Keywords: Extended classical solutions, domain walls

1. Introduction

It is believed that cosmic strings are fundamental bridges in the understanding of the Universe formation since several cosmological phenomena can be described by means of the cosmic strings properties. Besides of these there are other kind of cosmic objects possessing different properties of those inherited to ordinary cosmic strings, for example: domain walls, hybrid structures like domain walls bounded by strings and so forth [1]. They can arise in several Grand Unified Theories whenever exist an appropriate symmetry breaking scheme. However, there is other class of cosmological objects that can emerge with the ability to carry some sort of charge. For instance, as was suggested by Witten [2] in the middle of the eighties, cosmic strings could behave like superconductors.

Since that time the vast research on superconducting strings has thrown a new variety of cosmic objects. The cosmology resulting of supersymmetric theories (SUSY) was also considered yielding other class of cosmic strings, namely chiral cosmic strings. These objects are the result of a symmetry breaking in SUSY where a $U(1)$ symmetry is broken with a Fayet-Iliopoulos $D$ term, turning out a sole fermion zero mode traveling in only one direction in the string core [3]. In other words, when the current along the superconducting string shows a light-like causal structure then we have a chiral string. Carter and Peter [4] have made an exhaustive study of this kind of cosmic strings and later it has been continued by other authors [5,6].

The purpose of this contribution is to extend in an alternative geometrical way the dynamical results for chiral strings reported in [4, 5] using a Kaluza-Klein (KK) reduction mechanism [7] and following closely the variational techniques developed in [8]. Bearing in mind the KK idea, and assuming our original background spacetime to be 4-dimensional, the generalization to higher dimensional objects (membranes) tracing out worldsheets is possible. From this assumption we found that the dynamics of the chiral membranes resemble to a 5-dimensional DNG case. We describe now our membrane with an extended embedding. This description has the advantage of treating the new membrane on the same footing as an ordinary DNG membrane. It is often the case that while an existing theory admits a number of equivalent descriptions, one of them suggests generalizations and simplicities more readily than the others.

The equations of motion are a generalization of those of DNG type, i.e., the motion of the chiral membranes looks like minimal surfaces in a KK space but subject to a particular condition. In addition, we have the current conservation on the membrane which emerges as the remaining equation of motion for the extra variable.

This note is organized as follows: In Sec. 2 we develop the essential mathematical features to describe the superconducting chiral membranes. This is done by exploiting the theory of deformations achieved in [8]. In Sec. 3, we present a simplified version for the dynamics of chiral extended objects, in contrast with other approaches. In Sec. 4 we specialize to the chiral string model. We found a new method of integrability for chiral string model using a light-cone gauge. In Sec. 5 we found integrability for a simple chiral domain wall model. Finally, we give some conclusions.
2. Geometry for chiral membranes

In this section we describe both the intrinsic and extrinsic geometry for chiral membranes, i.e., possessing null currents on the worldsheet \( \omega = \gamma^{ab} \phi_{a} \phi_{b} = 0 \), based in the Kaluza-Klein approach achieved in Ref. 7. The present development is close to the conceptual framework made in Ref. 8. To begin with, we consider a relativistic membrane of dimension \( d \), whose worldsheet \( \{ m, \Gamma_{ab} \} \) is an oriented timelike \( d+1 \)-dimensional manifold, embedded in a \( 5 \)-dimensional extended arbitrary fixed background spacetime \( \{ M, g_{\mu \nu} \} \), \( \mu = 0, 1, \ldots, 4 \). We shall describe the worldsheet by the extended embedding

\[
X^\mu = \left( X^\mu(\xi^a), \phi(\xi^a) \right),
\]

where \( \phi \) is a field living on the worldsheet \( m \); \( a, b = 0, 1, 2 \), and \( \xi^a \) are coordinates on the worldsheet. With the former embedding, we can make contact with the Kaluza-Klein description for the background space-time metric

