Importance of symmetric spaces in the determination of masses

G. González-Martín
Departamento de Física, Universidad Simón Bolívar;
Apartado 89000, Caracas 1080-A, Venezuela,
e-mail: ggonzalm@usb.ve; web page: http://prof.usb.ve/ggonzalm/

Recibido el 11 de diciembre de 2001; aceptado el 24 de abril de 2002

The volumes of certain associated symmetric spaces determines the numerical ratio of masses of geometrical excitations (particles), calculated according to a previously proposed geometric definition of mass in terms of energy, in a geometrical unified theory.

Keywords: Mass ratios; geometry; groups; geometric mass.

Los volúmenes de ciertos espacios asociados determinan los cocientes numéricos de masas de excitaciones geométricas (partículas), calculadas de acuerdo a una definición geométrica de masa previamente propuesta en términos de energía, dentro de una teoría unificada geométricamente.

Descriptores: Cocientes de masa; geometría; grupos; masa geométrica.

PACS: 02.20.Qs; 04.50.+h; 14.60.-z

1. Bare masses

We have presented a definition of mass [1], within a geometric relativistic unified theory of gravitation and other interactions [2], in terms of the concept of selfenergy of the nonlinear selfinteraction in a geometric space, leading to the mass term in Dirac’s equation. The definition of the mass parameter $m$, in terms of a connection $\Gamma$ on the principal fiber bundle $(E, M, G)$, has been given in the fundamental defining representation of $\text{SL}(4, \mathbb{R})$ in terms of $4 \times 4$ matrices, but in general, may be written for other representations using the Cartan-Killing metric $g_C$:

$$m = \frac{1}{4} \text{tr} J^\mu \Gamma_\mu = \frac{\text{tr} J^\mu \Gamma_\mu}{\text{tr} I_A} \equiv \frac{g_C (J^\mu \Gamma_\mu)}{g_C (I_{D(A)})} \quad (1)$$

If we consider geometric excitations on a background, this mass may be expanded as a perturbation around the background in terms of the only small parameter, the coupling constant $\alpha$, indicating that the zeroth order term is given entirely by a background current and connection, with corrections depending on the excitation selfinteraction. As indicated in previous work [2, 3], these corrections correspond to a geometric quantum field theory. In this article we limit ourselves to the zeroth order term which we consider the bare mass of QFT.

The structure group $G$ is $\text{SL}(4, \mathbb{R})$ and the even subgroup $G_+$ is $\text{SL}(2, \mathbb{C})$. The subgroup $L$ is the subgroup of $G_+$ with real determinant in other words, $\text{SL}(2, \mathbb{C})$. There is another subgroup $H$ in the group chain $G \supset H \supset L$ which is $\text{Sp}(4, \mathbb{R})$. We are dealing with two cosets $G/G_+$ and $H/L$ which we shall designate respectively as $K$ and $C$. These groups have a principal bundle structure over the cosets and themselves carry representations.

The fields may be decomposed in terms of a set of basis functions labeled by a parameter $k$, the generalized spherical functions $Y_k$ on the symmetric space [4]. If $K$ were compact, the basis of this function space would be discrete, of infinite dimensions. Since the spaces under discussion are noncompact, the labels $k$ become continuous functions and the summation in matrix multiplication becomes integration over the continuous parameter $k$.

$$m = \frac{1}{4V(A_R)} \text{tr} \int dk \int J(k, k_2) \Gamma(k_2, k) dk_2. \quad (2)$$

2. Volumes of spaces

There is a constant local solution [5] for the nonlinear differential equations that provides a trivial connection. This local trivial solution may generate global scattering solutions with different topology according to the third homotopy group of the transition functions of the manifold. The excitations, although locally defined around a local trivial section (solution), are related by transition functions to associated topological global solutions. These transition functions are characterized by an entire number called winding number or wrapping number $n$.

