The inevitable consequence of this peculiarity is the particle-anti-particle metamorphosis giving rise to neutrinoless double beta decay, on the one side, and enabling spin-1/2 fields to act as gauge fields, gauginos, on the other side. The second part of the lecture notes is devoted to massive gravitinos. We argue that a spin measurement in the rest frame for an unpolarized ensemble of massive gravitinos, associated with the spinor-vector \([(1/2, 0) \oplus (0, 1/2)] \otimes (1/2, 1/2)\) representation space, would yield the results 3/2⁻ with probability one half, and 1/2 with probability one half. The latter is distributed uniformly, i.e. as 1/4, among the two spin-1/2+ and spin-1/2⁻ states of opposite parities. From that we draw the conclusion that the massive gravitino should be interpreted as a particle of multiple spin.

Keywords: Lorentz group; discrete symmetries; massive gauge fields

Se plantea que la profunda diferencia entre las partículas de Dirac y Majorana se debe a su diferente construcción en el espacio de representaciones \((1/2, 0) \oplus (0, 1/2)\). Al contrario que las partículas de Dirac, los propagadores de partícula de Majorana difieren de la estructura lineal simple \(\gamma^\mu p_\mu\). Además, ni las partículas de Majorana ni sus antipartículas se pueden asociar con una flecha del tiempo bien definida. La consecuencia inevitable de esta peculiaridad es la metamorfosis partícula-antipartícula que da lugar al decaimiento beta doble sin neutrino, por una parte, y permitiendo que los campos de espín 1/2 actúen como campos de norma, gauginos, por otra parte. La segunda parte de estas notas se dedica al gravitino masivo. Argumentamos que una medición del espín en el sistema de referencia en reposo para un conjunto no polarizado de gravitinos masivos asociados con el espacio de representaciones espín-vector \([(1/2, 0) \oplus (0, 1/2)] \otimes (1/2, 1/2)\), daría como resultado 3/2⁻ con probabilidad de un medio. Esta última es distribuida uniformemente, es decir, como 1/4 entre los estados de espín-1/2+ y espín-1/2⁻ de paridad opuesta. De lo cual concluimos que el gravitino masivo debe ser interpretado como una partícula de espín múltiple.

Descriptors: Grupo de Lorentz; simetrías discretas; campos masivos de norma

PACS: 14.20.Gk; 03.65.Pm; 11.30.Ly

1. Introduction

These lecture notes are essentially based on our recent work [1-3]. The goal is to review new insights into the spacetime properties of massive Majorana neutrinos, massive gauginos and gravitinos. We work out a method for deriving propagators of fundamental particles directly from the representation space and apply it to the description of the above indicated particles. Within this framework the metamorphosis of Majorana neutrinos to anti-neutrinos that is at the heart of the neutrinoless double beta decay finds a natural mathematical expression in terms of spinors that lack a well determined arrow of time. We further examine the nature of the spinor-vector \(\psi^\mu\) that appears in supersymmetric theories as a fermionic gauge field, the so called gravitino, and show that its single-spin interpretation is unjustified. Rather, this particle represents itself as a multiparticle object. The lecture notes are organized as follows.

In the next section we briefly outline the construction of the Dirac \((1/2, 0) \oplus (0, 1/2)\) representation space. In Sec. 3 same is done for the Majorana spin-1/2 field. There, we show that Majorana particles are reflection-less in the massless limit, insensitive to the arrow of time, and undergo a particle-antiparticle metamorphosis that is responsible for such processes as the neutrinoless double beta decay and the exchange of spin-1/2 gauge fields in supersymmetric theories. Section 4 is devoted to the massive gravitino. There we show that the so called auxiliary conditions to the spinor-vector \(\psi^\mu\) are nothing but defining conditions for the subspace of \(\psi^\mu\) that carries the maximal spin of 3/2⁻ in the rest frame. In particular, we derive an equation for \(\psi^\mu \gamma_0\) showing that the latter auxiliary condition is not arbitrary at all and can not be set equal to zero unconditionally. We suggest sets of such defining—"auxiliary"—conditions for each one of the spin-1/2⁻ and spin-1/2⁺ rest-frame sectors of \(\psi^\mu\) and show that none of them is more or less physical than the other two. From that we derive the conclusion about the multi-spin character of the massive gravitino. The lecture notes end with a brief outlook.

2. Dirac's \((1/2, 0) \oplus (0, 1/2)\) representation space

In this section we outline a general new procedure for obtaining the Dirac equation that relies on nothing more but the
boost operators. We begin with the rest-frame spinors and boost them using the following boosts
\[ \kappa(\frac{1}{2}, 0) \rightarrow \kappa(\frac{1}{2}, 0) \oplus \kappa(0, \frac{1}{2}), \]
with
\[ \kappa(\frac{1}{2}, 0) = \exp \left( +\frac{\sigma}{2} \cdot \varphi \right) = \sqrt{\frac{E + m}{2m}} \left( \mathbb{I}_2 + \frac{\sigma \cdot p}{E + m} \right) \quad (2) \]
\[ \kappa(0, \frac{1}{2}) = \exp \left( -\frac{\sigma}{2} \cdot \varphi \right) = \sqrt{\frac{E + m}{2m}} \left( \mathbb{I}_2 - \frac{\sigma \cdot p}{E + m} \right). \quad (3) \]

In Eqs. (2) and (3) the boost parameter is defined as:
\[ \cosh(\varphi) = \frac{E}{m}, \quad \sinh(\varphi) = \frac{|p|}{m}, \quad \varphi = \frac{p}{|p|}. \quad (4) \]

The boosts take a particle at rest to a particle moving with momentum \( p \) in the "boosted frame". We use the notation in which \( \mathbb{I}_n \) and \( 0_n \) represent \( n \times n \) identity and null matrices, respectively. The remaining symbols carry their usual contextual meaning. We define the spin-\( \frac{1}{2} \) helicity operator:
\[ \mathcal{H} = (\frac{\sigma}{2}) \cdot \mathbf{p}, \quad \text{where} \quad \mathbf{p} = p|p|, \quad \text{and} \quad p = |p| \sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta). \]

Keeping full freedom in the choice of phases, its positive and negative helicity states are:
\[ h^+ = N \exp \left( i\psi_+ \right) \begin{pmatrix} \cos(\theta/2) \exp(-i\phi/2) & \sin(\theta/2) \exp(i\phi/2) \\ -\sin(\theta/2) \exp(-i\phi/2) & \cos(\theta/2) \exp(i\phi/2) \end{pmatrix}, \]
\[ h^- = N \exp \left( i\psi_- \right) \begin{pmatrix} \sin(\theta/2) \exp(-i\phi/2) & -\cos(\theta/2) \exp(i\phi/2) \\ \cos(\theta/2) \exp(-i\phi/2) & \sin(\theta/2) \exp(i\phi/2) \end{pmatrix}. \quad (5) \]

The rest-frame \((1/2, 0) \oplus (0, 1/2)\) spinors are then chosen to be:
\[ (U_{+1/2}(0), V_{+1/2}(0)) = \left( \begin{array}{c} h^+ \hfill \cr h^- \end{array} \right), \quad (U_{-1/2}(0), V_{-1/2}(0)) = \left( \begin{array}{c} h^- \cr h^+ \end{array} \right). \quad (6) \]

The choice of the phases made in writing down these spinors has been determined by the demand of parity covariance [4]. The boosted spinors, \( u_{+1/2}(p) \) and \( v_{+1/2}(p) \) are obtained by applying the boost operator \( \kappa(\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) to the above spinors, yielding:

\[ u_{+1/2}(p) = \frac{N \exp(i\psi_+)}{\sqrt{2m(m + E)}} \begin{pmatrix} \exp(-i\phi/2)(m + |p| + E) \cos(\theta/2) \\ \exp(-i\phi/2)(m - |p| + E) \cos(\theta/2) \end{pmatrix}, \]
\[ u_{-1/2}(p) = \frac{N \exp(i\psi_-)}{\sqrt{2m(m + E)}} \begin{pmatrix} \exp(-i\phi/2)(m - |p| + E) \sin(\theta/2) \\ -\exp(-i\phi/2)(m - |p| + E) \sin(\theta/2) \end{pmatrix}, \]
\[ v_{+1/2}(p) = \frac{N \exp(i\psi_+)}{\sqrt{2m(m + E)}} \begin{pmatrix} \exp(-i\phi/2)(m + |p| + E) \sin(\theta/2) \\ -\exp(-i\phi/2)(m + |p| + E) \sin(\theta/2) \end{pmatrix}, \]
\[ v_{-1/2}(p) = \frac{N \exp(i\psi_-)}{\sqrt{2m(m + E)}} \begin{pmatrix} \exp(-i\phi/2)(m + |p| + E) \cos(\theta/2) \\ -\exp(-i\phi/2)(m + |p| + E) \cos(\theta/2) \end{pmatrix}. \quad (7) \]

