Shape phase transitions in algebraic nuclear models

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A review of the shape phase transitions within the IBA, is presented. This nuclear model depends on two control parameters, \( (r_2, r_1) \), and two order parameters, \( (\beta, \gamma) \). In the control parameter space, the accessible shapes and stability properties of the Consistent-Q nuclear model are established. The procedure of coherent states plus catastrophe formalisms to determine the shape phase diagram of an algebraic nuclear model is illustrated by considering the Meshkov-Glick-Lipkin nuclear model. The relevance of the separatrix to organize the classical orbits and the structure of the quantum energy levels is established.

Keywords: Algebraic nuclear models; shape phase transitions; coherent states and catastrophe theory.

Se presenta una revisión de las transiciones de fases nucleares que ocurren en el modelo de bosones interactuantes. Este modelo de los núcleos depende de dos parámetros de control, \( (r_2, r_1) \), y dos parámetros de orden, \( (\beta, \gamma) \). En el espacio de parámetros de control se establecen las formas y propiedades de estabilidad del Modelo Q-Consistente. El procedimiento de estados coherentes ayudado al formalismo de catástrofes determinan el diagrama de fases de modelos nucleares algebraicos, el cual se ilustra con el modelo nuclear de Meshkov-Glick-Lipkin. Se establece la importancia de la separatrix para organizar las órbitas clásicas y la estructura de los niveles de energía cuánticos.

Descriptores: Modelos nucleares algebraicos; transiciones fase de forma; estados coherentes y teoría de catástrofes.

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1. Introduction

Recently it was established that the energy surfaces of the Interacting Boson Approximation (IBA) nuclear model have only two essential control parameters, and that the associated energy surfaces can have shape phase transitions of orders zero, first and second [1]. This was done by means of the catastrophe formalism and following the bases introduced in Ref. [2]. It is important to notice that the classical theory of phase transitions fits naturally within that formalism. Thus, if one considers the deformation parameters \( \beta \) and \( \gamma \) as the order parameters, a shape phase transition occurs when the control parameters of the energy surface are varied and the deformation variables jump from one critical branch to another.

There are two important characteristics of the phase transitions according to the used convention about the behaviour of the physical system. If the Delay Convention is adopted a phase transition is happening when one crosses a bifurcation set on which local minima are created or destroyed whereas for the Maxwell convention a phase transition occurs when a Maxwell Set on which two or more global minima are degenerated (equally deep) is crossed.

Therefore a shape phase diagram in terms of the essential parameters \( (r_2, r_1) \) can be established when these control parameters are varied. Thus the regions of spherical shape phases, and prolate or oblate shape phases were determined, within the IBA framework. Besides it raises also a region of points where the shape coexistence phenomena is present.

For an algebraic nuclear model the procedure is the following: First of all, the associated coherent states are constructed. Thus the matrix elements of the Hamiltonian system with respect to these coherent states are evaluated, which are called energy surfaces. Second, by means of the catastrophe formalism the control parameters are defined and the bifurcation and Maxwell sets are determined. These sets define the Separatrix of the algebraic model which in essence constitutes the phase transition diagram of the variables of the so called energy surface.

In the present contribution we illustrate each one of the steps indicated above by considering the Meshkov-Glick-Lipkin (MGL) Nuclear Model, which traditionally has been used to test many fermion approximation methods.

In the Second Section a review of the stability and shape phase transitions of the IBA nuclear model is presented, besides of setting up the behaviour of the Consistent-Q Nuclear Model in the control parameter space of the IBA. In the Third Section the coherent states and energy surfaces associated to the MGL nuclear model are presented. Afterwards, in the Fourth Section we determine the critical points of the energy surface, the bifurcation and Maxwell sets to finally find the shape phase diagram of the MGL nuclear model. In the Fifth Section a study of the classical dynamics associated to the MGL model is made, and the relevance of the Separatrix in the behaviour of their orbits and trajectories is established. Finally a requantization of the classical system is made and notice that the behaviour of the energy levels evidences to be organized by the Separatrix of the MGL nuclear model [3].

