Nambu mechanics, odd dimensional phase space and gradient systems

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In the general setting of the Nambu dynamical system in odd dimensional phase space two simple cases are discussed: the Nambu oscillator and the conditions under which a gradient system coincides with a Nambu system.

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The dynamical system proposed by Nambu in [5] generalizes the Hamiltonian scheme allowing for even and odd phase space dimension N. Phase space is spanned by the N variables \( x = (x_1, \ldots, x_N) \). The evolution of a dynamical variable \( F(x) \), is determined once a set of \( N - 1 \) functions, \( H_1, \ldots, H_{N-1} \), is given. The evolution equation for \( F(x) \) is

\[
\frac{dF(x)}{dt} = \frac{\partial(F(x), H_1, \ldots, H_{N-1})}{\partial(x_1, \ldots, x_N)} = \{F, H_1, \ldots, H_{N-1}\},
\]

where \( \partial(\ldots)/\partial(\ldots) \) is a Jacobian of order N. The right hand side of (1) is the Nambu bracket which is antisymmetric and a derivation over the functions on phase space. The Hamiltonians \( H_k, k = 1, \ldots, N - 1 \) are constants of the motion, \( dH_k/dt = 0 \).

Various aspects of the Nambu dynamical system have been considered in Refs. 1-4,6-10. The present work concentrates on the study of an oscillating system and on conditions under which the Nambu system can be cast into the form of a gradient system, namely, one whose evolution equations for the coordinates that span phase space are

\[
\frac{dx_j}{dt} = -\partial_j \Phi,
\]

where \( \Phi \) is a function of the \( x_j \). An oscillating system is defined by the set of differential equations

\[
\frac{dx_j}{dt} = A_{jk} x_k
\]

and whose solutions are \( x_j = \exp(C_j \omega t) \). The eigenvalues \( \omega \) are easily determined. Oscillations will be present if the roots of the characteristic polynomial are all real -this corresponds to no damping. The main result is that one of the \( \omega \) vanishes due to the vanishing of the free term in the characteristic polynomial; its vanishing is a consequence of the antisymmetry of the matrix that defines the system (3) in any odd dimensional phase space.

Now we turn to gradient systems in three dimensional phase space. The tangent vector at \( x = (x_1, x_2, x_3) \) points in the direction of decreasing values of \( \Phi \) and the trajectory that satisfies (2) intersects the level surface \( \Phi = C, C \) constant, orthogonally. The time evolution of \( \Phi \),

\[
\frac{d\Phi}{dt} = -|\nabla \Phi|^2,
\]

shows that \( \Phi \) is decreasing along the trajectory attaining its minimum value when \( t \to \infty \) if the limit exists.

Remark When phase space is bidimensional the gradient system (2) is Hamiltonian if the Hamiltonian function \( K \) and \( \Phi \) are the real and imaginary parts of an analytic function. In fact, the equations to be satisfied are \( -\partial_1 \Phi = \partial_2 K \), \( -\partial_2 \Phi = \partial_3 K \) which are the Cauchy-Riemann equations.

If (1) for \( x \) is such that

\[
\frac{dx}{dt} = \nabla H(x) \times \nabla G(x) = -\nabla \Phi(x)
\]

it is called a Nambu-gradient system (NGS); (5) shows that \( H, G \) and \( \Phi \) determine a local coordinate system at each point of phase space which is undefined only at critical points of the dynamical system, a situation that occurs when the level surfaces of \( H \) and \( G \) are tangent to each other. If (5) is satisfied the Liouville condition \( \partial_i \partial_i = 0 \) implies \( \nabla^2 \Phi = 0 \). The questions answered in this part are: given \( H \) and \( G \) what conditions determine \( \Phi \)? Conversely, given \( \Phi \), under what conditions \( H \) and \( G \) can be determined? Stability is studied concentrating on the determination of a Liapunov function.
A Nambu system is a NGS if the integrability conditions
\[ \epsilon_{jkm} \partial_i [\partial_k H \partial_m G] = \epsilon_{ikm} \partial_j [\partial_k H \partial_m G] \] (6)
are satisfied and which, after simple manipulations, leads to
\[ \partial_i G \nabla H - \partial_i H \nabla G = \nabla \times A_i, \] (7)
with suitable vectors \( A_i \). Introducing \( U = (1, 1, 1) \) and \( D = A_1 + A_2 + A_3 \) it is found
\[ U \cdot \nabla G \nabla H - U \cdot \nabla H \nabla G = \nabla \times D, \] (8)
which after use of (5) gives for \( D \):
\[ D = \Phi U + \nabla F, \] (9)
with \( F \) an arbitrary function. Conversely, given \( \Phi \) and \( G \), from
\[ \nabla H \times \nabla G = -\nabla \Phi \] (10)
it follows ( \( \Lambda(x) \) arbitrary function)
\[ \nabla H = -\frac{\nabla G \times \nabla \Phi}{|\nabla G|^2} + \Lambda(x) \nabla G. \] (11)
This equation determines \( H \).

Now attention is paid to stability. Equilibrium points are defined by \( \nabla H \times \nabla G = -\nabla \Phi = 0 \); assume \( P \) is such a point. A Liapunov function \( V \) determines stability if in a neighborhood \( U \) of \( P \)
\( V(P) = 0 \) and in \( U - \{ P \} \), \( V > 0 \) and \( dV/dt \leq 0 \). In case \( dV/dt < 0 \) \( P \) is asymptotically stable
and \( V \) is a strict Liapunov function. Explicit computation of \( dV/dt \) leads to
\[ \frac{dV}{dt} = \partial_i V \frac{dx_i}{dt} = -\nabla V \cdot \nabla \Phi, \] (12)
so that if \( P \) is an isolated minimum of \( V \), then \( V = \Phi - \Phi(P) \) is a strict Liapunov function (this follows from (4)) and \( P \) is, therefore, asymptotically stable.

For a gradient system, linearization of \( -\nabla \Phi \) in a neighborhood of \( P \) defines a self-adjoint operator; its eigenvalues are real. When this result is used for a Nambu-gradient system conditions on \( H \) and \( G \) arise. These are described now. The linearized operator for \( -\nabla \Phi \) is given by the Jacobian matrix \( M \) whose matrix elements are \( M_{ij} = -\partial_i \partial_j \Phi \). The Liouville condition implies that \( \text{tr}(M) = 0 \) since \( \Phi \) is harmonic; the Jacobian matrix \( L \) associated to \( \nabla H \times \nabla G \) is also traceless. Symmetry of \( M \) (\( M_{ij} = M_{ji} \)) has a counterpart on \( L \) (\( L_{ij} = L_{ji} \)) which follows from the integrability conditions (6). Moreover, (5) ensures that both operators indeed coincide, \( L = M \). Once \( H \) and \( G \) satisfy (8) with \( D \) given in (9) the operator that linearizes \( \nabla H \times \nabla G \) is represented by a symmetric matrix which coincides with the linearization of \( -\nabla \Phi \).