These satisfy the orthonormality and completeness relations (in standard notation):
\[ \bar{u}_h(p)u_{h'}(p) = +2N^2\delta_{hh'}, \]
\[ \bar{v}_h(p)v_{h'}(p) = -2N^2\delta_{hh'}, \quad (8) \]
\[ \frac{1}{2N^2} \left[ \sum_{h=\pm 1/2} u_h(p)\bar{u}_h(p) - \sum_{h=\pm 1/2} v_h(p)\bar{v}_h(p) \right] = \mathbb{I}_4. \quad (9) \]

In order to obtain the wave equation satisfied by the \( u_h(p) \) and \( v_h(p) \) spinors we first note that (a)
\[ \frac{1}{2N^2} \left[ \sum_{h=\pm 1/2} u_h(p)\bar{u}_h(p) + \sum_{h=\pm 1/2} v_h(p)\bar{v}_h(p) \right] = \frac{\gamma_m p^\mu}{m}, \quad (10) \]
where we defined,
3. Majorana's (1/2, 0) \( \oplus (0, 1/2) \) representation space

Spacetime structures in supersymmetric theories find themselves deeply intertwined with the Majorana realization of fermionic representation spaces. In the simplest form, the algebra of supersymmetry is generated by fourteen operators. Ten are those of the Poincaré group. The remaining four anti-commuting generators, the supertranslations, are components of a Majorana spinor \[5\]. Thus, the major request for Majorana particles in physics arises in supersymmetric theories.

In this context, recent results of Klapdor-Kleingrothaus et al. \[6-8\], which provided a first direct evidence for neutrinoless double beta decay, exhibit particular interest.

In its simplest interpretation, the 0\( \nu \beta \beta \) experimental signal (apparently, an independent confirmation of the Klapdor-Kleingrothaus et al. result by other laboratories is required \[9\]) arises from the Majorana nature of massive \( \nu_e \). Even though the experiment by itself does not necessarily require a supersymmetric framework for its explanation, the indication towards a Majorana spacetime structure suddenly acquires a pivotal importance.\( ^6 \)

In this section we, therefore, undertake an \textit{ab initio} look at the spacetime structure of the massive Majorana particles—whether they be neutrinos or gaugons. In particular, we focus our attention on the \( \nu = \bar{\nu} \) metamorphosis.

In the first subsection, we present the standard picture of the 0\( \nu \beta \beta \), and raise the relevant question about the spacetime structure underlying the Majorana construct. Next, we present a self contained brief summary of self/antiself conjugate Majorana spinors. Furthermore, we examine to what extent it is possible to associate them with negative and positive energies. We find that the particle/antiparticle interpretation of self/antiself conjugate Majorana spinors fails as they turn out to be insensitive with respect to the arrow of time. In this way, a door opens to a \( \nu = \bar{\nu} \) metamorphosis which leads to 0\( \nu \beta \beta \) decay and exchange of spin-1/2 particles as gauge fields (gauginos) in supersymmetric theories. We explicitly construct the spacetime structure underlying the particle-antiparticle metamorphosis in terms of the momentum-space Feynman propagator. Finally, we show that Majorana self/antiself conjugate spinors are reflection-less in the massless limit, a property that is at the heart of parity violation in electroweak processes.

3.1. Conventional 0\( \nu \beta \beta \) description and questions it raises

In the simplest scenario, the recent results of Klapdor-Kleingrothaus et al. are interpreted as arising from the Majorana aspect of neutrinos. The essential physics of interest lies in the following. A virtual \( W^- \), which induces a weak \( n \to p \) transition, decays into e\(^-\) and \( \bar{\nu}_e \) ,

\[
W^- \to e^- + \bar{\nu}_e ,
\]

and the emitted \( \bar{\nu}_e \) is reabsorbed as an \( \nu_e \),

\[
\nu_e + W^- \to e^- ,
\]

where the second (virtual) \( W^- \) takes responsibility for the second \( n \to p \) transformation. The net result is the 0\( \nu \beta \beta \),

\[
A(N, Z) \to A(N - 2, Z + 2) + 2e^- ,
\]

where \( A(N, Z) \) represents a nucleus carrying \( N \) neutrons, and \( Z \) protons, see Fig. 1. Now, the Majorana nature of \( \nu_e \) does not mean \((\times)\) that one can naively identify the \( \bar{\nu}_e \) of (15) with the \( \nu_e \) of (16),

\[
\begin{aligned}
\times : \quad \bar{\nu}_e &\neq \nu_e .
\end{aligned}
\]

This is so because a neutrino carries a negative helicity in the \( m_{\nu_e}/E_{\nu_e} \to 0 \) limit, and transforms as a \((0, 1/2)\) object; while an antineutrino carries a positive helicity in the \( m_{\nu_e}/E_{\nu_e} \to 0 \) limit, and transforms as a \((1/2, 0)\) object. As such there is a \( m_{\nu_e}/E_{\nu_e} \) suppressed amplitude for the above identification \[11,12\]. This result arises in a framework in which the Majorana field \[13\] is obtained from the Dirac field,

\[
\psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{m}{2p_0} \sum_{h=t,\downarrow} \left[ a_h(p) u_h(p) e^{-ip \cdot z} + b_h^*(p) \nu_h(p) e^{ip \cdot z} \right] ,
\]

via the identification,

\[
b_h^*(p) = \zeta a_h(p) ,
\]

where \( \zeta \) is a "creation phase factor" \[14\]. As is apparent, the above Fock space construct ceases to place fundamentally
We refrain from identifying $\Theta$ with $-i\sigma_2$ because such identification does not exist for higher-spin $(j, 0) \oplus (0, j)$ representation spaces. The existence of Wigner time reversal operator for all $j$, allows, for fermionic $j$'s, the introduction of $(j, 0) \oplus (0, j)$ Majorana objects.

The above observations imply that:

1. If $\phi_L(p)$ transforms as a left handed spinor, then $\phi_L^T(0) \phi_L^*(p)$ transforms as a right handed spinor—where, $\zeta_\phi$ is an undetermined phase.

2. If $\phi_R(p)$ transforms as a right handed spinor, then $\phi_R^T(p) \phi_R^*(0)$ transforms as a left handed spinor—where, $\zeta_\rho$ is an undetermined phase.

As a consequence, the following spinors (in Weyl basis) belong to the $(1/2, 0) \oplus (0, 1/2)$ representation space:

$$\lambda(p) = \begin{pmatrix} (\zeta_\phi \Theta) \phi_L^T(p) \\ \phi_L(p) \end{pmatrix},$$

$$\rho(p) = \begin{pmatrix} \phi_R(p) \\ (\zeta_\rho \Theta)^* \phi_R^T(p) \end{pmatrix}. \quad (24)$$

Demanding $\lambda(p)$ and $\rho(p)$ to be self/anti-self conjugate, i.e.

$$C \lambda(p) = \pm \lambda(p), \quad C \rho(p) = \pm \rho(p), \quad (25)$$

under the operation of charge conjugation

$$C = \begin{pmatrix} 0_2 \\ i\Theta \\ -i\Theta \\ 0_2 \end{pmatrix} \equiv K, \quad (26)$$

where the operator $K$ complex conjugates any object that appears on its right, restricts the phases, $\zeta_\phi$ and $\zeta_\rho$, to two values:

$$\zeta_\phi = \pm i, \quad \zeta_\rho = \pm i. \quad (27)$$

The plus sign in the above equation yields self conjugate, $\lambda^A(p)$ and $\rho^A(p)$ spinors, while the minus sign results in the anti-self conjugate spinors, $\lambda^S(p)$ and $\rho^S(p)$. We shall now obtain explicit expressions for these spinors.

