New results concerning the so(2, 1) treatment for the hypergeometric Natanzon potentials

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The so(2, 1) analysis for the bound state sector of the hypergeometric Natanzon potentials (HNP) is extended to the scattering sector by considering the continuous series of the so(2, 1) algebra. As a result a complete algebraic treatment of the HNP by means of the so(2, 1) algebra is achieved. In the bound state sector we discuss a set of satellite potentials which arises from the action of the so(2, 1) generators. It is shown that the set of new potentials are not related to the one obtained by means of SUSYQM or of the potential algebra approach using the so(2, 2) algebra.

Keywords: Hypergeometric Natanzon potentials; so(2, 1) algebra; scattering; satellite potentials.

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1. Introduction

Algebraic techniques have been developed in the last decade to describe the bound state sector for Natanzon potentials [1] in their two forms; the confluent and the hypergeometric ones. The HNP was treated by means of an so(2, 2) algebra, the so called potential group approach developed in Ref. 2. More recently the HNP was analyzed using the so(2, 1) algebra [3], the confluent case also admits an a so(2, 1) algebra in the description of the bound sector [4].

The scattering sector for Coulomb problem has also been treated by using group theoretic methods long time ago [5]. Later on, in Ref. 6 a technique was developed for systems whose Hamiltonian may be written as a function of the Casimir invariant of an so(2, 1) algebra. They have developed a purely algebraic technique for the calculation of the $S$ matrix, which they call Euclidean connection, after noticing that scattering states are related to the eigenstates of the $c(2)$ algebra. The same authors, [2] where able to extend the Euclidean connection to deal with the scattering problem for the HNP. Another point of view for deal with systems whose Hamiltonian can be written in terms of an so(2, 1) algebra was developed in Ref. 7.

In this article we extend the algebraic treatment of the bound states of the HNP, developed in Ref. 3, to the scattering sector. Analyzing the asymptotic behavior of this particular realization of so(2, 1) and using the formalism developed in Ref. 8, we derive the $S$ matrix for the HNP. The simplicity of this treatment is stressed out.

The organization of this work is as follows:

a) A brief resume of the so(2, 1) analysis for the bound state sector for the HNP. An example is developed, the Pöschl-Teller potential.

b) The asymptotic algebra is shown to be an so(2, 1) algebra. Then the scattering of the Pöschl-Teller potential is analyzed.

c) The general case of the HNP is treated. Finally we discuss new satellite potentials related with the approach given in Ref. 3.

2. The bound state sector

The HNP are given by [1]

$$V(z) = f z^2 - (h_0 - h_1 + f) z + h_0 + 1 R(z)$$

$$+ \left( a + \frac{a + (c_1 - c_0) (2 z - 1)}{z (z - 1)} - \frac{5}{4} \frac{\Delta}{R(z)} \right) \times \frac{z^2 (1 - z)^2}{R(z)^2}, \quad (1)$$

where

$$R(z) = a z^2 + \tau z + c_0, \quad \tau = c_1 - c_0 - a, \quad \Delta = \tau^2 - 4 a c_0 \quad (2)$$

The constants $f$, $h_0$, $h_1$, $a$, $c_0$, $c_1$ are called Natanzon parameters. The function $z$ is supposed to depend on the vari-
able r and satisfies
\[ \frac{dz}{dr} = \frac{2z(z-1)}{\sqrt{R(z)}}, \]
the function z is restricted to [0, 1] and the transformation \( r(z) \) is assumed to carry \( r \to \infty \) to \( z = 1 \), \( r \to 0 \) to \( z = 0 \) [11].