\[
g_{\mu \nu} = \begin{pmatrix} g_{00} & 0 \\ 0 & g_{ij} \end{pmatrix},
\]

where \( g_{\mu \nu} \) is the metric of the original background spacetime, and \( g_{44} \) is a constant. The tangent basis for the worldsheet is defined by

\[
e_a := X^\mu_a, \quad X^\mu_a \partial_a = e^\mu_a \partial_a,
\]

where the prime denotes partial derivative with respect to the coordinates \( \xi^a \). The tangent vectors \( e^\mu_a \), associated with the embedding (1), can be written as

\[
e_a = \left( \phi_{a}, e_{a}^{\mu} \right),
\]

The metric induced on \( m \) is given by

\[
\Gamma_{ab} = g_{\alpha \beta} e^\alpha_a e^\beta_b = \gamma_{ab} + g_{44} \phi_{a} \phi_{b},
\]

where \( \gamma_{ab} = g_{\mu \nu} e^\mu_a e^\nu_b \) is the standard metric for the worldsheet without the field \( \phi \). The normal basis for the worldsheet is denoted by \( n^a \) which is intrinsically defined by

\[
g_{\mu \nu} n^\mu \gamma_{ab} n^\nu = \delta_{ab},
\]

where \( \delta_{ab} = g_{\mu \nu} e^\mu_a e^\nu_b \) is the metric of the orthonormal basis, which we label as \( n^\mu I \) as follows,

\[
n^\mu I = \begin{pmatrix} n^\mu i \\ 0 \end{pmatrix}, \quad n^\mu (4) = \sqrt{g_{44}} \begin{pmatrix} e^\mu_{a} \phi_{a} \\ -g_{44} \end{pmatrix},
\]

where we have assumed that \( n^\mu i \) satisfy \( g_{\mu \nu} n^\mu i n^\nu j = \delta_{ij} \), and \( i \) take the values \( i = 1, \ldots, 4 - d \). One of the more important quantities determining the extrinsic geometry is the extrinsic curvature \( K^I_{ab} \) along the normal basis is defined as

\[
K^I_{ab} = -n^\mu I D_a e^\mu_b,
\]

where \( D_a := e^\mu_a D_\mu \) is the gradient along the tangential basis, and \( D_\mu \) is the covariant derivative compatible with \( g_{\mu \nu} \). The last expression can be splitted as follows: i) For \( I = i \) we have, \( K^I_{ab} = -n^\mu \partial_a e^\mu_b \) which is the well known expression for the extrinsic curvature for the worldsheet of the membrane [8], and ii) for \( I = 4 \), \( K^{(4)}_{ab} = \sqrt{g_{44}} \nabla_a \phi \nabla_b \phi \), where \( \nabla_a \phi \) is the covariant derivative compatible with \( \gamma_{ab} \). The index \( (4) \) denotes the direction along the normal \( n^\mu \).

3. Chiral membrane dynamics

In this section we will show the equivalence between the chiral membrane dynamics and the DNG dynamics in an extended background spacetime plus a chirality condition. The starting point to discuss the dynamics of chiral membranes is the DNG like action which is invariant under reparametrizations of the worldsheet \( m \),

\[
S = -\mu_0 \int d^{d+1} \xi \sqrt{-\Gamma},
\]

where \( \Gamma \) is the determinant of the induced metric (2) from the spacetime by the embedding (1), and \( \mu_0 \) is a constant. The determinant is straightforward computed and given by \( \Gamma = \gamma (1 + g_{44} \omega) \), where \( \gamma \) is the determinant of the old induced metric on the worldsheet, \( \gamma_{ab} \). The action (7) turn into

\[
S = -\mu_0 \int d^{d+1} \xi \sqrt{-\Gamma} (1 + g_{44} \omega)^{1/2}.
\]

Observe that the resulting action from the DNG like action (7), is the one for superconducting strings involving the Nielsen model, where \( L(\omega) = \sqrt{1 + g_{44} \omega} \), [7]. In other words, the superconducting string theory with the Nielsen model is equivalent to DNG like action (7).