3.3. The $\lambda(p)$ spinors

To obtain explicit expressions for $\lambda(p)$, we first write down the rest spinors. These are:

$$\lambda^S(0) = \begin{pmatrix} +i\Theta \phi_L^T(0) \\ \phi_L(0) \end{pmatrix},$$

$$\lambda^A(0) = \begin{pmatrix} -i\Theta \phi_L^T(0) \\ \phi_L(0) \end{pmatrix}. \quad (28)$$

Next, we choose the $\phi_L(0)$ to be helicity eigenstates.

$$\sigma \cdot \hat{p} \phi_L^T(0) = \pm \phi_L^*(0), \quad (29)$$

FIGURE 1. $\nu_e = \bar{\nu}_e$ metamorphosis in $0\nu\beta\beta$ as $\rho_1^T = \lambda_2^T$ spin-flip of antise/ self-conjugate Majorana spinors.
and concurrently note that (see Appendix)
\[ \sigma \cdot \mathbf{p} \Phi [\phi_L^+(0)]^* = \mp \Phi [\phi_L^+(0)]^* \]

We are thus led to four rest spinors. Two of which are self conjugate
\[ \lambda_S^a(0) = \left(\frac{+i \Phi [\phi_L^+(0)]^*}{\phi_L^+(0)}\right), \]
\[ \lambda_A^a(0) = \left(\frac{-i \Phi [\phi_L^+(0)]^*}{\phi_L^+(0)}\right), \]
and the other two, which are anti-self conjugate
\[ \lambda_A^a(0) = \left(\frac{-i \Phi [\phi_L^-(0)]^*}{\phi_L^-(0)}\right), \]
\[ \lambda_S^a(0) = \left(\frac{+i \Phi [\phi_L^-(0)]^*}{\phi_L^-(0)}\right). \]
The boosted spinors are now obtained via the operation:
\[ \lambda(p) = \left(\begin{array}{cc} \kappa^{1/2,0} & 0 \\ 0 & \kappa^{0,1/2} \end{array}\right) \lambda(0), \]
where the \( \kappa^{1/2,0} \) and \( \kappa^{0,1/2} \) are in turn given by Eqs. (2) and (3). In the boost, we replace \( \sigma \cdot \mathbf{p} \) by \( \sigma \cdot \mathbf{p} \mathbf{e}_\mathbf{p} \), and then exploit Eq. (30). After simplification, Eq. (33) yields:
\[ \lambda_S^a(p) = \sqrt{\frac{E+m}{2m}} \left(1 - \frac{|\mathbf{p}|}{E+m} \right) \lambda_S^a(0), \]
which, in the massless limit, identically vanishes, while
\[ \lambda_A^a(p) = \sqrt{\frac{E+m}{2m}} \left(1 + \frac{|\mathbf{p}|}{E+m} \right) \lambda_A^a(0), \]
does not. We hasten to warn the reader that one should not be tempted to read the two different prefactors to \( \lambda^S(0) \) in the above expressions as the boost operator that appears in Eq. (33). For one thing, there is only one (not two) boost operator(s) in the \( (1/2,0) \oplus (0,1/2) \) representation space. The simplification that appears here is due to a fine interplay between Eq. (30), the boost operator, and the structure of the \( \lambda^S(0) \). Similarly, the anti-self conjugate set of the boosted spinors reads:
\[ \lambda_A^a(p) = \sqrt{\frac{E+m}{2m}} \left(1 - \frac{|\mathbf{p}|}{E+m} \right) \lambda_A^a(0), \]
\[ \lambda_S^a(p) = \sqrt{\frac{E+m}{2m}} \left(1 + \frac{|\mathbf{p}|}{E+m} \right) \lambda_S^a(0). \]
In the massless limit, the first of these spinors identically vanishes, while the second does not. Choosing \( \phi_L^+(0) \) as follows,
\[ \phi_L^+(0) = \sqrt{m} e^{i \theta_1} \left(\begin{array}{c} \cos(\theta/2) e^{-i \theta/2} \\ \sin(\theta/2) e^{i \theta/2} \end{array}\right), \]
\[ \phi_L^-(0) = \sqrt{m} e^{i \theta_2} \left(\begin{array}{c} \sin(\theta/2) e^{-i \theta/2} \\ -\cos(\theta/2) e^{i \theta/2} \end{array}\right), \]
and on setting \( \theta_1 \) and \( \theta_2 \) to be zero (a fact that we explicitly note), we find the following bi-orthonormality relations,
\[ \bar{\lambda}_S^S(p) \lambda_S^S(p) = 0, \quad \bar{\lambda}_S^S(p) \lambda_A^S(p) = +2im, \]
\[ \bar{\lambda}_A^S(p) \lambda_S^S(p) = -2im, \quad \bar{\lambda}_A^S(p) \lambda_A^S(p) = 0, \]
and
\[ \bar{\lambda}_S^A(p) \lambda_A^A(p) = 0, \quad \bar{\lambda}_S^A(p) \lambda_A^A(p) = -2im, \]
\[ \bar{\lambda}_A^A(p) \lambda_A^A(p) = +2im, \quad \bar{\lambda}_A^A(p) \lambda_A^A(p) = 0, \]
while all combinations of the type \( \bar{\lambda}_A^A(p) \lambda_S^S(p) \) and \( \bar{\lambda}_S^A(p) \lambda_A^A(p) \) identically vanish. We take note that the bi-orthogonal norms of the Majorana spinors are intrinsically imaginary. The associated completeness relation is:
\[ -\frac{1}{2im} \left\{ [\lambda_S^S(p) \bar{\lambda}_S^S(p) - \lambda_A^S(p) \bar{\lambda}_A^S(p)] - [\lambda_A^A(p) \bar{\lambda}_A^A(p) - \lambda_S^A(p) \bar{\lambda}_S^A(p)] \right\} = \mathbb{I}_4. \]

3.4. The \( \lambda(p) \) and \( \rho(p) \) spinors

Now, \( (1/2,0) \oplus (0,1/2) \) is a four dimensional representation space. Therefore, there cannot be more than four independent spinors. We find that the \( \rho(p) \) spinors are related to the \( \lambda(p) \) spinors via the following identities:
\[ \rho_S^S(p) = -i \lambda_A^A(p), \quad \rho_A^S(p) = +i \lambda_A^S(p), \]
\[ \rho_S^A(p) = +i \lambda_S^A(p), \quad \rho_A^A(p) = -i \lambda_S^S(p). \]
Using these identities, one may immediately obtain the bi-orthonormality and completeness relations for the \( \rho(p) \) spinors. Thus, in the self/antitself conjugate \( (1/2,0) \oplus (0,1/2) \) representation space there are at least two physically equivalent sets of basis spinors: \( \{\lambda^S(p), \lambda^A(p)\} \), and \( \{\rho^S(p), \rho^A(p)\} \).

In the massless limit, \( \rho_S^S(p) \) and \( \rho_A^A(p) \) identically vanish.

A particularly simple orthonormality, as opposed to bi-orthonormality, relation exists between the \( \lambda(p) \) and \( \rho(p) \) spinors:
\[ \bar{\lambda}_S^S(p) \rho_S^S(p) = -2m = \bar{\lambda}_A^A(p) \rho_A^S(p) \]
\[ \bar{\lambda}_S^A(p) \rho_A^A(p) = -2m = \bar{\lambda}_A^S(p) \rho_S^A(p). \]
An associated completeness relation also exists, and it reads:
\[ -\frac{1}{2m} \left\{ [\lambda_S^S(p) \bar{\rho}_S^S(p) + \lambda_A^S(p) \bar{\rho}_A^A(p)] - [\lambda_A^A(p) \bar{\rho}_A^A(p) + \lambda_S^A(p) \bar{\rho}_S^S(p)] \right\} = \mathbb{I}. \]
Note, that while the majority of the relations reported in this section are new, they take their origin from Refs. 15–18.
3.5. Reflection-less Majorana spinors

In the massless limit, the Majorana spinors are reflection-less. This can be proved by taking note of the fact that, e.g., under the Parity operation $\Lambda_{\lambda}^{\mp}(p)$ transforms to $i\Lambda_{\lambda}^{\pm}(p)$. The latter, in the indicated limit, identically vanishes, as already explained in subsection (C). The consequence of this is that, if, for instance, in the ordinary beta decay

$$n \rightarrow p + e^- + \nu_e,$$  

one identifies $\nu_e$ with $\rho_\pm(p)$ [which equals $i\Lambda_{\lambda}^\mp(p)$, according to the first equation in Eq. (46)], then its Parity transform $-\Lambda_{\lambda}^\mp(p)$, identically vanishes in the massless limit. This circumstance is the source of parity violation as it provides the reason for which empirically observed ordinary beta decays do not have a mirror image. For massive neutrinos, this would be no longer true.

