Symplectic structure and time reversal operation for TQFT

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We present a general approach to study TQFT in terms of general gauge fixing functions. The theories are formulated on a generalized phase space and a symplectic structure is introduced. Following Witten's ideas we construct a time reversal operation T and the corresponding inner product is defined. Then we present a non-covariant gauge fixing that gives rise to a manifestly time reversal invariant Lagrangian and a positive definite Hamiltonian. First we discuss the Witten’s TQFT and then a topological noncommutative theory.

Keywords: Topological field theories; BRST formulation.

Presentamos un tratamiento general para TQFT en términos de una fijación general de calibre. Las teorías son formuladas en un espacio de fase generalizado e introducimos una estructura simpléctica. Siguiendo ideas de Witten se construye una operación de inversión temporal T y definimos el correspondiente producto. Se muestra una selección de calibre no covariante que permite obtener un lagrangiano manifestamente invariante bajo la inversión temporal y hamiltoniano definido positivo. Analizamos primero la TQFT de Witten y luego una teoría topológica no commutativa.

Descriptores: Teorías topológicas; formulación BRST.

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1. BRST charge and Hamiltonian of the TQFT

We begin with the gauge invariant action introduced in [1]

\[ S = \frac{1}{4} \int \text{Tr}\,(B + F) \wedge (B + F) \]

\[ = \frac{1}{4} \int d^4x \varepsilon^{\mu\nu\rho\sigma}(F_{\mu\nu} + B_{\mu\nu})(F_{\rho\sigma} + B_{\rho\sigma}), \]

where \( F \) is the curvature two form of the gauge connection \( A \), and \( B \) is an independent two form.

The action (1) may be reformulated in a canonical form that have the following first class constraints:

\[ \phi = D_i \pi^i = \partial_i \pi^i + (A_i \times \pi^i) = 0, \quad \phi^i = \pi^i = 0. \]

Nevertheless they are not linearly independent since they satisfy the following identity

\[ (D^*_i \phi^i) = \phi, \]

and thus we have to deal with a reducible theory with one level of reducibility.

To construct the BRST charge we follow Ref. [2] and [3] and introduce the minimal sector of the extended phase space expanded by the conjugate pairs:

\[ (A_i, \pi^i); (C_i, \mu^i); (C_{1i}, \mu^{1i}); (C_{11}, \mu^{11}), \]

where \( A_i, \pi^i \) are the original canonical coordinates and \( (C, \mu) \) are the canonical ghost and antighost associated to the constraints (2).

The off-shell nilpotent BRST charge associated with (2) is then given by

\[ \Omega = (C_1(D_i \pi^i) + C_{1i} \pi^i + C_{11}[(D_i \mu^{1i}) - \mu^{1i}] - \frac{1}{2} \]

\[ \times C_1(C_1 \times \mu^i) - C_1(C_{1i} \times \mu^{1i}) - C_{11}(C_1 \times \mu^{11})), \]

where \( \{ \cdots \} \) stands for integration on the three dimensional continuous index.

The effective action is then given by

\[ S_{\text{eff}} = \int d^4x \sqrt{g} \left[ \pi^i \dot{A}_i + \mu^i \dot{C}_i + \mu^{1i} \dot{C}_{1i} + \mu^{11} \dot{C}_{11} \right. \]

\[ + \dot{\delta}(\lambda^0_0 \mu^1 + \lambda^0_1 \mu^{11} + \lambda^0_{11} \mu^{1}) \]

\[ \left. + \dot{\delta}(C_2 \chi_2 + C_2' \chi_{21}) + \dot{\delta}(C_1 \chi_{12}) \right], \]

We now introduce an extended phase space in order to define a symplectic structure over it. The new canonical conjugate pairs we define are:

\[ (C_2, P_2), (C_3, \lambda^0_0), (C_2', P_2'), (C_3', \lambda^0_{11}), \]

\[ (C_3', \lambda^{11}_0), (C_{12}, P_{12}), (C_{13}, \lambda^{1}_{11}), \]

The BRST charge in the extended phase space is

\[ Q = \Omega + \langle -C_2 P_2 - C_3' P_{21} + C_{13} P_{12} \rangle. \]