2. Shapes and stability within the IBA

The IBA model considers the nucleus as formed by bosons which can occupy two levels one of angular momentum two
and the other with angular momentum zero which are separated by an energy \( \epsilon \) and where the total number of bosons is a constant, this constant is associated to the number of pairs of valence nucleons.

The most general IBA-1 model Hamiltonian of one and two boson interactions can be written in terms of the Casimir operators of the chain of groups [4]

\[
U(6) \supset \left\{ \frac{U(5)}{O(6)} \text{ or } \frac{O(6)}{SU(3) \text{ or } SU(3)} \right\} \supset O(3) \supset O(2) .
\]

In these chains the \( O(6) \) together with \( SU(3) \), are indicating two different realizations used in the literature of the generators of these groups. These groups are important to describe the available shapes in the model. The Hamiltonian is diagonal if it can be written in terms of the Casimir operators associated to one of the chains of groups and thus there are three dynamical symmetries, which have played an important role in the developing of the IBA model. It is well known that the \( U(6) \) symmetry describes an anharmonic vibrator, the \( SU(3) \) is an axial rotor, and the \( O(6) \) limit is a deformed \( \gamma \)-unstable rotor.

The corresponding energy surface is given by [4]

\[
E(N, \beta, \gamma) = \frac{N \epsilon \beta^2}{(1 + \beta^2)^2} + \frac{N(N - 1)}{2(1 + \beta^2)^2} \times \left( a_1 \beta^4 + a_2 \beta^3 \cos 3 \gamma + a_3 \beta^2 + \frac{u_0}{2} \right)
\]

where \( \epsilon \equiv \epsilon_d - \epsilon_s \) and the constant \( N \epsilon_s \) was subtracted. The parameters \( a_1, a_2 \) and \( a_3 \) are given in terms of the strengths of the two boson interactions

\[
a_1 = \frac{c_0}{10} + \frac{c_2}{7} + \frac{9c_4}{50},
\]

\[
a_2 = -\frac{2}{\sqrt{35}} v_2,
\]

\[
a_3 = \frac{1}{\sqrt{5}} (v_0 + u_2).
\]

From the catastrophe theory formalism it is found that the energy surface is overdetermined if three parameters are fixed and for this reason, the energy surface can be rewritten in terms of a magnification plus a shift [1]:

\[
E(N, \beta, \gamma) = M \epsilon(\beta, \gamma) + S ,
\]

where

\[
\epsilon(\beta, \gamma) = \frac{1}{(1 + \beta^2)^2} \left( \beta^4 + r_1 \beta^2 (\beta^2 + 2) - r_2 \beta^3 \cos 3 \gamma \right),
\]

\[
M = \frac{N(N - 1)}{2} \left[ 2a_1 - a_3 + \frac{\epsilon}{(N - 1)} \right],
\]

\[
S = \frac{N(N - 1)}{2} u_0 ,
\]

where the magnification must be a positive number to preserve the nature of the critical points, e.g., minima of \( E(N, \beta, \gamma) \) are minima of \( \epsilon(\beta, \gamma) \). Therefore in Eq. (7) we take the absolute value, and the expressions

\[
r_2 = \frac{-2a_2}{2a_1 - a_3 + \frac{\epsilon}{(N - 1)}},
\]

\[
r_1 = \frac{a_3 - u_0 + \frac{\epsilon}{(N - 1)}}{2a_1 - a_3 + \frac{\epsilon}{(N - 1)}} ,
\]

define the essential or control parameters of the system.