Note a rather subtle aspect of the Majorana framework: Parity violation occurs despite the apparent parity covariance of the Majorana construct. No such kinematic conclusion can be drawn if neutrinos are described by Dirac spinors without explicitly invoking parity violation via the standard projectors $(1/2)(1 \pm \gamma_5)$.

3.6. Wave equations for Majorana spinors

To analyze the time-evolution of the $\lambda(p)$ and $\rho(p)$ Majorana spinors one needs to know the equations of motion. After taking into account the observations of Ref. 4 and 21, these can be obtained following a text-book procedure similar to that of Ryder [22]. The result is:

$$\begin{pmatrix} -\Pi_2 & 0^+ \cr 0_\lambda^- & -\Pi_2 \end{pmatrix} \lambda(p) = 0,$$

$$\begin{pmatrix} +\Pi_2 & 0^+ \cr 0_\rho^+ & +\Pi_2 \end{pmatrix} \rho(p) = 0.$$  

(51)

In the above equations we have used the following abbreviations:

$$\begin{align*}
0_\lambda^\pm & \equiv \zeta_\lambda \exp(\pm \sigma \cdot \varphi/2) \Theta \exp(\pm \sigma \cdot \varphi/2), \\
0_\rho^\pm & \equiv \zeta_\rho \exp(\pm \sigma \cdot \varphi/2) \Theta \exp(\pm \sigma \cdot \varphi/2),
\end{align*}$$

(52, 53)

where

$$\eta = \begin{pmatrix} e^{i\phi} & 0 \\
0 & e^{-i\phi} \end{pmatrix}.$$  

(54)

These wave equations cannot be reduced to the Dirac equation.

3.7. Indiscernibility between negative and positive energy solution for Majorana particles

We note that for self-, as well as self-conjugate Majorana spinors (i.e., irrespective of whether $\zeta_\lambda = +i$, or $\zeta_\lambda = -i$)

$$\begin{pmatrix} -\Pi_2 & 0^+ \cr 0_\lambda^- & -\Pi_2 \end{pmatrix} = \begin{pmatrix} (3m^2 + 4Em - p^2 + E^2)^2 & 0 \\
0 & (m^2 + p^2 - E^2)^2 \end{pmatrix}.$$  

(55)

That the determinant must vanish for non-trivial solutions, $\lambda(p)$, translates into

$$(m^2 + p^2 - E^2) = 0.$$  

The $\lambda(p)$ spinors are thus consistent with the causal dispersion relation. To check the hypothesis if the sign of $\zeta_\lambda$ can determine the sign of energy associated with the $\lambda(p)$ spinors, it suffices to look at the wave equations for the latter, irrespective of the helicity, or the sign of $\zeta_\lambda$, then reduces to the conditions of the type

$$\begin{pmatrix} ie^{-i\phi/2} (m^2 - E^2) \sin(\theta/2) \\
-ie^{i\phi/2} (m^2 - E^2) \cos(\theta/2) \\
e^{-i\phi/2} (m^2 - E^2) \cos(\theta/2) \\
e^{i\phi/2} (m^2 - E^2) \sin(\theta/2) \end{pmatrix} = \begin{pmatrix} 0 \\
0 \\
0 \\
0 \end{pmatrix}.$$  

(56)

Clearly, this condition cannot determine the sign of energy associated with the $\lambda_\lambda(0)$ to be either positive, or negative. The same holds true for the remaining $\lambda(0)$ spinors, and results readily extend to all $\lambda(p)$.

The above discussion is equally valid for the $\rho(p)$ spinors.

3.8. Insensitivy of Majorana spinors to the direction of time

In configuration space Majorana spinors are, $\lambda(x,t) = \exp(iep \cdot x)\lambda(p)$ and $\rho(x,t) = \exp(iep \cdot x)\rho(p)$, where $\epsilon = \pm 1$ (depending upon whether the propagation is forward in time, or backward in time).

To determine $\epsilon$ it suffices to study the wave equations in the rest frame of the particles. In that frame, the $\lambda(x,t)$ satisfies the following (simplified) differential equation:

$$\begin{pmatrix} O_a & 0 & 0 \\
0 & O_b & 0 \\
0 & O_c & 0 \end{pmatrix} \lambda(0) e^{-iemt} = 0.$$  

(57)

where

$$\begin{align*}
o_a & \equiv -2m \left( \frac{\partial}{\partial t} + m \right), \\
o_b & \equiv -\zeta_\lambda e^{-i\phi} \left( m^2 - \frac{\partial^2}{\partial t^2} + 2im \frac{\partial}{\partial t} \right), \\
o_c & \equiv \zeta_\lambda e^{i\phi} \left( m^2 - \frac{\partial^2}{\partial t^2} + 2im \frac{\partial}{\partial t} \right).
\end{align*}$$

(58, 59, 60)
This equation does not fix the sign of \( \epsilon \). It only determines \( \epsilon^2 \) to be unity. That is, Majorana spinors are insensitive to the "forward" and "backward" directions in time so important in the Feynman-Stückelberg interpretation of particles and antiparticles. Same result, and similar analysis, holds true for the \( \rho \) spinors.

We thus find that for the self/anti-self conjugate \((1/2, 0) \oplus (0, 1/2)\) representation space conventional distinction between particles and antiparticles disappears.

3.9. Metamorphosis of Majorana \( \bar{v} \) to \( v \): Propagator for the Majorana field

The completeness relation (44) suggests that we define the Majorana field operator as:

\[
\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \sum_h \left[ \lambda_h^S(p) a_h(p) e^{-ip \cdot x} + \lambda_h^A(p) a_h^\dagger(p) e^{ip \cdot x} \right].
\]  

Parenthetically, we take note of the fact that, using Eqs. (45), it may be re-written entirely in terms of the self conjugate \( \lambda_h^S(p) \) and \( \rho_h^S(p) \) spinors. Any relative signs that different helicity Majorana spinors may carry in the indicated sum are implicitly assumed to be absorbed in the annihilation, \( a_h(p) \), and creation, \( a_h^\dagger(p) \), operators. The latter are assumed to satisfy the bi-orthogonality respecting fermionic anticommutators,

\[
\{ a_h(p), a_{h'}(p') \} = (2\pi)^3 2p_0 \delta^3(p - p') \delta_{h,-h'}.
\]

By postulating this field-theoretic structure we hope to avoid negative norm states, while preserving the essential bi-orthonormality of the underlying \((1/2, 0) \oplus (0, 1/2)\) Majorana representation space. In Eq. (62), if, \( h = + \), then \(-h\) represents \( + \), and if, \( h = - \), then \(-h\) represents \( - \).

With these definitions in mind, the configuration-space Feynman propagator is obtained as

\[
\langle x | S_F | y \rangle = \langle \text{vac} | T [\mu(x) \bar{\mu}(y)] | \text{vac} \rangle,
\]

in the standard contextual meaning of the used symbols. A straightforward calculation yields:

\[
\langle x | S_F | y \rangle = \sum_h \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \left[ \lambda_h^S(p) \bar{\lambda}_{(-h)}^S(p) e^{-ip \cdot (x - y)} \theta (x^0 - y^0) - \lambda_h^A(p) \bar{\lambda}_{(-h)}^A(p) e^{ip \cdot (x - y)} \theta (y^0 - x^0) \right].
\]

As a consequence, the momentum-space propagator is obtained to be,

\[
\langle p' | S_F | p \rangle = \int \frac{d^3x}{(2\pi)^3} \int \frac{d^3y}{(2\pi)^3} e^{ip' \cdot x} e^{-ip \cdot y} \langle y | S_F | x \rangle = i\delta^4(p' - p) \sum_h \left[ \frac{\lambda_h^S(p) \bar{\lambda}_{(-h)}^S(p)}{p_0 - E(p) + i\epsilon} - \frac{\lambda_h^A(p) \bar{\lambda}_{(-h)}^A(p)}{p_0 + E(p) - i\epsilon} \right],
\]

where \( E(p) = \sqrt{m^2 + p^2} \). As the above propagator explicitly exhibits, an exchanged Majorana particle is emitted with one helicity, at one vertex, and is absorbed with opposite helicity, at the second vertex.