The integrand $J\Gamma$ is a local constant, equal for all values of $n$. Integration is on a subspace $K_R \subset C_K$ of relativistic inequivalent points of $K$ for the group $G$ and on a subspace $C_R \subset C_K$ for the group $H$. When integrating over trivial local states $n = 0$, the expressions for the masses is [5],

$$m_G = \frac{m_g}{4V(A_R)} \text{tr} \int_{K_R} F(k, k) dk = \frac{V(K_R)}{4V(A_R)} m_g \text{tr} (F_0), \quad (3)$$

$$m_H = \frac{m_g}{4V(A_R)} \text{tr} \int_{C_R} F(k, k) dk = \frac{m_g}{4V(A_R)} \text{tr} (F_0) V(C_R) = \frac{m_g V(C_R)}{V(K_R)}. \quad (4)$$
For a topological excitation, the physical count should be over states with \( n \neq 0 \). A topological excitation may be considered as a wrapping of \( n + 1 \) sections over \( C_R \), homeomorphic to \( S^3 \), in the group \( H \) fiber bundle as will be discussed in a paper under preparation. Define the space \( C^* \) as the product of \( n + 1 \) spaces, copies of the original \( C_R \) space, corresponding to the wrapping \( n \). The group \( U(1) \) acts on the manifold charts varying \( \beta \). We should integrate over all possible inequivalent \( C^* \), determined by the volume of the \( U(1) \) group. The total space \( C^T \) for an \( n \)-excitation \( (n \neq 0) \) is

\[
C^T = U(1) \times C^* .
\]  

(5)

The total number of states (different values of \( k \)) is proportional to the volume of this total space \( C^T \), given by

\[
V(C^T) = V(U(1)) \times \left(V(C_R)^{n+1}\right) n \neq 0.
\]  

(6)

3. Physical mass ratios

The volumes of \( C \) an \( K \) are previously calculated \([5]\). We have to eliminate the equivalent states by dividing by the equivalence relation \( R \) under the boosts of \( \text{SO}(3,1) \). Equivalent points are related by a Lorentz boost transformation of magnitude \( \beta \). There are as many equivalent points as the volume of the orbit developed by the parameter \( \beta \). The respective inequivalent volumes are,

\[
V(C_R) = \frac{V(C)}{V(R(\beta))} = \frac{16\pi}{3} I_C(\beta) = \frac{16\pi}{3}, \quad V(K_R) = \frac{V(K)}{V(R(\beta))} = \frac{2^5\pi^6 I_K(\beta)}{I_K(\beta)} = 2^5\pi^6. \]  

(7)

The ratio of masses for these geometric excitations has the finite exact value,

\[
\frac{m_C}{m_H} = \frac{V(K_R)}{V(C_R)} = 6\pi^5 = 1836.1181 \approx \frac{m_\mu}{m_e}, \quad (9)
\]

which is a very good approximation for the ratio of the experimental physical values for the proton mass and electron masses. This geometrical expression for this mass ratio has been known \([6–8]\) but not physically explained.

We should relate the \( G \) group to the proton and the \( H \) group to the electron. The only other dynamical subgroup \( L= \text{SL}(2,\mathbb{C}) \) of \( G \) should lead to a similar mass ratio. Previously we have related \( L \) to the neutrino. In this case the quotient space is the identity and we get for the bare mass,

\[
\frac{m_L}{m_H} = \frac{V_R(L/L)}{V_R(C)} = \frac{V_R(I)}{V_R(C)} = 0 = \frac{m_\mu}{m_e}, \quad (10)
\]

The bare mass of the trivial \( H \)-excitation \( (n = 0) \) is proportional to the volume of \( C_R \), Eq. (7). The bare masses of the \( n \)-excitations \( (n \neq 0) \) depend on powers of this volume. For excitations with wrapping numbers 1 and 2 we obtain

\[
\frac{m_2}{m_1} = \frac{16\pi}{3} = 16.76 \approx \frac{m_\mu}{m_\mu} = 16.88, \quad (11)
\]

\[
m_1 = m_0 \left(\frac{16\pi}{3}\right)4\pi = 0.5110041 \times 210.5516 \approx 107.5927 \text{ MeV} \approx m_\mu + O(\alpha) \quad (12)
\]

\[
m_2 = m_0 \left(\frac{16\pi}{3}\right)^24\pi = 0.5110041 \times 3527.825 \approx 1802.7 \text{ MeV} \approx m_\tau + O(\alpha), \quad (13)
\]

which correspond to the leptons \( \mu \) and \( \tau \) as was previously suggested \([9]\). The bare masses for the corresponding neutrinos are zero. The necessary corrections are of order \( \alpha \).