In the algebraic description of the HNP using an \( so(2,1) \) algebra [3] a two-variable realization of the algebra is used. The Hamiltonian \( H \) is related to the Casimir invariant \( \nu \) via \( (Q - q)\Psi(r, \phi) = G(r)(E - H)\Psi(r, \phi) \), where \( q \) and \( E \) are the eigenvalues of \( Q \) and \( H \). \( G(r) \) is a function fixed by consistency. The eigenfunctions of the Hamiltonian have the form \( \Psi(r, \phi) = e^{im\phi} \Phi(r) \). The realization of \( so(2,1) \) used is
\[
J_0 = -i \frac{\partial}{\partial \phi}, \\
J_\pm = e^{\pm i\phi} \left( \pm \frac{\sqrt{z}(z-1)}{z'} \frac{\partial}{\partial r} - \frac{i}{2} \frac{\partial}{\partial \phi} + \frac{1}{2} \frac{\partial}{\partial \phi} \right) + \frac{1}{2} \frac{\partial}{\partial \phi} + \frac{1}{4} (z-1)^2 \\
\times \left[ \frac{z^2(2z^3z'-3z^2) - z^4(p^2-1)}{z z'^4} \right],
\]
where \( z' = dz/dr \) and \( p \) is a function of the Natanzon parameters. These generators satisfy the usual commutation relations of the \( so(2,1) \) algebra: \( [J_0, J_\pm] = \pm J_\pm \), \( [J_+, J_-] = -2J_0 \).

The Casimir operator \( Q \) turns out to be
\[
Q = \frac{z(z-1)}{z'^2} \frac{\partial^2}{\partial r^2} + \frac{1}{4} \frac{\partial^2}{\partial \phi^2} + \frac{1}{z'} \frac{\partial}{\partial r} \left( \frac{i}{2} \frac{\partial}{\partial \phi} \right) + \frac{1}{4} (z-1)^2 \\
\times \left[ \frac{z^2(2z^3z'-3z^2) - z^4(p^2-1)}{z z'^4} \right],
\]
where \( \nu \) is an integer, the same one that occurs in the energy spectra. The eigenvalues of the compact generator \( J_0 \) are known to be
\[
m_\nu = \nu + \frac{1}{2} + \sqrt{q_\nu + \frac{1}{4}}, \quad \nu = 0, 1, \ldots,
\]
where \( q_\nu \) are the eigenvalues of the Casimir operator \( Q \). The energy spectra is given by
\[
2\nu + 1 = \alpha_\nu - \beta_\nu - \delta_\nu,
\]
where
\[
\alpha_\nu = \sqrt{-aE_\nu + f + 1} = p_\nu + m_\nu \\
\beta_\nu = \sqrt{-aE_\nu + h_0 + 1} = p_\nu - m_\nu \\
\delta_\nu = \sqrt{-c_1 E_\nu + h_1 + 1} = \sqrt{4q_\nu + 1}.
\]
The set \( \{p_\nu, q_\nu, m_\nu\} \) are called group parameters, these label the states of the system. The carrier space for the given representation is
\[
\Phi_{p_q,m} = K z^{\beta_\nu/2}(1-z)^{(\delta_\nu+1)/2} \\
\times_2F_1(-\nu,\alpha_\nu - 1, 1 + \beta_\nu, z),
\]
where \( K \) is a normalization constant.

To fix ideas let us consider as an example the Pöschl-Teller potential given by
\[
V_{pt} = -A(A+1)sech(r)^2 + B(B-1)csch(r)^2.
\]
It is a simple task to verify that the Natanzon parameters given by
\[
a = c_0 = 0, \quad c_1 = 1 \\
h_0 = \frac{(2B+1)(2B-3)}{4} \\
h_1 = -1, \quad f = \frac{(2A-1)(2A+3)}{4},
\]
where \( c_\nu = \text{inpart}(A-B)/2 \), we assume \( A > B \) in order to have bound states. The result given in (12) is obtained by a careful study of the ambiguities occurring in (8) due of the signs of the square roots involved, the main point is that the energy should increase with \( \nu \). The group parameters are
\[
p_\nu = \frac{A + B}{2}, \quad m_\nu = \frac{A - B + 1}{2} \\
q_\nu = \frac{(2\nu + 1 - A + B)(2\nu - 1 - A + B)}{4}.
\]
The generators and the Casimir operator are obtained from (4) and (11), we obtain
\[
J_{pt,\pm} = \left( \pm \frac{\partial}{\partial r} - \frac{i(1 + \tan(r)^2)}{2\tan(r)} \frac{\partial}{\partial \phi} \right) + \frac{(2p \pm 1)\tan(r)^2 - 2p \pm 1}{4\tan(r)} \exp(\pm i\phi)
\]
and
\[
Q_{pt} = \frac{1}{4} \frac{\partial^2}{\partial r^2} - \frac{i p_\nu(1 - \tan(r)^4)}{2\tan(r)^2} \\
+ \frac{(1 - \tan(r)^2)^2}{4\tan(r)^2} \frac{\partial^2}{\partial \phi^2} + \frac{(1 - 4p_\nu^2)(1 + \tan(r)^4)}{16\tan(r)^2} + \frac{(8p_\nu^2 - 6)\tan(r)^2}{16\tan(r)^2}.
\]