This is the spacetime picture of \( \bar{v} \Rightarrow v \) metamorphosis for Majorana particles. Using Eqs. (45) and (46), the preceding statement can be made more transparent. Stated differently, exploiting the identities (45) and (46), one may re-express \( \langle p' | S_F | p \rangle \) by replacing the \( \bar{\lambda}_{(-h)}^S(p) \) and \( \bar{\lambda}_{(-h)}^A(p) \) in terms of \( \bar{\rho}_{(h)}^A(p) \) and \( \bar{\rho}_{(h)}^S(p) \), respectively.

\[
\langle p' | S_F | p \rangle = \delta^4(p' - p) \sum_h \left[ \frac{-\lambda_h^S(p) \bar{\rho}_{(h)}^A(p) + \lambda_h^A(p) \bar{\rho}_{(h)}^S(p)}{p_0 - E(p) + i\epsilon} - \frac{\lambda_h^A(-p) \bar{\rho}_{(-h)}^S(-p) - \lambda_h^S(-p) \bar{\rho}_{(-h)}^A(-p)}{p_0 + E(p) - i\epsilon} \right],
\]

The emergent picture is delicately beautiful as it shows how in the massless limit the propagator identically vanishes.\(^{(6)}\)

For comparison, the standard Dirac propagator can be cast [23] into a form similar to (65)

\[
\langle p' | S_F | p \rangle = i\delta^4(p' - p) \sum_h \left( \frac{u_h(p) \bar{u}_h(p)}{p_0 - E(p) + i\epsilon} - \frac{v_h(-p) \bar{u}_h(-p)}{p_0 + E(p) - i\epsilon} \right).
\]
The conclusion is that the bi-orthogonality of the Majorana spinors spin-flips $\lambda^a$ to $\rho^a$ and acts thereby as the major designer of the particle-antiparticle metamorphosis (see Fig. 1).

To summarize, we described the metamorphosis of Majorana particles to antiparticles (and vice versa) at both the spinorial spin-1/2 and the Fock space levels, rather than, as usually done, solely at level of the Fock space states.

Within the Standard Model, with the restriction to left-handed fields, the hybrid formalism in Eqs. (19) and (20), on the one side, and Eqs. (61) and (62), on the other side, represent equivalent languages for describing same physical reality.

This equivalence no longer holds for left-right symmetric theories, i.e. theories beyond the Standard Model. There, the profound differences between Dirac and Majorana particles will inevitably show up.

4. Spacetime properties of massive gravitino

A massive gravitino is described by $\psi^\mu$. As far as its spacetime properties are concerned, it transforms as a finite dimensional non-unitary representation of the Lorentz group,

$$\psi^\mu : \left[(1/2, 0) \oplus (0, 1/2)\right] \otimes (1/2, 1/2) \ . \ (68)$$

The unitarily transforming physical states are built upon this structure [24].

We enumerate two circumstances that motivate us to take an ab initio look at this representation space.

1. For the vector sector, it has recently been called to attention that the Proca description of the $(1/2, 1/2)$ representation space is incomplete [1]. An ab initio construction of this sector reveals that the Stickelberg contribution to the propagator, so important for the renormalization of the gauge theories with massive bosons [27], is found to naturally reside in the $(1/2, 1/2)$ representation space.

2. At the same time, the properties of the $(1/2, 1/2)$, along with that of the $(1/2, 0) \oplus (0, 1/2)$, representation space determine the structure of $\psi^\mu$. In order to impose a single-spin, i.e., spin $3/2$, interpretation on the latter, the lower spin-1/2* and spin-1/2- components of $\psi^\mu$ are considered as redundant, unphysical, states that are claimed to be excluded from consideration by means of the two supplementary conditions: $\gamma_\mu \psi^\mu(x) = 0$, and $\partial_\mu \psi^\mu(x) = 0$, respectively. However, this time-honored framework was questioned by a recent empirical observation regarding the $N$ and $\Delta$ resonances [25]. The available data on high-spin resonances reveal an unexpected and systematic clustering in terms of the $(j/2, j/2) \otimes [(1/2, 0) \oplus (0, 1/2)]$ representations with $j = 1, 3$ and 5 without imposition of the supplementary conditions. For the $N$ and $\Delta$ resonances these results are summarized in Fig. 2.

In regard to the latter of the two enumerated circumstances, we take the position that any solution of the QCD Lagrangian for particle resonances must carry well-defined transformation properties when looked upon from different inertial frames. This forces these resonances to belong to one, or the other, of various representation spaces of the Lorentz group. For this reason the data on particle resonances may furnish hints on physical interpretation of various Lorentz group representations that one needs in gauge theories, or theories of supergravity.

For exploring the spacetime structure of massive gravitinos the charge conjugation properties play an important role. Under the operation of charge conjugation, one may choose the spinor sector to behave as a Dirac object, and implement the Majorana nature of the massive gravitino at the level of the Fock space. This is standard, see, e.g., Ref. 28. Or, from the very beginning choose the spinor sector to behave as a Majorana object. Since we wish to stress certain non-trivial aspects of massive gravitino that do not—at least qualitatively—depend on this choice, we shall here treat the spinor sector to be of Dirac type.

Very nature of our ab initio look at the representation space defined in Eq. (68), obliges us to present sufficient pedagogic details so that by the end of the lecture much that is needed to form an opinion on the arrived results is readily available. At the same time, length constraints of this manuscript would prevent us from delving into subtle details which are, for present, of secondary importance (but have been studied and are planned to be presented elsewhere).

We shall work in the momentum space. The notation will be essentially that introduced in Ref. 1.
4.1. (1/2, 1/2) representation space—an ab initio construct

We have constructed in Sec. 2 the spinorial sector entering $\psi^\mu$ in Eq. (68). Therefore, our next task is to construct the (1/2, 1/2) representation space. As such, now we introduce the rest-frame vectors for the (1/2, 1/2) representation space,

$$\xi_1(0) = h^+ \otimes h^+,$$
$$\xi_2(0) = \frac{1}{\sqrt{2}} (h^+ \otimes h^- + h^- \otimes h^+),$$
$$\xi_3(0) = h^- \otimes h^-,$$
$$\xi_4(0) = \frac{1}{\sqrt{2}} (h^+ \otimes h^- - h^- \otimes h^+).$$

The boosted vectors are thus: $\xi_\zeta(p) = \kappa^{(1/2)} \xi_\zeta(0), \zeta = 1, 2, 3, 4,$

$$\xi_1(p) = \frac{N^2}{2} \begin{pmatrix}
2 \exp(-i\phi) \cos^2(\theta/2) \\
\sin(\theta) \\
\sin(\theta) \\
2 \exp(i\phi) \sin^2(\theta/2)
\end{pmatrix},$$
$$\xi_2(p) = \frac{N^2}{\sqrt{2}m} \begin{pmatrix}
\exp(-i\phi) E \sin(\theta) \\
-|p| + E \cos(\theta) \\
|p| - E \cos(\theta) \\
-\exp(i\phi) E \sin(\theta)
\end{pmatrix},$$
$$\xi_3(p) = \frac{N^2}{2} \begin{pmatrix}
2 \exp(-i\phi) \sin^2(\theta/2) \\
-\sin(\theta) \\
-\sin(\theta) \\
2 \exp(i\phi) \cos^2(\theta/2)
\end{pmatrix},$$
$$\xi_4(p) = \frac{N^2}{\sqrt{2}m} \begin{pmatrix}
\exp(-i\phi)|p| \sin(\theta) \\
-|p| + E \cos(\theta) \\
E - |p| \cos(\theta) \\
-\exp(i\phi)|p| \sin(\theta)
\end{pmatrix}. \quad (73)$$

Here $\kappa^{(1/2,1/2)} = \kappa^{(1/2,0)} \otimes \kappa^{(0,1/2)}$.