The integration is defined with the scalar volume element \( \sqrt{\gamma} d^4x \). Since

\[ \sqrt{g} D_i \pi^i = \partial_i (\sqrt{g} \pi^i) + A_i \times \sqrt{g} \pi^i, \]

we may thus remove the dependence on the metric by redefinition of the momenta. This is possible because the BRST charge \( Q \) is linear in the momenta, this is a particular feature of the topological field theory. The charge \( Q \) is off-shell nilpotent \( \{Q, Q\} = 0 \).
In order to express the Hamiltonian of the effective action (6) as a Q-anticommutator we introduce the following functional $\tilde{Q}$ which will turn out to be the time reversal of the BRST charge when reduced to the minimal sector of phase space. We define $\tilde{Q}$ in a very general way in terms of general gauge fixing functions $\chi$’s, and step by step we impose some consistency conditions over them. At the end of the process we give the general structure of the gauge fixing functions solving all consistency restrictions. The $\tilde{Q}$ is given by

$$\tilde{Q} = \langle C_2 \tilde{\chi}_2 + C_4 \tilde{\chi}_2 \rangle + C_{12} \tilde{\chi}_{12} + \lambda_2^0 \mu^1 + \lambda_{11}^0 \mu^{11}. \quad (10)$$

We have thus expressed the action (6) completely in terms of canonical conjugate pairs with the Hamiltonian having the form

$$H = \{Q, \tilde{Q} \}, \quad (11)$$

with $Q$ and $\tilde{Q}$ off-shell nilpotent charges. From (11) one obtains

$$\{Q, H\} = 0, \quad \{\tilde{Q}, H\} = 0. \quad (12)$$

### 2. Time reversal operation and positivity of the Hamiltonian

In order to discuss the problems related to the positivity of the inner product in the space of quantum states, we will be interested in analysing the quadratic part of the Hamiltonian, considered as a polinomy on the fields. We will denote it $H_2$. It may be expressed as

$$H_2 = \{Q_2, \tilde{Q}_2 \}, \quad (13)$$

where $Q_2$ and $\tilde{Q}_2$ are the quadratic parts, when considered as a polinomy on the fields, of the BRST and anti-BRST charges:

$$\{Q_2, Q_2\} = 0, \quad \{\tilde{Q}_2, \tilde{Q}_2\} = 0. \quad (14)$$

We may now introduce the time reversal operation $T$:

$$P_2 \xrightarrow{T} C_2, \quad P_4 \xrightarrow{T} C_2, \quad P_{12} \xrightarrow{T} C_{12}, \quad C_{13} \xrightarrow{T} -\tilde{\chi}_2, \quad C_{13} \xrightarrow{T} \tilde{\chi}_{12} \quad (15)$$

where $\partial_i$ denotes the covariant derivative with respect to the background metric. The time reversal operation $T$ by definition satisfies $T^2 = 1$.

For the BRST charge we have

$$Q_2 \xrightarrow{T} \tilde{Q}_2 \xrightarrow{T} Q_2. \quad (16)$$

We will denote $\phi^\dagger \equiv T\phi$. Following we will interpret $\dagger$ as the adjoint under the internal product we will introduce.

We define the inner product on the Hilbert space of physical states $\mathcal{H}$ as

$$(\psi, \varphi)_+ \equiv (T\psi, \varphi), \quad (17)$$

where $(\cdot, \cdot)_+$ is the $L^2$ inner product. The same happens to $Q$ and $Q^\dagger$, which are adjoint under the $(\cdot, \cdot)_+$ inner product and this is precisely the property needed in the argument used to show that all ground states of the Hamiltonian are physical states [4].

### 3. Effective action. Operatorial approach

We will choose the following gauge fixing conditions that allows us to write the Hamiltonian in a manifestly positive form:

$$\chi_2 = \lambda_2^0 - (D_i \pi^i - \frac{1}{2} C_1 \times \mu^i)^\dagger, \quad \chi_{2i} = \lambda_{2i}^0 - (\pi^i - C_1 \times \mu^{1i})^\dagger, \quad \chi_{12} = \lambda_{12}^0 - (D_i \mu^{1i} - \mu^i - C_1 \times \mu^{1i})^\dagger. \quad (18)$$

After a canonical reduction we end with a description of the theory in terms of the minimal sector of phase space. After several calculations we obtain

$$S = \int dt \{[\pi^i \dot{A}_i + \mu^i \dot{C}_1 + \mu^{1i} \dot{C}_{11} - H]. \quad (19)$$

The Hamiltonian (19) may be expressed as the Poisson bracket of the off-shell nilpotent BRST charge $\Omega$ (5), and anti-BRST charge $\Omega^\dagger$:

$$\Omega^\dagger = T\Omega, \quad \{\Omega, \Omega\} = 0, \quad \{\Omega^\dagger, \Omega^\dagger\} = 0, \quad H = \{\Omega, \Omega^\dagger\}. \quad (20)$$