An important issue that deserve attention is that of the equations of motion which are already known [4, 5]. We want to restate here the geometrical framework introduced in Sec. 2 in the attainment of chiral membrane dynamics. Using similar variational techniques to that developed in Ref. 8 we can get the equations of motion from the action (7). It is worth to mention that this method is very graceful because reobund the geometrical nature of the worldsheet. The variation of the action (7) gives

\[
\delta S = -\mu_0 \int d^{d+1} \xi \frac{1}{2} \sqrt{-\Gamma} \Gamma^{ab} \delta \Gamma_{ab} \]

\[
= -\mu_0 \int d^{d+1} \xi \sqrt{-\Gamma} g_{ab} \delta K^I_{ab} = 0,
\]

where we have considered only normal deformations to the worldsheet, \( \delta K^I_{ab} \) are the deformation normal vector fields and \( g_{ab} \) is the inverse metric of \( \Gamma_{ab} \) given by \( \Gamma_{ab} = \gamma_{ab} - g_{44} \nabla_a \phi \nabla_b \phi \). We can immediately read the equations of motion

\[
\mu_0 \Gamma^{ab} K^I_{ab} = 0.
\]
It is worth noticing the similarity of these equations with those ones arising for minimal surfaces, namely, $\gamma^{ab} K^i_{ab} = K^i = 0$ [8]. In fact, in our description $\Gamma^{ab}$ plays the role of a metric. Let us now decode the several cases involved in Eq. (10). a) $I = i$. The equations of motion take the form,

$$\mu_0 \gamma^{ab} K^i_{ab} - \mu_0 g_{ab} \nabla^a \phi \nabla^b \phi K^i_{ab} = 0. \quad (11)$$

On other hand, in the generic superconducting membranes picture, the stress-energy-momentum tensor acquires the form $T_{ub} = \mathcal{L}(\omega) \gamma_{ab} - 2 (d \mathcal{L}/d\omega) \nabla_a \phi \nabla_b \phi$, where $\mathcal{L}(\omega)$ is a function of $\omega$, depending on the particular models [10]. When the current limit is taken into account, the quantities $\mathcal{L}(\omega)$ and $d \mathcal{L}/d\omega$, acquire constant values. If we define $g_{ab} := 2 (d \mathcal{L}/d\omega)|_{\omega=0}$ and $\mu_0 = \mathcal{L}(\omega)|_{\omega=0}$, we can identify Eq. (11) with the standard equations of motion, namely: $T^{ab} K^i_{ab} = 0$, [10]. b) $I = (4)$. In this case we have now directly,

$$\gamma^{ab} K^i_{ab} = 0 = \nabla_a \nabla^a \phi, \quad (12)$$

which is a wave equation for $\phi$, corresponding to a conserved current carrying onto the worldsheet for chiral currents.

4. Chiral string model

We specialize now to the case of chiral strings. We illustrate the chiral string model from a Lagrangian point of view.

The presence of the gauge symmetry in a field theory means that not all of the field components, $X^\bar{a}(\xi^a)$, are dynamical. In our case, the reparametrization invariance allow us to choose a gauge in which the dynamical equations are tractable. Since we have considered a DNG like action, (7), we have the freedom to choose an acceptable gauge condition. Several authors have reported solutions of the chiral string model using specific ans"atze [4–6]. In this contribution we get a solution for the mentioned model using a different gauge. For a review of those solutions using our description, see Ref. 11.

We consider the light-cone gauge over the spacetime coordinates adapted to our description. It is well known that the orthonormal gauge do not fully fix the gauge because there is residual reparametrization invariance. A favorite gauge choice that fix the gauge and allow us to solve the constraints is the light-cone gauge. This gauge was used by Hoppe in the search for explicit solutions for the classical equations of motion of relativistic membranes [12]. We shall use this orthonormal light-cone gauge in the search of integrability for the chiral string model.

