New solitonic structures in nonlinear model with interparticle anharmonic interaction

Maximo A. Aguero and Mauricio Paulin
Facultad de Ciencias, Universidad Autónoma del Estado de México,
Instituto Literario 100, 50000 Toluca, Edomex, Mexico

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The purpose of the present paper is to try to find new analytical structures and determine the velocity for each of them in a Klein-Gordon model with anharmonic nearest-neighbor interparticle interaction. These strong nonlinear analytic objects were obtained without disregarding nonlinear terms in the equation of motion and considering only topological invariances and new combinations of the main parameters.

Key words: Solitons, nonlinear dynamics

El propósito de este artículo es encontrar nuevas estructuras analíticas y determinar la velocidad de cada una en un modelo de Klein-Gordon con interacción anhármonica de primeros vecinos. Estos objetos analíticos fuertemente no lineales se obtuvieron sin despreciar términos no lineales en la ecuación de movimiento y considerando sólo invariancias topológicas y combinaciones nuevas de los principales parámetros.

Descriptors: Solitones, dinámica no lineal

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1. Introduction

The inclusion of anharmonicity in the study of lattice models can produce the variety of new features [1-3]. Rosenau and Hyman [4] found solutions of the solitary type with the absence of infinity tails termed "compactons". Recently, S. Dusuel, P. Michaux, and M. Remoissenet (DMR) in Ref. 5, the generalized \( \Phi \)-four or double well model with anharmonic interparticle interaction in the continuum limit was studied and various types of kink compactons were found. We try to explore the possibility of obtaining other types of analytic soliton-like structures for the nonlinear evolution Eq. (3) written below, by taking into consideration the physical possible boundary conditions and subsequently define the parameter domains of existence for these solutions.

2. Anharmonic interaction

The model proposed in Ref. 5 start with the Hamiltonian

\[
H = \sum_n \left[ \frac{1}{2} \frac{d\Phi_n}{dt} \right]^2 + V_0 \left( 1 - \Phi_n^2 \right) + U(\Phi_{n+1} - \Phi_n) \]

(1)

with the anharmonic interparticle interaction potential

\[
U(\Phi_{n+1} - \Phi_n) = \frac{C_1}{2} (\Phi_{n+1} - \Phi_n)^2 + \frac{C_{nl}}{4} (\Phi_{n+1} - \Phi_n)^4 \]

(2)

Here \( \Phi_n \) denotes the position of the \( n \)-th particle measured from the \( n \)-th lattice site, \( V_0, C_1 \) are constant parameters and \( C_{nl} \) is the parameter that controls the strength of the nonlinear coupling. In the continuum limit the equation of motion of the discrete system represented by (1) in 1 + 1 space-time manifold is

\[
\Phi_{tt} - C_1 \Phi_{xx} + 3C_{nl} \Phi_x^2 \Phi_{xx} - 2V_0 (\Phi - \Phi^3) = 0. \]

(3)

As usual the subindexes indicate partial derivatives with respect to time \( t \) and space \( x \) respectively. When the interactions \( U'(x) = dU/dx \) between particles are linear, one obtains kink solitons.

2.1. Condensate type of boundary condition.

A. New travelling wave solutions \( \Phi(\zeta) = \Phi(x - ut) \) with constant velocity \( u \) could be obtained by using the condensate boundary condition

\[
\Phi \to \pm 1, \Phi_x \to 0 \quad \zeta \to \pm \infty \]

(4)

in the equation of motion (3). As a result of this procedure for a specific parameter values determined by the following relation

\[
(u^2 - C_1)^2 + 6V_0 C_{nl} = 0, \]

(5)

with \( A = 2(u^2 - C_1)/(3C_{nl}) \), \( B = -2V_0/(3C_{nl}) \) and \( C_{nl} \neq 0 \), new solitonic structures are possible to find. By obvious reasons the signs of \( V_0 \) and \( C_{nl} \) have to be opposite each other. We aim to use this restriction to find solitonic structures represented by the set of inverse functions \( \Phi(\zeta)^{-1} \) that can be conveniently written as follows:

\[
\pm \sqrt{A}(\zeta_{qps} - \zeta_0) = \arcsin \left( \frac{p \Phi}{\sqrt{q}} \right) + \ln \left( q \sqrt{1 - \frac{\Phi^2}{2} + \frac{s \Phi}{\sqrt{2}}} \right), \]