The separatrix of the IBA model is constituted by the bifurcation sets

\[
r_{12}^\pm = \pm \frac{(9r_2^2 + 16)^{3/2}}{54r_2^2} - \frac{32}{27^2 r_2^2} - 1 ,
\]

\[
r_2 = 0 , \quad \text{with} \quad r_1 < 0 ,
\]

and the Maxwell sets

\[
r_{13}^\pm = \frac{3}{2} \pm \frac{1}{2} \sqrt{1 + \frac{r_2^2}{36}},
\]

\[
r_2 = 0 , \quad \text{with} \quad r_1 < 0 .
\]

Therefore the shape phase diagram within the IBA model is shown in Fig. 1. By means of the Ehrenfest classification of the classical phase transitions, we can determine the order of the shape phase transitions. The phase shape transition takes place between \( p \) and \( q \) branches of critical points, and is of nth-order, if

\[
\lim_{\epsilon \to 0} \frac{\partial^i E(p) (s)}{\partial s^i} \bigg|_{s_0 + \epsilon} = \lim_{\epsilon \to 0} \frac{\partial^i E(q) (s)}{\partial s^i} \bigg|_{s_0 - \epsilon} ,
\]

for \( i = 0, 1, 2, \ldots, n - 1 \), but fails for \( i = n \) [5].

![Figure 1](image_url)
In Fig. 1 the bifurcation and Maxwell sets, the exact limits associated to the symmetries U(5), SU(3), SU(3), and O(6), and the existent shape phases within the IBA, are indicated. The O(6) symmetry yields a constant energy surface [1]. For \( r_1 > r_{12}^+ \) we have the spherical shape phase where the U(5) limit is localized. Below the \( r_2 \)-axis the deformed shape phases are obtained; at the left of the \( r_1 \)-axis there are oblate shape phases, while to the right are localized the prolate ones. For \( 0 \leq r_1 \leq r_{12}^+ \), one has the presence of shape phases coexistence phenomena. In the Maxwell set \( r_{13}^+ \), the spherical and deformed minima are equally deep. Above this set, a spherical absolute minimum coexists with a local deformed one while below the set, a deformed absolute minimum coexists with a local spherical one.

If the Delay Convention [5, 6] is adopted there are shape phase transitions when:

i) The bifurcation set \( r_{12}^+ \) is crossed there is a zero order transition, and an excited deformed shape phase is appearing or disappearing.

ii) For the bifurcation set \( r_1 = 0 \), its crossing in points with \( r_2 \neq 0 \) yields a zero order transition, and an excited spherical shape phase is appearing or disappearing. If \( r_2 = 0 \), then it gives rise to a second order transition, and we have a change from a spherical shape phase to deformed \( \gamma \)-unstable shape phase. Notice that at the point \( (r_2, r_1) = (0, 0) \) spherical, oblate, prolate and \( \gamma \)-unstable phases coexist.

If the Maxwell Convention [5, 6] is adopted there are phase transitions when:

i) The Maxwell set \( r_{13}^+ \) is crossed there is a first order transition, and a change from a spherical shape phase to a deformed shape phase is happening or vice versa, depending of the direction of the crossing.

The locus of points \( r_2 = 0 \) with \( r_1 < 0 \), is a bifurcation and Maxwell set, and for the Maxwell and Delay conventions its crossing yields a first order transition from oblate to prolate shape phases.

Finally a model Hamiltonian within the Consistent-Q formalism introduced in [7] is considered,

\[
H = \eta N_d - \frac{1 - \eta}{N} Q^\chi \cdot Q^N,
\]

which contains only two parameters. The parameter \( \eta \) is varying in the range \( 0 \leq \eta \leq 1 \), and \( \chi \) typically takes the values \( -\sqrt{7}/2 \leq \chi \leq \sqrt{7}/2 \). However, it has provided several predictions that agree with the experimental data. The strengths of the interactions are chosen to coincide with the expressions recently used in Ref. [8], where quantum phase transitions within this model are studied. This Hamiltonian is a particular case of that presented above and so it is straightforward to determine the values of the essential parameters

\[
r_2 = \frac{-8 \sqrt{7} \chi}{\xi - \chi^2 + 4},
\]

\[
r_1 = \frac{-\chi^2 + 4}{\xi N - 1} - 4
\]

where the parameter \( \xi = \frac{\eta}{1-\eta} N \) is defined. Solving for \( N \) the ratio \( r_2/r_1 \), and substituting again this result in Eq. (14), the locus of points in the control parameter space is given by the straightline

\[
r_1 = -\frac{\sqrt{2}}{\sqrt{7}} \left( \frac{\chi^2 - 14}{4\chi} \right) r_2 \pm 1.
\]