To proceed further, we assume the original background metric to be flat, $g_{\mu\nu} = \eta_{\mu\nu}$, with signature $(-, +, +, +)$ and the embedding (1). For the KK spacetime we define light-cone coordinates, $X^+$ and $X^-$, as

$$X^+ = \frac{1}{\sqrt{2}} (X^0 + \sqrt{g_{44}} \phi), \quad (13)$$

$$X^- = \frac{1}{\sqrt{2}} (X^0 - \sqrt{g_{44}} \phi), \quad (14)$$

$$\vec{X} = (X^1, X^2, X^3). \quad (15)$$

The light-cone gauge points $\tau$ along $X^+$,

$$X^+ = X_0^+ + P^+ \tau, \quad (16)$$

where $X_0^+$ and $P^+$ are constants. The idea is solve for $X^-$ leaving the $X^i$ variables, where $i = 1, 2, 3$. In the orthonormal light-cone gauge [12],

$$\Gamma^{ab} = \begin{pmatrix} \Gamma_{\tau\tau} & 0 \\ 0 & \Gamma_{AB} \end{pmatrix}, \quad (17)$$

where $\Gamma^{ab}$ is given by (2) and $A, B = 1, \ldots, d$, besides of $\sqrt{-\Gamma} \Gamma_{\tau\tau} = -1$, we can simplify the equations of motion, $\partial_a (\sqrt{-\Gamma} \Gamma^{ab} X^b) = 0$ in the set of equations

$$\mathcal{D} X^B = 0, \quad (18)$$

$$2P^+ \hat{X}^- = \hat{X} \cdot \hat{X} + \hat{\Gamma}, \quad (19)$$

$$P^+ X^\bar{a} = \hat{X} \cdot \hat{X}^\bar{a}, \quad (20)$$

where we have defined $\hat{\Gamma} := \det(\Gamma_{AB}) = -\Gamma_{\tau\tau}$ and we have defined the differential operator

$$\mathcal{D} := -\partial^\tau + \partial_A (\hat{\Gamma} \Gamma^{AB} \partial_B). \quad (21)$$

Equation (18) represents the equations of motion in this gauge, (19) and (20) are the constraints relations for the system. Deriving with respect to $\tau$ the Eq. (19), we can rewrite it as $P^+ \mathcal{D} X^- = \hat{X} \cdot \mathcal{D} \hat{X}$, so if

$$\mathcal{D} \hat{X} = 0, \quad (22)$$

we get the condition $\mathcal{D} X^- = 0$, which we can observe from (18). Thus, we have reduced the problem to solve the set (19), (20) and (22). So far the results are general for minimal surfaces of arbitrary dimension. Now we specialize to the case of chiral strings. In order to get integrability for the chiral string model, besides of Eqs. (19), (20) and (22), the condition $\omega = 0$ must be considered. Using the stringy notation, the appropriate expressions are

$$\Gamma^{ab} = \begin{pmatrix} -\hat{X}^i \cdot \hat{X}^i & 0 \\ 0 & \hat{X}^i \cdot \hat{X}^i \end{pmatrix}, \quad (23)$$

with $\Gamma_{\tau \tau} = -\Gamma_{\sigma \sigma}$. The equations of motion (22) transform into the wave equation: $\mathcal{D} \hat{X} = (\partial^\tau + \partial^\sigma) \hat{X} = -\hat{X} + \hat{X}'' = 0$, whose general solution is given by