With these results the bound sector has a complete group description. The next step is an algebraic treatment of the scattering sector for the example given.
3. Scattering sector

We have seen in the previous section that the algebra for describing the bound state sector is an \( \text{so}(2,1) \) one. For the scattering sector we first analyze the case of the Pöschl-Teller potential. Following the ideas developed in Ref. 8, one can ask for the asymptotic limit, \( r \to \infty \), of the bound state algebra. One can guess that the limiting algebra could be suitable to describe the scattering sector, we are going to see that indeed this is the case. We define the asymptotic algebra as the limits of the one given in (4), after a straightforward calculation we obtain

\[
J^\pm_\alpha = \exp(\pm i\phi) \left[ -i \frac{\partial}{\partial \phi} \mp \frac{\sqrt{c_{1}}}{2} \frac{\partial}{\partial r} \pm \frac{1}{2} \right]
\]

\[
J^\alpha_0 = -i \frac{\partial}{\partial \phi}.
\]

The operators given in (16) close in an \( \text{so}(2,1) \) algebra. Their Casimir operator is

\[
Q^\infty = \frac{1}{4} \left[ c_1 \frac{\partial^2}{\partial \phi^2} - 1 \right],
\]

we notice that the asymptotic generators obtained from (4) are \( p \) independent as one expects. The example developed in the previous section correspond to \( c_1 = 1 \). This results are easily obtained from (14) and (15) when \( r \to \infty \). From (1) we obtain that the asymptotic behavior of the HNP is given by

\[
V^\infty_{\infty}(z) = \frac{b_1 + 1}{c_1},
\]

then it is necessary to choose \( b_1 = -1 \), in order to satisfy the conditions given in Ref. 1, this condition is satisfied as seen from (11).

Let us consider the continuous series of \( \text{so}(2,1) \) for which the eigenvalues of the Casimir operator, \( q = j(j + 1) \) are such that \( j \) is given by [9]

\[
j = \frac{1}{2} + \frac{\lambda}{2}, \quad \lambda \text{ real}
\]

and the compact generator has eigenvalues: \( m = m_0 \pm \sigma, \quad \sigma = \text{integer} \).

The asymptotic states [6,10] are given by

\[
|j, m\rangle^\infty = A_m \exp(\lambda r + m \phi) + B_m \exp(\mp \lambda r + m \phi),
\]

where the coefficients or the Jost functions \( A_m \) and \( B_m \) has to be evaluated. This can be done if we use the expression for \( J^\infty_+ \) given in (16) and then act on the asymptotic state (20). The result obtained should be compared with the general expression for the action of generators of an \( \text{so}(2,1) \) algebra, namely

\[
J_\pm |j, m\rangle = \sqrt{(m \mp j)(m \pm j + 1)} |j, m \pm 1\rangle.
\]

Recursion relations are obtained for \( A_m \) and \( B_m \), the reflection coefficient, \( R_m = A_m / B_m \), is found to be

\[
R_m = \frac{\Gamma\left(-\frac{1}{2} \frac{\lambda}{2} + m + \frac{1}{2}\right) \Gamma\left(\frac{1}{2} \frac{\lambda}{2} + \frac{1}{2}\right) A_0}{\Gamma\left(-\frac{1}{2} \frac{\lambda}{2} + m + \frac{1}{2}\right) \Gamma\left(\frac{1}{2} \frac{\lambda}{2} + \frac{1}{2}\right) B_0}.
\]

Assuming that \( A_0 \) and \( B_0 \) are holomorphic functions of \( \lambda \), it is then easy to verify that the poles of (22) indeed reproduce the spectra given in (12), after using (13). The general case for the algebraic treatment of the HNP is the same, since it is obtained by scaling the variable \( r \) as is seen from (16) and (17). Concerning to an Euclidean connection analysis for the scattering of the HNP using an \( \text{so}(2,1) \) algebra see Ref. [11].