In Fig. 2 the behaviour of \( \chi \) and \( \eta \) in the control parameter space is displayed, when the \( \chi \) varies from \( -\sqrt{7}/2 \) to \( \sqrt{7}/2 \), and \( \eta \) from 0 to 1. In all the plots, each line represents a locus of points with \( N \) constant. The case \( \eta = 1 \) represents the limit U(5), i.e., (0, 1). It is very interesting that for \( \eta = 0 \) and \( N \to \infty \), the line crosses the SU(3), SU(3), and O(6) exact symmetries. Let us remark that for the selected range of \( \chi \), the consistent Q-model yields only shapes inside the double triangle formed by the exact limits. This double triangle is formed by the straightlines

\[
r_1 = \pm \frac{7}{4\sqrt{2}} r_2 + 1,
\]

\[
r_1 = \pm \frac{1}{4\sqrt{2}} r_2 - 1,
\]

\[
r_2 = 0.
\]

In the plots of Fig. 2, for fixed \( \eta \), when the number of quanta increases from 2 to 10, the lines move to the bottom of the double triangle, i.e., to the case \( N \to \infty \), indicated by the dashed lines. Besides when \( \eta \) is growing, the deformed shape phase is becoming less important. For the case \( \eta = 5/6 \), all the lines belong to a spherical shape phase.

3. Energy surfaces in the MGL model

The MGL nuclear model is simple enough to be solved exactly but is yet non-trivial. For that reason, since it was established [9] has been used to compare the behaviour of many fermion approximation methods like for example the random phase approximation (RPA), the renormalized RPA and the self-consistent RPA [10]. The MGL model assumes that the nucleus is a system of fermions which can occupy two \( N \)-folded levels, which are separated by an energy \( \epsilon \). Besides, the nucleons have residual interactions which scatters pairs of particles between the two levels without changing the particular degenerate states within the shells. Then the Hamilt-
associated with a different value of Hamiltonian matrix breaks up into submatrices, each associated state within each shell. It is also immediate that the Hamiltonian can be written in the form

\[ H = \epsilon J_0 + \frac{\lambda}{2} (J_+^2 + J_-^2) + \frac{\gamma}{2} (J_+ J_- + J_- J_+) \]  

(17)

The \( \lambda \)-term annihilates pairs of particles in one level and creates pairs in the other level. The \( \gamma \)-term scatters one particle up while another is scattered down.

The angular momentum operators are realized in terms of fermion creation \( a^\dagger_{\pm \rho} \) and annihilation operators \( a_{\pm \rho} \)

\[ J_+ = \sum_{\rho=1}^{N} a^\dagger_{+ \rho} a_{- \rho} \]  

(18)

\[ J_- = \sum_{\rho=1}^{N} a^\dagger_{- \rho} a_{+ \rho} \]  

\[ J_0 = \frac{1}{2} \sum_{\rho=1}^{N} (a^\dagger_{+ \rho} a_{+ \rho} + a^\dagger_{- \rho} a_{- \rho}) \]}

where the quantum number \( \rho \) denotes the particular degenerate state within each shell. It is also immediate that the Hamiltonian commutes with the operator \( J^2 \) and thus the Hamiltonian matrix breaks up into submatrices, each associated with a different value of \( J \) and of order \( 2J + 1 \).

Next the case with maximum symmetry is considered, i.e., when \( J = N/2 \), and the notation \( \epsilon = 2\omega \) is used.