We notice that the complete effective action (19) is invariant under the time reversal operation. We have constructed then gauge fixing functions (18) that yield an effective BRST invariant action, consistent with a positive inner product in the space of Hilbert states. The quantum equivalence between the effective action (19) and Witten’s effective action arises from the independence of the functional integral on the gauge fixing functions. This latest point is based on the BRST invariance of the canonical action [2], [3]. In order to ensure this point, the requirement of nilpotency of $Q$ has to be raised, to the quantum level.
4. A TQFT over a noncommutative space

In the previous sections we presented the Hamiltonian structure of Witten's topological quantum field theory where the fields are $SU(2)$ valued. We will consider now a different theory defined on a noncommutative space. The geometrical objects involved are the same as before, an antisymmetric field $B_{\mu\nu}$ and a gauge vector $A_\mu$, which will now be abelian fields however since they are defined over a noncommutative background the expression of the curvature is now

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_M,$$  \hspace{1cm} (21)

where

$$[A_\mu, A_\nu]_M = \omega^{\alpha\beta} \partial_\alpha A_\mu \partial_\beta A_\nu + \cdots \hspace{1cm} (22)$$

is the Moyal bracket with a simplectic form $\omega^{\alpha\beta}$. We will assume that the background admits a flat simplectic connection over it, and $\partial_\alpha$ is the covariant derivative constructed with that connection which, since it is simplectic, it satisfies

$$\partial_\alpha \omega^{\mu\nu} = 0. \hspace{1cm} (23)$$

The first term in the expansion of the Moyal bracket is the Poisson bracket as shown in the previous expression, everything that follows is also valid by considering only the Poisson bracket in the expansion but in that case the bracket does not arise from a noncommutative product.

The action we propose is

$$S = \int Tr (B - F) \ast \wedge (B - F)$$

$$= \int Tr (B - F) \wedge (B - F). \hspace{1cm} (24)$$

We notice that the wedge product between the two $(B - F)$ factors may also be composed with the Moyal product, however the integration annihilates almost all terms on the noncommutative product leaving only the one we wrote which corresponds to the first term in the expansion of the noncommutative product.

Although the theory we are proposing is geometrically and physically very different from the Witten’s TQFT, the canonical and BRST structure as well as the positivity of the Hamiltonian may be analysed on the lines we presented in the previous sections.

The constraints are as before

$$\pi^i = 0, \quad D_i \pi^i = 0, \hspace{1cm} (25)$$

where $D_i$ is the covariant derivative

$$\partial_i + [A_{\alpha i}, \cdot]_M,$$  \hspace{1cm} (26)

with $\partial_i$ being the symplectic covariant derivative.

The result for the BRST charge is

$$\Omega = \{C_1(D_1 \pi^1) + C_1, \pi^1 \} + C_1 \{D_\mu^{1\dagger} - \mu^{1\dagger}\}$$

$$- \frac{1}{2} C_1 [C_1, \mu^{1\dagger}]_M - C_1 \{C_1, \mu^{1\dagger}\}_M$$

$$- C_1 \{[C_1, \mu^{1\dagger}]_M\}, \hspace{1cm} (27)$$

where the ghosts and antighosts are introduced in a similar way as for the Witten’s TQFT.

It can be shown that a similar time reversal operation as the one considered in Sec. 2 may be defined. The internal product on the Hilbert space can be then introduced allowing to prove the positivity of the Hamiltonian of the noncommutative theory we have introduced.

5. Conclusions

We obtained the canonical structure for Witten’s TQFT. The Hamiltonian is expressed in the form $\{Q, Q^\dagger\}$ where $Q$ and $Q^\dagger$ are the BRST and anti-BRST nilpotent charge. The $Q^\dagger$ is expressed in a general form in terms of gauge fixing functions that satisfy the necessary requirements to obtain the nilpotency of the anti-BRST charge. The explicit expression of $Q$ is metric independent, in distinction to the one given in Ref. [4]. A time reversal operation is introduced. It is then shown that the necessary conditions raised in Ref. [4] to have a positive definite inner product in the Hilbert space of states are satisfied provided that suitable admissible gauge fixing functions are chosen. The resulting TQFT with those gauge fixing functions is not manifestly covariant but satisfies the positivity requirement.

We introduced a new noncommutative field theory over a symplectic manifold with a flat symplectic connection over it. The action depends only on the symplectic structure on the manifold. Although the geometrical and physical interpretation are very different from $SU(2)$ TQFT, its canonical structure may be obtained from the same approach briefly presented in Secs. 2 and 3. We explicitly constructed its BRST charge and from it the BRST effective action may be directly obtained. The time reversal operation can be introduced as in Sec. 2 and 3 and thus a proof of the positivity of the corresponding Hamiltonian should be possible. We expect to report on this problem soon.