$$\hat{X} = \vec{a}(\tau + \sigma) + \vec{b}(\tau - \sigma). \quad (24)$$

The constraints (19) and (20) acquire the form

$$P^+ \hat{X}^- = |\vec{a}|^2 + |\vec{b}|^2, \quad (25)$$

$$P^+ X^\bar{a} = |\vec{a}|^2 - |\vec{b}|^2, \quad (26)$$

where we have used the notation $\vec{a}$ and $\vec{b}$ to denote derivatives with respect to their arguments. From Eqs. (13) and (14) as well as Eq. (2), we can separate the metric $\gamma_{ab}$.
\[
\gamma_{ab} = \begin{pmatrix} -\dot{X}' \cdot \dot{X}' - \frac{1}{2}(P^+ - \dot{X}^-)^2 & \frac{1}{2}(P^+ - \dot{X}^-)X^- \cr \frac{1}{2}(P^+ - \dot{X}^-)X^- & \dot{X}' \cdot \dot{X}' - \frac{1}{2}(X^-)^2 \end{pmatrix}.
\] (27)

Now, the chirality condition becomes

\[
\omega = \gamma^{ab} \partial_a \phi \partial_b \phi = \frac{1}{2\gamma_{44}} \left( (P^+ - \dot{X}')^2 - (X^-)^2 \right) (\dot{X}' \cdot \dot{X}') = 0,
\] (28)

from which is deduced that

\[
X^- = \pm \left( P^+ - \dot{X}'^2 \right). \tag{29}
\]

Plugging Eq. (29) in the constraint (20) we get the conditions that should satisfy \( \alpha \) and \( \beta \) in the chiral string solution, namely,

\[
\pm(P^+)^2 = \left( \alpha^2 + \beta^2 \right) \left[ \alpha - \beta \pm P^+ \left( \alpha + \beta \right) \right]. \tag{30}
\]

Thus, this equation suggest to consider some cases.

For instance, if we assume \( P^+ = 1 \), we get

\[
1 = 2 \left( \alpha^2 + \beta^2 \right) \cdot \alpha, \tag{31}
\]

which is different for the values for \( \alpha \) and \( \beta \) reported in Ref. 5.

It is worthy to mention that the integrability for the last cases was reached in absence of electromagnetic field coupled to superconducting strings. From the equations defining \( X^+ \), \( X^- \) and the constraints (25) and (26) we can find the value for \( \phi \). Again the solitons states are obtained when \( \beta^+ = 0 \) and in such case \( \alpha \) satisfy the relation \( \pm P^+ = \alpha^2 \cdot [\alpha (1 \pm P^+)] \).

5. Chiral domain wall model

In this section applying a similar mechanism to that for conformal gauge chiral string model we get integrability for a chiral domain wall model by means of a special ansatz. For such intention, with all the former ingredients, we study a worldsheet described by the embedding [1],

\[
X^\bar{\mu}(\tau, \xi^1, \xi^2) = \begin{pmatrix} \tau \\ \bar{X} \end{pmatrix}, \tag{32}
\]

where

\[
\bar{X} = \begin{pmatrix} \bar{X} \\ \phi \end{pmatrix}. \tag{33}
\]

Now we can choose a gauge similar to the conformal gauge for strings, as follows

\[
\Gamma_{ab} = \begin{pmatrix} \Gamma_\tau \tau & 0 \\ 0 & \Gamma_{AB} \end{pmatrix}, \tag{34}
\]

where \( \Gamma_{AB} = g_{\mu \nu} X^\mu A X^\nu B \) and \( \Gamma_\tau A = 0 \). We assume the special form for the embedding (32), as

\[
\bar{X} = \hat{n} \xi^2 + \bar{X}_\perp (\tau, \xi^1), \tag{35}
\]

with the conditions on \( \hat{n} \) to be a unit vector and perpendicular to \( \bar{X}_\perp \). So, in this conformal gauge we have the constraints

\[
\bar{X}_\perp \cdot \dot{X}_\perp = 0, \tag{36}
\]

\[
\dot{X}_\perp \cdot \hat{n} + \dot{X}_\perp \cdot \bar{X}_\perp = 1, \tag{37}
\]

and the condition \( \sqrt{-\Gamma} \Gamma^\tau \tau = -1 \). It is straightforward to demonstrate that \( \Gamma := \det(\Gamma_{AB}) = -\Gamma^\tau \tau \). In this part of the work, \( \cdot \) and \( ' \) denote derivatives with respect to \( \tau \) and \( \xi^1 \), respectively.