(6)

where \( p, q, s = \pm, \) i.e. these symbols simply indicate the signs of each denoted term in the right hand side of the Eq. (6) and \( \zeta_0 \) is defined by chosen initial conditions. It is readily observed that for this case we have four pairs (possibilities) of defined boundary conditions according to the definition (4). Each pair produces two branches to construct the solutions.
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FIGURE 1. Typical kink-like structure without small perturbation near its center. This solution corresponds to the configuration (a) for the boundary condition Eq. (8).

FIGURE 2. Kink soliton with small hump near its center like a travelling well domain. This is the (b) configuration for the condition (8).

One of them is mainly situated in the negative sector of $\zeta$ while the other branch is in the positive one. Since each branch of the solutions (for each pair of determined boundary conditions) travels with the same velocity $u$, the necessary gap $\Delta$ in the space $\zeta$ for coalescing solutions, determined by the initial conditions, can be presented in two types

$$\Delta_1 = 0, \quad \Delta_2 = \arcsin[1] = \pi/2$$

(7)

B. Let us first take the solutions when they merge in the “space” with $\Delta_1 = 0$.

1. Thus, for

$$\Phi \to -1 \text{ while } \zeta \to -\infty \text{, } \Phi \to 1 \text{ while } \zeta \to \infty$$

(8)

The solitonic structure can be defined by

(a) $\sqrt{A} \zeta_{++,} \text{ if } \sqrt{A} \zeta \in (-\infty, 0]$ and

$-\sqrt{A} \zeta_{--,} \text{ if } \sqrt{A} \zeta \in [0, \infty)$

(b) $\sqrt{A} \zeta_{++,} \text{ if } \sqrt{A} \zeta \in (-\infty, \pi/2]$ and

$-\sqrt{A} \zeta_{--,} \text{ if } \sqrt{A} \zeta \in \left[\pi/2, \infty\right)$

The corresponding pattern structures are represented in Fig. 1 for (a) and in Fig. 2 for (b). These are kink-like solitons, the second one has a little peak near its center. They can be interpreted as domain walls travelling along the medium.

2. For

$$\Phi \to 1 \text{ while } \zeta \to -\infty \text{ and } \Phi \to 1 \text{ while } \zeta \to \infty$$

(9)

The solutions are determined by

(c) $\sqrt{A} \zeta_{--,} \text{ if } \sqrt{A} \zeta \in (-\infty, 0]$, and

$-\sqrt{A} \zeta_{--;} \text{ if } \sqrt{A} \zeta \in [0, \infty)$

(d) $\sqrt{A} \zeta_{++;} \text{ if } \sqrt{A} \zeta \in (-\infty, 0)$, and

$-\sqrt{A} \zeta_{++;} \text{ if } \sqrt{A} \zeta \in [0, \infty)$

These structures live as excitations in the condensate state, like a rarefaction of the field. Their forms (see Fig. 3 for (c) and Fig. 4 for (d) suggest us to give them the name peak-bubble and cusp-bubble respectively. Additionally we have

(e) $\sqrt{A} \zeta_{--;} \text{ if } \sqrt{A} \zeta \in (-\infty, -\pi/2]$, and

$-\sqrt{A} \zeta_{++;} \text{ if } \sqrt{A} \zeta \in [-\pi/2, \infty)$

This is a usual peak soliton 5 and lives as an excitation to the nonzero background, see Fig. 5 for (e).

NEW SOLITONIC STRUCTURES IN NONLINEAR MODEL WITH INTERPARTICLE ANHARMONIC INTERACTION

1. For boundary condition \( \Phi \rightarrow 1 \) while \( \zeta \rightarrow -\infty \) and \( \Phi \rightarrow -1 \) while \( \zeta \rightarrow \infty \).