First of all, the non-normalized spin coherent states are defined \[ |\zeta \rangle = \exp (\zeta^* J_+ - J_-) \]  

(19)
as trial state. The \( \zeta \) is a point in the complex plane mapped by the stereographical projection of a point on the sphere from its south pole,

\[ \zeta = \frac{x + iy}{1 + z} = \tan (\theta/2), \exp (i\phi), \]  

(20)

where \( x, y, \) and \( z \) are the Cartesian coordinates of a point in the unit sphere. The expectation value of the Hamiltonian with respect to these states is given by

\[ E(\theta, \phi) = \frac{\langle \zeta | H | \zeta \rangle}{\langle \zeta | \zeta \rangle} = \gamma J - 2\omega J \cos \theta + \frac{1}{2} (2J - 1) \times (\gamma + \lambda \cos 2\phi) \sin^2 \theta. \]  

(21)

By means of the catastrophe formalism, it is found that the energy surface can be written in terms of a magnification times a function \( \epsilon(\theta, \phi) \) plus a shift

\[ E(\theta, \phi) = M \epsilon(\theta, \phi) + S, \]  

(22)

with

\[ \epsilon(\theta, \phi) = -2 \cos \theta + \gamma_x \sin \theta \cos\phi \]  

\[ \gamma_x = 2J - 1 \]  

\[ \gamma_y = \frac{2J - 1}{2\omega} (\gamma - \lambda), \]  

(23)

are defined.

4. Separatrix

The most general behavior of the system is obtained when the control parameters, \( \gamma_x \) and \( \gamma_y \), are varied, defining thus a whole family of functions. A complete analysis of a family of functions can be given by means of the application of catastrophe formalism, and so we shall be able to determine: the critical points, the degeneracy of critical points, the bifurcation sets of the energy surface, and the locus of points in the space of control parameters at which a phase transition occurs from one local critical point to another. Critical points are the ones determined by

\[ \tilde{\nabla} \epsilon = 0. \]  

(24)

These critical points are given in Table I. They are located in the poles, in the parallel \( \phi_c = 0 \), and \( \pi \), and in the parallel \( \theta_c = \arccos (-1/\gamma_x) \) in the directions \( \phi_x = 0 \), and \( \pi \), and in the parallel \( \theta_c = \arccos (-1/\gamma_y) \), in the directions \( \phi_y = \pi/2 \), and \( 3\pi/2 \). If \( \gamma_x = \gamma_y \), then the critical points are independent of \( \phi \); i.e., they are whole parallel \( \theta_c = \arccos (-1/\gamma_x) \).
The corresponding critical points in terms of $\theta$ and $\phi$ are established.

\[
\begin{array}{|c|c|}
\hline
(x_c, y_c) & (\theta_c, \phi_c) \\
(0, 0) & (0, 0), (\pi, 0) \\
(\pm x_c, 0) & (\theta_c, 0) \quad (\theta_c, \pi) \\
x_c = \sqrt{1 - 1/\gamma_x^2} & \theta_c = \arccos(-1/\gamma_x) \\
(0, \pm y_c) & (\theta_c, \pi/2) \quad (\theta_c, 3\pi/2) \\
y_c = \sqrt{1 - 1/\gamma_y^2} & \theta_c = \arccos(-1/\gamma_y) \\
(\pm x_c, y_c) & \gamma_x = \gamma_y = \gamma \\
x_c^2 + y_c^2 = 1 - 1/\gamma^2 & \theta_c = \arccos(-1/\gamma) \\
\hline
\end{array}
\]

One can also get the organization of all the critical points, particularly for the equilibrium points, and their stability within the control parameter’s space. Bifurcation sets are obtained from the conditions

- $\nabla \epsilon = 0$,
- $\det \epsilon_{ij} = 0$, $i, j = 1, 2$,

with the matrix of second derivatives of energy $\epsilon_{ij} = \partial^2 \epsilon / \partial x_i \partial x_j$ evaluated at the critical points. The set of parameters that satisfy these conditions are

\[
\begin{align*}
\gamma_x &= \pm 1, \\
\gamma_y &= \pm 1, \\
\gamma_x &= \gamma_y, \quad |\gamma| \geq 1.
\end{align*}
\]

In these locus of points in the control parameter space the function $\epsilon$ changes because equilibria are either created or destroyed.