Assuming \( g_{\mu \nu} \) to be Minkowski's metric and resting in the constraints (36) and (37), the induced metric \( \Gamma_{ab} \) takes the form,

\[
\Gamma_{ab} = \begin{pmatrix} -|\bar{X}_\perp|^2 & 0 & 0 \\ 0 & |\bar{X}_\perp'|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{38}
\]

According to the standard DNG equations of motion, in our present case the corresponding ones are promoted as \( \dot{X}_\perp - \dot{X}_\perp'' = 0 \), whose solutions have the form

\[
\bar{X}_\perp = \frac{1}{2} \dot{a}(t + \xi_1) + \frac{1}{2} \frac{\dot{b}}{b}(t - \xi_1). \tag{39}
\]

Imposition of the chirality for superconducting domain walls, lead us to the relation

\[
|\bar{X}_\perp'|^2 (n^4)^2 = \phi^2 - \phi''^2, \tag{40}
\]

where \( n^4 \) is the four component of the vector \( \hat{n} \). Furthermore, the constraints (36) and (37) read as \( |\dot{\alpha}|^2 = 1 \) and \( |\dot{\beta}|^2 = 1 \), or explicitly they are given by

\[
|\dot{\alpha}|^2 + g_{44} \phi^2 = 1, \tag{41}
\]

\[
|\dot{\beta}|^2 + g_{44} \phi'^2 = 1, \tag{42}
\]

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where we have considered the notation \( \vec{a} = (a, \phi) \) and \( \vec{b} = (\vec{b}, \vec{\phi}) \), i.e.,

\[
\vec{X}_\perp = \left( \frac{1}{2} \vec{a}(t + \xi^1) + \frac{1}{2} \vec{b}(t - \xi^1) \right).
\]

(43)

Plugging (43) in the condition (40), the chirality condition is expressed now as

\[
\phi' \vec{\phi} \left[ 1 + g_{44} (n^4)^2 \right] = (n^4)^2 \left( 1 - \vec{a} \cdot \vec{b} \right).
\]

(44)

In a similar way as in the chiral string model case, the last equation suggests some cases. For example, the solution considering \( n^4 = 0 \) and \( \phi = 0 \) (or \( \vec{\phi} = 0 \)), correspond to a straight superconducting domain wall with a carrying current arbitrary cross section, but not including current along the \( \xi^2 \) direction.

6. Conclusions

In this work we have developed the dynamics of chiral membranes using geometrical techniques. Using our description we are able to reproduce the results of [4–6] in order to show consistency of our description. In fact, our scheme resembles DNG theory in five dimensions using a Kaluza-Klein approach. Integrability for both chiral string model and chiral domain wall model was obtained using a simple ansatz. Besides of this description, the Hamiltonian analysis for the chiral model is part of an extensive work [11]. The full physical description is not over yet because a deep understanding on integration of equations of motion has not been accomplished. The search for new solutions in the case of chiral superconducting membranes is part of a forthcoming paper.

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(a) The tangential deformations can be identified with the actions of worldsheet diffeomorphisms so we can ignore them since we are interested in quantities invariant under reparametrizations of the worldsheet. These tangential deformations are important in the study of composite objects [8, 9].

(b) As it is well known, in such case the worldsheet is parametrized by the coordinates \( \xi^0 = r \) and \( \xi^1 = \sigma \). The symbols \( \cdot \) and \( ' \) denote partial derivatives with respect to \( \xi^0 \) and \( \xi^1 \), respectively.