The two branches

\[
\begin{align*}
(f) \quad \sqrt{A}\zeta_{+++} & \quad \text{if} \quad \sqrt{A}\zeta \in (-\infty, \frac{\pi}{2}] \quad \text{and} \\
& \quad -\sqrt{A}\zeta_{+--} & \quad \text{if} \quad \sqrt{A}\zeta \in [-\frac{\pi}{2}, \infty).
\end{align*}
\]

merge together and produce the configuration named loop soliton, see Fig. 6 for (f).

Let us now consider the case of solutions that are specially obtained by coalescing two branches of each solution pair in Eq. (6) for the second type of nonzero gap \( \Delta_2 \) in Eq. (7).

1. For the boundary condition (8), the solutions are

\[
\begin{align*}
(g) \quad \sqrt{A}\left(\zeta_{+++} + \frac{\pi}{2}\right) & \quad \text{if} \quad \sqrt{A}\zeta \in (-\infty, 0] \quad \text{and} \\
& \quad -\sqrt{A}\left(\zeta_{+--} - \frac{\pi}{2}\right) & \quad \text{if} \quad \sqrt{A}\zeta \in [0, \infty),
\end{align*}
\]

\[
\begin{align*}
(h) \quad \sqrt{A}\left(\zeta_{++-} - \frac{\pi}{2}\right) & \quad \text{if} \quad \sqrt{A}\zeta \in (-\infty, 0] \quad \text{and} \\
& \quad -\sqrt{A}\left(\zeta_{--+-} + \frac{\pi}{2}\right) & \quad \text{if} \quad \sqrt{A}\zeta \in [0, \infty).
\end{align*}
\]

These relations represented as pictures in Fig. 7 for (g) and Fig. 8 for (h) are usually referred to as shock waves because of the abrupt discontinuities that are observed when they are travelling. This suggests the interpretation of such solutions as propagating defects in the lattice. Shock waves are universal entities that are observed in great diversities of nonlinear systems. However, it is surprising that they can appear analytically also in the second order nonlinear equations (3).

2. For boundary conditions (9) the solutions can be constructed using the following two branches

\[
\begin{align*}
(i) \quad \sqrt{A}\left(\zeta_{++-} - \frac{\pi}{2}\right) & \quad \text{if} \quad \sqrt{A}\zeta \in (-\infty, 0] \quad \text{and} \\
& \quad -\sqrt{A}\left(\zeta_{+--} - \frac{\pi}{2}\right) & \quad \text{if} \quad \sqrt{A}\zeta \in [0, \infty),
\end{align*}
\]

\[
\begin{align*}
(j) \quad \sqrt{A}\left(\zeta_{++-} + \frac{\pi}{2}\right) & \quad \text{if} \quad \sqrt{A}\zeta \in (-\infty, 0] \quad \text{and} \\
& \quad -\sqrt{A}\left(\zeta_{+--} + \frac{\pi}{2}\right) & \quad \text{if} \quad \sqrt{A}\zeta \in [0, \infty).
\end{align*}
\]

Both solutions exist as excitations above the condensate state and the first one takes the form of a hump soliton Fig. 9 for (i) while the Fig. 10 for (j) represents a peckon.

3. For boundary condition (10) the soliton structures can be obtained utilizing the following relations
FIGURE 9. Other type of peakon structure and emerges when the relations (i) takes place.

FIGURE 10. Plot showing the form of a peakon obtained merging two branches as is explained in the relations (j).

FIGURE 11. Representation of an exotic loop soliton for the expressions (k). After the evaluation of the energy of this configuration one finds that this diverges.

FIGURE 12. Peak-soliton for the structure labeled by the letter (l).

\[
\begin{align*}
\text{(k)} & \quad \sqrt{A} \left( \zeta_{++} + \frac{\pi}{2} \right) \quad \text{if} \quad \sqrt{A} \zeta \in (-\infty, \frac{\pi}{2}] \quad \text{and} \quad -\sqrt{A} \left( \zeta_{++} + \frac{\pi}{2} \right) \quad \text{if} \quad \sqrt{A} \zeta \in [-\frac{\pi}{2}, \infty) \\
\text{(l)} & \quad \sqrt{A} \left( \zeta_{+++} - \frac{\pi}{2} \right) \quad \text{if} \quad \sqrt{A} \zeta \in (-\infty, 0] \quad \text{and} \quad -\sqrt{A} \left( \zeta_{+++} - \frac{\pi}{2} \right) \quad \text{if} \quad \sqrt{A} \zeta \in [0, \infty)
\end{align*}
\]

The first pair is a loop-soliton see Fig. 11 for (k) and the second is another representative of the peakon see Fig. 12 for (l).