In the notation of Ref. 1(b), these satisfy the orthonormality and completeness relations, along with a new wave equation. The orthonormality and completeness relations are:

$$\xi_\zeta(p)\xi_{\zeta'}(p) = -N^4 \delta_{\zeta\zeta'}, \zeta = 1, 2, 3,$n$$
$$\xi_\zeta(p)\xi_{\zeta'}(p) = +N^4 \delta_{\zeta\zeta'}, \zeta = 4,$n$$

$$\frac{1}{N^4} \left[ \xi_4(p)\overline{\xi}_4(p) - \sum_{\zeta=1,2,3} \xi_\zeta(p)\overline{\xi}_\zeta(p) \right] = \mathbb{I}_4, \quad (74)$$

where $\overline{\xi}_\zeta(p) \equiv \xi_\zeta(p)^\dagger \lambda_{00}$,

$$\lambda_{00} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}. \quad (76)$$

The parity operator for the (1/2, 1/2) representation space is:

$$\mathcal{P} = \lambda_{00} \exp[i\alpha] \mathcal{R}, \mathcal{R}: \{ \theta \rightarrow \pi - \theta, \phi \rightarrow \pi + \phi \} \quad \alpha = \text{a real number},$$

while the helicity operator for this space is, $J \cdot \tilde{p}$, with $J$ given by:

$$J_x = \frac{1}{2} \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix},$$
$$J_y = \frac{1}{2} \begin{pmatrix}
i & -i & -i & 0 \\
i & 0 & 0 & -i \\
i & -i & 0 & 0 \\
i & 0 & 0 & 1
\end{pmatrix},$$
$$J_z = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}. \quad (78)$$

We now must take a small definitional detour towards the notion of the dragged Casimirs for spacetime symmetries. It arises in the following fashion. The second Casimir operator, $C_2$, of the Poincaré group is defined as the square of the Pauli-Lubanski pseudovector:

$$\mathcal{W}^\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} M_{\nu\rho} \Pi_{\sigma},$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the standard Levi-Civita symbol in four dimensions, while $M_{\mu\nu}$ denote generators of the Lorentz group,

$$M_{0i} = K_i, \quad M_{ij} = \epsilon_{ijk} J^k,$$ (80)

where each of the $i, j, k$ runs over 1, 2, 3. The $P_\mu$ are generators of the spacetime translations. In general, these have non-vanishing commutators with $M_{\mu\nu}$.

$$[P_\mu, M_{\rho\sigma}] = i (\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho). \quad (81)$$

On using Eq. (81), we rewrite $C_2$ as
The squared bracket vanishes due to antisymmetry of the Levi-Civita symbol. As such, the space-time translation operators entering the definition of $C_2$ can be moved to the very right. This observation allows for introducing the dragged Casimir $C_2$ as an operator with the same form as $C_2$—the difference being that the commutator in Eq. (81) is now set to zero [as is appropriate for finite dimensional $SU_R(2) \otimes SU_L(2)$ representations]. Consequently, while $C_2$ and $\tilde{C}_2$ carry same invariant eigenvalues when acting upon momentum eigenstates, their commutators with the Lorentz group generators are no longer identical.\(^{(1)}\) For the $(1/2,0) \oplus (0,1/2)$ representation space, $[C_2, J^2]$ vanishes. For the $(1/2,1/2)$ representation space, $[\tilde{C}_2, J^2]$ does not vanish (except when acting upon rest states), and equals $-4Ep \cdot K$. This leads to the fact that while the former representation space is endowed with a well-defined spin, the latter is not:

As an immediate application, $\tilde{C}_2$ for the $(1/2,1/2)$ representation space bifurcates this space into two sectors. The three states $\xi_{\zeta}(p)$ with $\zeta = 1, 2, 3$ are associated with the $\tilde{C}_2$ eigenvalue, $-2m^2$; while the, $\zeta = 4$, corresponds to eigenvalue zero.

Thus, all the $\xi_{\zeta}(p)$, except for the rest frame, cease to be eigenstates of the $(1/2,1/2)$'s $J^2$ and do not carry definite spins. This contrasts with the situation for the $(1/2,0) \oplus (0,1/2)$ representation space, where the $\psi_{\chi}(p)$ are eigenstates of the corresponding $J^2$.

Now in order that the $\xi_{\zeta}(p)$ carry the standard contravariant Lorentz index, we introduce a rotation in the $(1/2,1/2)$ representation space via [1]:

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & -i & 0 \\ -i & 0 & 0 & i \\ 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \end{pmatrix}$$

Then, the $(1/2,1/2)$ representation space is spanned by four Lorentz vectors:

$$A^\mu_{\zeta}(p) = S^{\alpha\mu} [\xi_{\zeta}(p)]_\alpha, \quad \zeta = 1, 2, 3, 4, \quad (84)$$

and the superscript $\mu$ is the standard Lorentz index. Note that the $\mathcal{W}_{\mu\nu}$'s by themselves are $4 \times 4$ matrices in Lorentz index space, i.e. $\mathcal{W}_{\mu\nu}^{\alpha\beta}$. Following the procedure established in Sec. 2, they can be shown to satisfy a new wave equation [1],

$$\left(A_{\mu\nu} p^\nu p^\mu - m^2 I_4 \right) A_{\zeta}(p) = 0, \quad (85)$$

where the plus sign is to be taken for $\zeta = 1, 2, 3$, while the minus sign belongs to, $\zeta = 4$. The $A_{\mu\nu}$ matrices are: $A_{00} = \text{diag}(1, -1, -1, -1)$, $A_{11} = \text{diag}(1, -1, 1, 1)$, $A_{22} = \text{diag}(1, 1, -1, 1)$, $A_{33} = \text{diag}(1, 1, 1, -1)$, and $A_{12} = \text{diag}(0, 0, 0, 0, 0)$.

The remaining $A_{\mu\nu}$ are obtained from the above expressions by noting: $A_{\mu\nu} = A_{\nu\mu}$. Parenthetically, we note that the $S$-transformed $A_{00}$ equals $A_{00}$ and is nothing but the standard spacetime metric (for flat spacetime).

The massive $(1/2,1/2)$ propagators that follow from the completeness relation within the $(1/2,1/2)$ representation space in Eq. (74) read [in the notations of Eq. (84)]

$$[A_{4}(\vec{p}) \tilde{A}_{4}(\vec{p}) A_{00}]_{\mu\nu} = \frac{P_\mu P_\nu}{m^2}, \quad (87)$$

$$- \sum_{\zeta=1}^{3} [A_{\zeta}(\vec{p}) \tilde{A}_{\zeta}(\vec{p}) A_{00}]_{\mu\nu} = g_{\mu\nu} - \frac{P_\mu P_\nu}{m^2}. \quad (88)$$

The massive $(1/2,1/2)$ propagator contains both the Stickelberg (87) and the Proca (88) terms [27].

This feature of the completeness relation within the $(1/2,1/2)$ representation space appears quite appealing to us as it automatically leads to a well behaved propagator of a massive gauge boson as arising in a spontaneously broken local gauge theory. Within the context of the scenario presented above, the Proca sector is characterized by vanishing of $P_\mu A_{\mu} = 0$, while the Stickelberg sector is characterized by vanishing of $P_\mu \mathcal{W}_{\mu} = 0$ (see Table 1).


One immediately realizes that the massive $(1/2,1/2)$ propagator contains both the Stickelberg (87) and the Proca (88) terms [27].

This feature of the completeness relation within the $(1/2,1/2)$ representation space appears quite appealing to us as it automatically leads to a well behaved propagator of a massive gauge boson as arising in a spontaneously broken local gauge theory. Within the context of the scenario presented above, the Proca sector is characterized by vanishing of $P_\mu A_{\mu} = 0$, while the Stickelberg sector is characterized by vanishing of $P_\mu \mathcal{W}_{\mu} = 0$ (see Table 1).
It can also be seen that \( \xi_1(p) \) for \( \zeta = 1, 2, 3 \), coincide with the solutions of Proca framework (and are divergence-less); whereas \( \xi_4(p) \), that gives the Stückelberg contribution to the propagator, lies outside the Proca framework (2) (see Table I).

In Table I we have introduced the \( \lambda_\tau \) via the equation:

\[
\tilde{C}_2^{(1/2,1/2)} \mathcal{A}(p) = -\lambda_\tau m^2 \mathcal{A}(p).
\]  

we obtain:

\[
C^{(1/2,0)@0(1/2)} : \begin{cases}
  u_{+1/2}(p) \rightarrow -v_{-1/2}(p), \quad v_{-1/2}(p) \rightarrow v_{+1/2}(p), \\
  u_{+1/2}(p) \rightarrow u_{-1/2}(p), \quad v_{-1/2}(p) \rightarrow -u_{+1/2}(p).
\end{cases}
\]

In the spirit outlined, the massive gravitino lives in a space spanned by sixteen spinor-vectors defined in items A, B, C below:

A. Of these, eight spinor-vectors have \( \tilde{C}_2 \)—but not \( J^2 \)—eigenvalues, \(-15/4m^2\). These can be further subdivided into particle,

\[
\psi_{+}^{\mu}(p) = \begin{cases}
  \psi_{+}^{\mu}(p) = u_{+1/2}(p) \otimes A_{+}^{\mu}(p), \\
  \psi_{+}^{\mu}(p) = \sqrt{2} u_{+1/2}(p) \otimes A_{+}^{\mu}(p) + \frac{1}{3} u_{-1/2}(p) \otimes A_{+}^{\mu}(p), \\
  \psi_{+}^{\mu}(p) = \sqrt{3} u_{+1/2}(p) \otimes A_{+}^{\mu}(p) + \frac{2}{3} u_{-1/2}(p) \otimes A_{+}^{\mu}(p), \\
  \psi_{+}^{\mu}(p) = u_{-1/2}(p) \otimes A_{+}^{\mu}(p),
\end{cases}
\]

and antiparticle sectors:

\[
[\psi_{+}^{\mu}(p)]^C : \begin{cases}
  \psi_{+}^{\mu}(p) = -v_{-1/2}(p) \otimes [A_{-}^{\mu}(p)]^*, \\
  \psi_{+}^{\mu}(p) = -\sqrt{2} v_{-1/2}(p) \otimes [A_{-}^{\mu}(p)]^* + \frac{1}{3} v_{+1/2}(p) \otimes [A_{+}^{\mu}(p)]^*, \\
  \psi_{+}^{\mu}(p) = -\frac{1}{3} v_{-1/2}(p) \otimes [A_{-}^{\mu}(p)]^* + \frac{2}{3} v_{+1/2}(p) \otimes [A_{+}^{\mu}(p)]^*, \\
  \psi_{+}^{\mu}(p) = v_{+1/2}(p) \otimes [A_{+}^{\mu}(p)]^*.
\end{cases}
\]

Here, \([\psi_{+}^{\mu}(p)]^C = C^{(1/2,0)@0(1/2)} \otimes C^{(1/2,1/2)} \psi_{+}^{\mu}(p), \) \( \tau = a, b, c. \)

B. Four spinor-vectors have \( \tilde{C}_2 \)—but not \( J^2 \)—eigenvalues, \(-3/4m^2\):

\[
\psi_{+}^{\mu}(p) = \begin{cases}
  \psi_{+}^{\mu}(p) = \sqrt{2} u_{-1/2}(p) \otimes A_{+}^{\mu}(p) - \frac{1}{3} u_{+1/2}(p) \otimes A_{+}^{\mu}(p), \\
  \psi_{+}^{\mu}(p) = \frac{1}{3} u_{-1/2}(p) \otimes A_{+}^{\mu}(p) - \frac{2}{3} u_{+1/2}(p) \otimes A_{+}^{\mu}(p),
\end{cases}
\]


TABLE II

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( p_\mu \psi_\mu^I(p) )</th>
<th>( \gamma_\mu \psi_\mu^I(p) )</th>
<th>( \tilde{W}<em>\mu^{1/2,1/2} \psi</em>\mu^I(p) )</th>
<th>( \lambda_\tau )</th>
<th>( \beta_\tau )</th>
<th>( \alpha_\tau )</th>
<th>Remarks</th>
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<td>0</td>
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<td>2</td>
</tr>
<tr>
<td>( b )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>3/4</td>
<td>2</td>
<td>(-2)</td>
</tr>
<tr>
<td>( c )</td>
<td>( \neq 0 )</td>
<td>( \neq 0 )</td>
<td>0</td>
<td>0</td>
<td>3/4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
[\psi_\mu^I(p)]^c = \begin{cases} 
\psi_{11}^I(p) = \sqrt{\frac{2}{3}} v_{+1/2}(p) \otimes [A_{+2}^I(p)]^* + \sqrt{\frac{1}{3}} v_{-1/2}(p) \otimes [A_{+1}^I(p)]^*, \\
\psi_{12}^I(p) = \sqrt{\frac{1}{3}} v_{+1/2}(p) \otimes [A_{0}^I(p)]^* + \sqrt{\frac{2}{3}} v_{-1/2}(p) \otimes [A_{-1}^I(p)]^*.
\end{cases}
\]

**C.** Another set of four spinor-vectors with \( C_2 \)—but not \( J^2 \)—eigenvalues, \(-\frac{3}{2}m^2\):

\[
\psi_\mu^I(p) = \begin{cases} 
\psi_{13}^I(p) = u_{+1/2}(p) \otimes A_{+2}^I(p), \\
\psi_{14}^I(p) = u_{-1/2}(p) \otimes A_{+1}^I(p),
\end{cases}
\]

\[
[\psi_\mu^I]^c = \begin{cases} 
\psi_{15}^I(p) = -v_{-1/2}(p) \otimes [A_{0}^I(p)]^*, \\
\psi_{16}^I(p) = v_{+1/2}(p) \otimes [A_{-1}^I(p)]^*.
\end{cases}
\]

We have evaluated \( \gamma_\mu \psi_\mu(p), \) \( p_\mu \psi_\mu(p), \) and \( \tilde{W}_\mu^{1/2,1/2} \psi_\mu(p), \) for all of the above sixteen spinor vectors. The \( p_\mu \psi_\mu(p), \) when transformed to the configuration space, tests the divergence of \( \psi_\mu(x). \)

For \( \eta = 1, 4, 5, 8, \) \( \gamma_\mu \psi_\mu(p) \) identically vanishes. Requiring it to vanish for \( \eta = 2, 3, 6, 7 \) results in \( E^2 = |p|^2 + m^2. \)

The \( \tau = b, c \) sectors, if (wrongly) imposed with the vanishing of \( \gamma_\mu \psi_\mu(p) \) and \( p_\mu \psi_\mu(p), \) results in kinematically acausal dispersion relation (i.e., in \( E^2 \neq |p|^2 + m^2). \) This could be the source of the well-known problems of the Rarita-Schwinger framework as noted in works of Johnson and Sudarshan [19], and those of Velo and Zwanziger [20]. In this context one may wish to recall that interactions can induce transitions between different \( \tau \) sectors.

The analysis for all the \( \tau \) sectors of the \( \psi_\mu(p) \) can be summarized in Table II.

Table II clearly illustrates that there is no particular reason—except (the unjustified) insistence that each particle of nature be associated with a definite spin—to favor one \( \tau \) sector over the other. Each of the \( \tau \) sectors is endowed with specific properties. The Rarita-Schwinger sector has no more, or no less, physical significance than the other two sectors. While, for instance, the Rarita-Schwinger sector can be characterized by vanishing of the \( p_\mu \psi_\mu(p) \) and \( \gamma_\mu \psi_\mu(p); \) the \( \tau = c \) sector is uniquely characterized by vanishing of \( \tilde{W}_\mu^{1/2,1/2} \psi_\mu(p). \) The \( \tau = b \) sector allows for vanishing of \( p_\mu \psi_\mu(p) \) only.

Except for the rest frame, the \( \psi_\mu^I(p) \), in general, are not eigenstates of the \( J^2 \) for representation space (68). Instead, the three \( \tau \) sectors of the representation space under consideration correspond to the following inertial-frame independent values of the associated dragged second Casimir invariant:

\[
\tilde{C}_2^{[1/2,0)[0(1/2)]}(1/2,1/2) \psi_\tau(p) = -m^2 \lambda_\tau \psi_\tau(p). \quad (93)
\]

Note that the latter equation is a matrix equation in the Lorentz index space. For this reason the index \( \mu \) of the spinor vector does not show up. Yet, \( \psi_\tau(p) \) is still the vector spinor and should not be confused with an ordinary Dirac spinor. Also there is no summation over the index \( \tau. \) In fact, the defining Eq. (93) of the various \( \psi^I_\mu(p) \) sectors of \( \psi_\mu(p) \) translates into a condition on \( \gamma_\mu \psi_\mu^I(p); \) to establish it, we note that \( \tilde{C}_2^{[1/2,0)[0(1/2)]}(1/2,1/2) \) is given as the squared sum of the (dragged) Pauli-Lubanski pseudovectors \( \tilde{W}_\mu^{1/2,0)[0(1/2)]} \) and \( \tilde{W}_\mu^{1/2,1/2}[0(1/2)] \). In other words, one has

\[
\tilde{C}_2^{[1/2,0)[0(1/2)]}(1/2,1/2) = (\tilde{W}_\mu^{1/2,0)[0(1/2)]} + \tilde{W}_\mu^{1/2,1/2}[0(1/2)])^2. \quad (94)
\]

The action of \( \tilde{C}_2^{[1/2,0)[0(1/2)]}(1/2,1/2) \) upon \( \psi_\tau(p) \) (with \( \tau = a, b, c \)) takes the form

\[
\tilde{C}_2^{[1/2,0)[0(1/2)]}(1/2,1/2) \psi_\tau(p) = \left( \tilde{C}_2^{[1/2,0)[0(1/2)]} + \tilde{C}_2^{[1/2,1/2)[0(1/2)]} + 2\tilde{W}_\mu^{[1/2,0)[0(1/2)][1/2,1/2]} \right) \psi_\tau(p)
\]

\[
= -m^2 \left( \frac{1}{2} + 1 \right) - m^2 \beta_\tau - m^2 \alpha_\tau) \psi_\tau(p)
\]

\[
= -m^2 \lambda_\tau \psi_\tau(p), \quad \beta_\tau = 0, 2. \quad (95)
\]