The Maxwell sets must satisfy the conditions

- $\nabla \epsilon = 0$,
- $\epsilon^{(p)} = \epsilon^{(p+1)}$, $p = 1, 2, \ldots$,
- $\{ \partial \epsilon^{(p)} / \partial \gamma_\alpha - \partial \epsilon^{(p)} / \partial \gamma_\nu \} \delta \gamma_\alpha = 0$, $\alpha = x, y$.

Therefore we obtain

\[
\begin{align*}
(\pm x_c, 0), (\gamma_x, \gamma_y) \text{ such that } |\gamma_x| > 1, \\
(0, \pm y_c), (\gamma_x, \gamma_y) \text{ such that } |\gamma_y| > 1, \\
\gamma_x = \gamma_y = \gamma, (\gamma, \gamma), |\gamma| > 1.
\end{align*}
\]

In these critical points $p = 1, 2, \ldots$ the function $\epsilon$ takes the same value, and the function jumps from one minimum (maximum) to another.

In Fig. 3 the separatrix of the MGL model and the shape phase diagram are shown. The spherical phase is bounded by the conditions

\[
\{ (\gamma_x, \gamma_y) | \gamma_x > -1 \text{ and } \gamma_y > -1 \} ;
\]

by the other hand, the deformed phase is found in two regions:

\[
\begin{align*}
\{ (\gamma_x, \gamma_y) | \gamma_x < -1 \text{ and } \gamma_y > \gamma_x \}, \\
\{ (\gamma_x, \gamma_y) | \gamma_y < -1 \text{ and } \gamma_y < \gamma_x \}.
\end{align*}
\]

In the region (27), the absolute minima are found in the azimuthal angles 0 and $\pi$, while for the region (28), they are on the angles $\pi/2$ and $3\pi/2$.

This is in agreement with previous results indicating when occurs a second order phase transition [12]. In terms of the strength parameters of the residual interactions, the deformed phase is obtained if

\[
|\lambda| > \gamma + \frac{\epsilon}{2J - 1},
\]

and the spherical phase in the opposite case.
It is interesting to remark that crossing the point \((-1, -1)\) through the straightline \(\gamma_y = -\gamma_x - 2\) gives rise to a third order transition, and it is the only way to get this order for the transitions.

5. Dynamical behaviour

The dynamical equations of motion for the classical system associated to our Hamiltonian are

\[
\dot{\theta} = -i \{\theta, E(\theta, \phi)\},
\]
\[
\dot{\phi} = -i \{\phi, E(\theta, \phi)\},
\]

where the Poisson brackets are defined by the expression

\[
\{F, G\} = -i \left( \frac{\partial F}{\partial \theta} \frac{\partial G}{\partial \phi} - \frac{\partial F}{\partial \phi} \frac{\partial G}{\partial \theta} \right).
\]

From these one obtains

\[
\frac{d\theta}{d\tau} = \frac{1}{2} (\gamma_x - \gamma_y) \sin \theta \sin 2\phi,
\]
\[
\frac{d\phi}{d\tau} = 1 + \frac{1}{2} \cos \theta [(\gamma_x + \gamma_y) + (\gamma_x - \gamma_y) \cos 2\phi],
\]

with \(\tau = 2\omega t\).

The orbit is determined through the intersection of the two surfaces

\[
z = \frac{1}{2} (\gamma_x x^2 + \gamma_y y^2 - \epsilon),
\]
\[
x^2 + y^2 + z^2 = 1,
\]

which are constants of the motion.

The energy surfaces of the system determine the possible orbits of the system. As the nature and number of the critical points of the energy surfaces is determined by the separatix. Therefore it plays a fundamental role to organize the characteristic orbits of the system.