C. Let us now present the results for evaluated energies that can be obtained from

\[
E_{\text{total}} = \int_{-\infty}^{+\infty} \left[ \frac{1}{2} \Phi_{t}^{2} + \frac{1}{2} C_t \Phi_{x}^{2} + \frac{1}{4} C_{nl} \Phi_{x}^{4} + \frac{1}{2} V_0 \left( 1 - \Phi_{x}^{2} \right)^{2} \right] dx.
\]

The final form of the energies of all the structures found above are represented by the following expressions

\[
\begin{align*}
a, c) & \quad \frac{1}{8} \pi (u^2 + C_t) \left( \frac{1}{A} \right)^{1/2} + \frac{3}{64} (\pi - 2) C_{nl} \left( \frac{1}{A} \right)^{\frac{3}{2}} + \frac{1}{8} (6 + \pi) \sqrt{AV_0} \\
b) & \quad \frac{1}{2} \left( \frac{2}{8 \pi + 1} \right) (u^2 + C_t) \left( \frac{1}{A} \right)^{1/2} + \frac{1}{32} \left[ 5 + \frac{3}{2} \right] C_{nl} a \left( \frac{1}{A} \right)^{\frac{3}{2}} + \frac{1}{4} \left( \frac{1}{2} \pi + 3 \right) \sqrt{AV_0}
\end{align*}
\]

\[
e) \quad \frac{E_{\text{total}}^i}{2} + \frac{E_{\text{total}}^f}{2}
\]
The energies \( d, f, k \) diverge. The values of energies of all soliton structures are restricted by their velocities. Then if we have to avoid singularities in energy values, from the Eq. (12) it is necessary to impose:

\[
A = \frac{4V_0}{u^2 - C_1} > 0
\]  

(13)

3. Properties of the soliton solutions

For solutions satisfying condensate boundary conditions, their strong localizations are determined by the topological invariant charge concerning their field values at infinity. On the other hand, both parts of the coalescing solutions travel with equal velocity \( u \) in the same direction. All these requirements satisfy the restriction imposed on \( A \) in the Eq. (13) to be positive. Additionally, if \( u^2 - C_1 = 0 \) we should need infinity energy values because of the Eq. (12). This situation can be avoided since it represents a nonphysical situation.

It is noticeable that the energy formulae of all solutions encountered above, have the following typical form in terms of three parameters \( u, C_1 \) and \( V_0 \) see Eq. (12)

\[
E = \alpha \left( u^2 + C_1 \right) \sqrt{C_1 - u^2} + \frac{1}{16} \left[ 5 - \frac{3}{4} \pi \right] C_{nl} a \left( \frac{1}{A} \right)^{\frac{3}{2}} + \left( \frac{1}{8} + \frac{3}{4} \pi \right) a\sqrt{AV_0}
\]

(14)

being \( \alpha, \beta \) and \( \gamma \) the respectively numerical coefficients of each member of the energy expression [see Eq. (12)].

4. Conclusions

Since soliton-like structures are very special solutions, in this paper we have obtained new conditions determined by combinations of the parameters of the model in order to integrate the corresponding nonlinear evolution equations. We have demonstrated that the anharmonicity in the interparticle interactions can allow the appearance of a rich variety of static and travelling solitonic structures. When the boundary condition is of the condensate type and the relevant parameters satisfy the Eq. (5), new solitonic structures like defects, kink, bubble, peakons and cusps were found. In this case from all of the variety of structures found here, the most probable to exist are the structures like bell and peak solitons as nonlinear excitations above the constant field, while the defects are those that carry less energies when the field has different vacuum at infinity.

Checking the formulae for the velocities of both of these types of structure, one can easily find that there is a crucial value of velocity that divides the region in the parameter space of existence for both types of solutions. This value corresponds to those structures found by Remoissenet and co-workers in Refs. 5–7. For our solutions, this velocity \( u = \pm \sqrt{C_1} \) is inaccessible.