The latter equation shows that the values of $\alpha_\tau$ and $\beta_\tau$, which in turn determine the eigenvalues of $2\tilde{W}^{(1/2,1/2)}\tilde{W}^{(1/2,0)}\tilde{W}^{(1/2,0)}(0,1/2)$ and $\tilde{C}^{(1/2,1/2)}$ with respect to $\psi_\tau(p)$ are well suited to label the various $\tilde{C}_2^{(1/2,0)(0,1/2)(1/2,1/2)}$ sectors. Indeed, one can replace Eq. (93) by

$$2\tilde{W}^{(1/2,1/2)}\tilde{W}^{(1/2,0)}\tilde{W}^{(1/2,0)}(0,1/2)(0,1/2)\psi_\tau(p) = -m^2\alpha_\tau\psi_\tau(p),$$

$$\alpha_\tau = \lambda_\tau - \beta_\tau - \frac{3}{4}.$$

(96)

Insertion of the explicit expression for $\tilde{W}^{(1/2,0)(0,1/2)}$ into (96) and usage of the Dirac equation leads to

$$2\tilde{W}_\gamma^{(1/2,1/2)}\left(-\frac{1}{4}\gamma_\sigma[p, \gamma_\rho]\right)\psi_\tau(p) = -m^2\alpha_\tau\psi_\tau(p),$$

$$\tilde{W}_\gamma^{(1/2,0)}\gamma_5(p\gamma_\rho - \gamma_\rho p)\psi_\tau(p) = 2m^2\alpha_\tau\psi_\tau(p),$$

$$\tilde{W}_\gamma^{(1/2,1/2)}\gamma_5(2\gamma_\rho p_\lambda - 2\gamma_\rho p)\psi_\tau(p) = 2m^2\alpha_\tau\psi_\tau(p),$$

$$\left(\tilde{W}^{(1/2,1/2)}:p + \tilde{W}^{(1/2,1/2)}::\gamma\right)\gamma_5m\psi_\tau(p) = m^2\alpha_\tau\psi_\tau(p).$$

(97)

In taking Lorentz contraction of both sides of the last equation with $\gamma^i$ and in accounting for $\tilde{W}^{(1/2,1/2)}:p = 0$, one arrives at the following equation for the $\gamma \cdot \psi_\tau(p)$ spinor

$$\frac{1}{m_\alpha}\gamma^i\left(\tilde{W}_{\gamma\eta}^{(1/2,1/2)}:\gamma\gamma_{\eta\tau}\right)\psi_\tau^\mu(p) = \gamma \cdot \psi_\tau(p).$$

(98)

The non-relativistic counterpart of Eq. (98) reads

$$(1 + \frac{2}{m_\alpha})\Sigma \cdot \psi_\tau(p) = 0.$$  

(99)

For $\alpha_\alpha = 1$ one finds $\Sigma \cdot \psi_\alpha(p) = 0$ (corresponding to $\gamma \cdot \psi_\alpha = 0$) while for $\alpha_\beta = -2$, where the numerical factor in (99) vanishes, one encounters $\Sigma \cdot \psi_\beta(p) \neq 0$ (corresponding to $\gamma \cdot \psi_\beta \neq 0$).

For each of the $\tau$ sectors, the $\lambda_\tau$, $\beta_\tau$, and $\alpha_\tau$ are given in the table above. Stated differently, the boosted $\tau = b$, $c$ sectors do not carry spin one half. Similarly, the $\tau = a$, sector is not a spin three half sector. The consequence is that the boosted $\tau = b$, $c$ sector, in particular, should not be treated as a Dirac representation space. The correct wave equation for $\psi^\mu(p)$ is:

$$[(\gamma^\mu p^\rho \pm mI_4) \otimes (A_{\mu\nu}p^\rho p^\tau \pm m^2I_4)]_{\eta\tau} \psi_\tau^\mu(p) = 0.$$  

(100)

In the standard Rarita-Schwinger framework $\partial_\mu \psi^\mu(x)$ and $\gamma_5\gamma_\mu\psi^\mu(x)$ do indeed behave like Dirac spinors, and do indeed satisfy the Dirac equation. However, they are not identical to the $\tau = b$, $c$ sectors (which do not carry a characterization in terms of spin one half). If one (mistakenly) makes this identification, and sets $\partial_\mu \psi^\mu(x)$ and $\gamma_5\gamma_\mu\psi^\mu(x)$ to zero, one introduces an element of kinematic causality. The quantum numbers that are appropriate for labeling the basis vectors of the spinor-vector are

$$|\lambda_\tau\beta_\tau h\rangle,$$

(101)

where $h$ is the eigenvalue of the helicity ($J \cdot \hat{p}$) operator in the $\tau$ sector under consideration.

4.3. Interpretation of the massive gravitino as a particle of multiple spin

If one is to respect the mathematical completeness of the spinor-vector representation space associated with $\psi^\mu(x)$, the Rarita-Schwinger framework cannot be considered to describe the full physical content of the representation space associated with a massive gravitino. This circumstance is akin to Dirac's observation that a part of a representation space [which would have violated the completeness of the $(1/2,0) \oplus (0,1/2)$] cannot be "projected out" without introducing certain mathematical inconsistencies, and loosing its physical content (i.e. antiparticles, or particles)$^{(k)}$. Further, the same qualitative remarks apply to the $(1/2,1/2)$ representation space when in the Proca framework one only confines to the divergence-less vectors. The "projecting out" of the divergence-full vector, throws away the Stückelberg contribution to the propagator, and in addition leaves the $(1/2,1/2)$ representation space mathematically incomplete. Now, we suggest that for the representation space defined by Eq. (68), one needs to consider all three $\tau$ sectors of $\psi^\mu(x)$ as physical, and necessary for its mathematical consistency. The suggested framework already carries consistency with the known data on the $N$ and $\Delta$ resonances, and asks that massive gravitino be considered as an object that is better described by the eigenvalues of the dragged second Casimir operator. In its rest frame it is endowed with a spin three half, and two spin half, components. A spin measurement for unpolared ensemble of massive gravitinos at rest would yield the results 3/2 with probability one half, and 1/2 with probability one half. The latter probability is distributed uniformly, i.e. as one quarter, over each of the, $\tau = b$, and, $\tau = c$, sectors.

5. Outlook

The systematic and self-contained description of the spacetime structure of fundamental particles presented in this lecture notes calls for a more detailed analysis of the Majorana aspects of the particles participating the supermultiplets. Especially for left-right symmetric theories beyond the Standard Model, the profound difference between Dirac and Majorana spinors is expected to show up. There, the supersymmetric partners of the known particles can be expected to undergo particle-antiparticle metamorphosis and to give rise to physics beyond the standard model. Of further interest is to explore consequences of the multi-spin character of the massive gravitino for such phenomena like the renormalizability of supergravity and the phenomenology of the early universe.

Appendix

Derivation of Eq. (23)

Complex conjugating Eq. (29) gives,

\[ \sigma^* \cdot \hat{\mathbf{p}} \left[ \phi^+_L(0) \right]^* = \pm \left[ \phi^+_L(0) \right]^*. \]  

(102)

Substituting for \( \sigma^* \) from Eq. (22) then results in,

\[ \Theta \sigma \Theta^{-1} \cdot \hat{\mathbf{p}} \left[ \phi^+_L(0) \right]^* = \mp \left[ \phi^+_L(0) \right]^*. \]  

(103)

But \( \Theta^{-1} = -\Theta \). So,

\[ -\Theta \sigma \Theta \cdot \hat{\mathbf{p}} \left[ \phi^+_L(0) \right]^* = \mp \left[ \phi^+_L(0) \right]^*. \]  

(104)

Or, equivalently,

\[ \Theta^{-1} \sigma \Theta \cdot \hat{\mathbf{p}} \left[ \phi^+_L(0) \right]^* = \mp \left[ \phi^+_L(0) \right]^*. \]  

(105)

Finally, left multiplying both sides of the preceding equation by \( \Theta \), and moving \( \Theta \) through \( \hat{\mathbf{p}} \), yields Eq. (30).

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