As a final comment concerning to the scattering sector, the deformed scattering [12], can be done using the method developed here and in Ref. 11 in simple way, work is in progress.

4. Satellite potentials

Let us see an interesting feature concerning the algebraic description of the bound state sector mentioned before. Let us denote by \( H_{p_{\nu}q_{\nu}} \), the carrier space of \( \text{so}(2,1)_{p_{\nu}q_{\nu}} \). Thus the eigenfunctions of HNP belong to the direct sum of this spaces with \( \nu = 0 \) to \( \nu = \nu_{\max} \). For a specific state, \( \Psi_{p_{\nu}q_{\nu}m_{\nu}} \), which belongs to a carrier space label by \( p_{\nu} \) and \( q_{\nu} \), one can ask for the result of the ladder operators given in (4) acting on this state, the result is [13]

\[
J_\pm \Psi_{p_{\nu}q_{\nu}m_{\nu}} = \frac{\mathrm{e}^{\pm \nu |\alpha_{\nu} - \beta_{\nu}|}}{1 + |\beta_{\nu}|} \Psi_{p_{\nu}q_{\nu}m_{\nu} \pm 1},
\]

We obtain states corresponding to different group parameters, these states are eigenfunctions of a different HNP, we call them satellite potentials. The problem then is to find their Natanzon parameters. From (23) we see that the new state has the same \( z \), thus \( \{a, c_1\} \), \( p_{\nu} \) and \( q_{\nu} \) are unchanged while \( m_{\nu} \to m_{\nu} \pm 1 \). To be more specific, let us deal with the action of \( J_+ \). From (8) we obtain

\[
\alpha_{\nu+1} = \alpha_{\nu} + 1, \quad \beta_{\nu+1} = \beta_{\nu} - 1, \quad \delta_{\nu+1} = \delta_{\nu}.
\]

The index \( s \) is used to label the satellite potentials. Complicated relations for the new set of Natanzon parameters \( f_s \), \( h_{b_{\pm}}, h_{a_{\pm}} \), are obtained. For simplicity we study the potential given in (10) and we add a constant for convenience, namely

\[
V_{PT} = -A(A+1)\mathrm{sech}(r)^2 + B(B-1)\mathrm{sech}(r)^2 + (A-B)^2.
\]

In this case the Natanzon parameters are the same ones given in (11) except that now \( h_1 \) is given by \( h_{1a} = -(A+B-1)(-A+B+1). \) The group parameters are the same as in (13). For the energy spectra it is found

\[
E_{PT}(\nu) = 4\nu(\nu - A + B).
\]
From (8) we obtain
\[ \alpha_\nu = \frac{2A + 1}{2}, \quad \beta_\nu = \frac{2B - 1}{2}, \quad \delta_\nu = \frac{A - B - 2\nu}{2} \]
(27)

Let us define \( A_s \) and \( B_s \) as the parameters of the satellite potential, then from (27) and (24) we have
\[ A_s = A + 1, \quad B_s = B - 1 \]
(28)

The energy spectra of the satellite potential is obtained from the last equation in (8) and the result is
\[ E_s(\nu + 1) = E_{PT}(\nu) - h s 1 - (A - B)^2 \]
(29)

We notice that the change of parameters obtained in (28) are not the same occurring in SUSYQM [14], neither in the so(2, 2) potential approach [6]. To see clearly this effect, consider the case where \( B = 0 \). Thus, with this technique one generates HNP potentials from a seed that can be analyzed by the algebraic technique developed in Ref. 4. The other cases of shape invariant potentials can easily done by the same method developed in this note.

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