In Fig. 4 the orbits in the \((x, y)\) plane for the parameters \(\gamma_x = -2\) and \(\gamma_y = -4\) corresponding to a deformed phase are shown. The orbits are evaluated for six energy values running from -4.0 to 1.75. For the lowest energy there are two orbits around global minima; i.e., bistability in the \(y\) axis. When the energy increases, this property disappears.

6. Requantization

The energy surfaces evaluated in the critical points give rise to six different functions. For the case \(\gamma_y = \gamma_x \equiv \gamma\) they are shown in Fig. 5a. One can see that the global minimum of the system is deformed for \(\gamma < -1\) and spherical for \(\gamma > -1\).

For the case \(\gamma_y = -\gamma_x - 2\) they are displayed in Fig. 6a. For \(\gamma_x < -1\) there are minima in \(x\)-axis, while for \(\gamma_x > -1\), there are minima in \(y\)-axis. Notice that in both figures there is coexistence between deformed and spherical phases at the point \(\gamma_x = -1\).

\[
J_x = J \sin \theta \cos \phi,
\]
\[
J_y = J \sin \theta \sin \phi,
\]
\[
J_z = -J \cos \theta.
\]

They satisfy the Poisson bracket relations of angular momentum components

\[
\{J_x, J_y\} = iJ_z,
\]

and cyclic permutations of the indices. Then the classical function \(\epsilon(\theta, \phi)\), which determines the stability and shape phase transitions of the energy surface can be rewritten as

\[
\epsilon(J_x, J_y, J_z) = \frac{2J_z}{J} + \frac{\gamma_x J_x^2}{J^2} + \frac{\gamma_y J_y^2}{J^2}
\]
\[
= \frac{2J_z}{J} + \gamma_x + \gamma_y \left( J_+ J_- + J_- J_+ \right) + \frac{\gamma_x - \gamma_y}{4J^2} \left( J_+^2 + J_-^2 \right),
\]

which agrees with the original quantum Hamiltonian divided by the constant \(J\omega\) in the limit \(J \gg 1\).
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Next we determine the energy levels of the quantum version of the Hamiltonian (33) for the cases $\gamma_y = \gamma_x$, and $\gamma_y = -\gamma_x - 2$. The energy levels for $J = 10$ are plotted in Fig. 5b and 6b. The energy levels are organized by the separatrix. In Fig. 5b, for $|\gamma_x| < 1$ there is not degeneracy between the levels while outside there is. In Fig. 6b, the same phenomena are present, but for the region $-3 < \gamma_x < 1$.

7. Conclusions

In the first part of this contribution a review of the shape phase transitions within the IBA is presented. A redefinition of the parameters $r_1$ and $r_2$ used in Ref. [1] was done to guarantee that the nature of the critical points of $E(N, \beta, \gamma)$ is preserved by the function $\epsilon(\beta, \gamma)$. The regions of the control parameter space available within the Consistent-Q nuclear model are established. Thus we conclude that for $\eta$ close to zero, this model describes deformed nuclei, while for $\eta$ near to one realizes spherical shapes. It can be seen that if the parameter $\chi$ is restricted from $-\sqrt{7}/2$ to $\sqrt{7}/2$, all the shapes accessible fall inside the double triangle shown in Fig. 2. However it presents the same shape phase transitions that the most general Hamiltonian of the IBA.

In the second part, the MGL nuclear model is studied to give a simple example of how to determine the Maxwell and Bifurcations sets; these constitute the separatrix of the model. It is shown that the separatrix is fundamental to determine stability properties and shape phase transitions of the model. In this case, a third order shape phase transition is happening when the control parameters are changing along the straightline $\gamma_y = -\gamma_x - 2$. Notice that in the point $(\gamma_x, \gamma_y) = (-1, -1)$ there is a convergence of second order shape phase transitions. The separatrix organizes the classical orbits of the associated Hamiltonian function and is also relevant for the structure of their quantum energy levels.

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