Unfolding a degeneracy point: Crossings and anticrossings of unbound states in parameter space

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Some geometric and topological properties of a degeneracy of unbound states are investigated in the scattering of a beam of particles by a double barrier potential well with two regions of trapping. We find that, near and at a degeneracy of unbound states, the surfaces representing the resonance eigenwave numbers as functions of the control parameters of the system have the topology of the Riemann surfaces of the square root of the difference of two complex, regular functions of the control parameters.

\textbf{Keywords}: Multiple resonances; Non-relativistic scattering theory; Phases: topological; Berry’s phase.

Se investigan algunas propiedades geométricas y topológicas de una degeneración de estados no ligados en la dispersión de un haz de partículas por una barrera doble de potencial con dos regiones de atrapamiento. Encontramos que, en la vecindad de una de generación de estados no ligados, las superficies que representan los autovalores del número de ondas como función de los parámetros de control del sistema tienen la topología de las superficies de Riemann de la raíz cuadrada de la diferencia de dos funciones complejas regulares de los parámetros de control.

\textbf{Descriptores}: Resonancias múltiples; teoría de la dispersión no relativista; fases topológicas; fase de Berry.

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1. Introduction

In this paper, by means of a simple and elementary example, we will exhibit some characteristic properties of the singularities of the surfaces representing the complex resonance energy eigenvalues as functions of the control parameters of the system in the vicinity of and at a degeneracy of unbound states.

Many years ago von Neumann and Wigner [1] explained the now familiar phenomenon of energy level repulsion and avoided level crossings of bound states observed in many quantum systems driven by Hermitean Hamiltonians depending on external parameters. E. Teller [2] gave a geometric interpretation of level repulsion of bound states in terms of the shape of the surfaces representing the energy eigenvalues as functions of the control parameters near and at a degeneracy of two bound states. He found that the two degenerating levels correspond to the two sheets of an elliptic double cone. Since then, it has been realized that accidental degeneracy and level crossings, true or avoided, are important for the understanding of a wide variety of quantum phenomena [3]. For instance, a quantum system acquires a topological phase - the Berry phase - when transported adiabatically around a path in parameter space [4, 5] which includes an accidental degeneracy of bound states, sometimes called a conical or diabolical point [6].

More recently, a great deal of attention has been given to the avoided level crossing phenomena of quantum energy eigenvalues in the case of unbound states [7]. Novel effects have been found which attracted considerable theoretical [8, 9], and very recently, also experimental interest.
the Jordan blocks in the complex energy representation of the resolvent operator associated with them was developed by Hernández et al. [25] in the framework of the theory of the analytic properties of the radial wave function.

2. Scattering by a double barrier potential

Doublets of resonances and accidental degeneracies of unbound states may occur in the scattering of a beam of particles by a potential with two regions of trapping. A simple example is provided by a spherically symmetric potential \( V(r) \) in which the two regions of trapping are two potential wells defined by two concentric potential barriers located between the origin of coordinates and the outer region where \( V = 0 \). In order to make the analysis as simple and explicit as possible, we take the wells and barriers to be square as shown in Fig. 1.

In what follows, we will consider the conditions for the occurrence of a degeneracy of unbound states in this simple system and we will be interested in the geometric and topological properties of the surfaces that represent the complex energy eigenvalues as functions of the control parameters of the system in the neighborhood of and at a degeneracy of unbound states.

2.1. The regular solution

The s-wave radial Schrödinger equation is

\[
\frac{d^2u(k,r)}{dr^2} + (k^2 - U(r))u(k,r) = 0
\]  

(1)

the potential \( U(r) = 2mV(r)/\hbar^2 \) is a double barrier such that between the origin of coordinates, and the outer region, \( r > r_4 \), where the particles propagate freely, there are two square potential wells separated by two square potential barriers, as shown in Fig. 1. The system has seven parameters, the positions \( r_i \) (i = 1,2,3,4) and heights \( V_i \) (i = 2,3,4) of the four discontinuities of the potential. In this work, we will keep the five parameters \( (V_2, V_4, r_1, r_3 - r_2, r_4 - r_3) \) fixed and will vary the depth of the outer well \( V_3 \) and the thickness of the inner barrier \( d = r_2 - r_1 \). In the following, we will refer to the pair of parameters \( (d, V_3) \) as the control parameters of the system.

The Jost regular solution of (1) normalized to unit slope at the origen, \( \phi(k,r) \), is as follows:

In the wells,

\[
\phi_1(k,r) = \frac{1}{k} \sin kr, \quad 0 \leq r \leq r_1, \quad (2)
\]

and

\[
\phi_3(k,r) = \phi_2(k,r_2) \left[ \cos \left( K_3(k)(r - r_2) \right) + \alpha_2(k,d) \times \sin \left( K_3(k)(r - r_2) \right) \right], \quad r_2 \leq r \leq r_3. \quad (3)
\]

In the barriers,

\[
\phi_i(k,r) = \phi_{i-1}(k,r_{i-1}) \left[ \cosh \left( K_i(k)(r - r_{i-1}) \right) \right] + \alpha_{i-1}(k,d) \sinh \left( K_i(k)(r - r_{i-1}) \right), \quad r_{i-1} \leq r \leq r_i, \quad i = 2, 4, \ldots \quad (4)
\]

and, in the outer region,

\[
\phi_5(k,r) = \phi_4(k,r_4) \left[ \cos k(r - r_4) + \alpha_4(k,d) \times \sin k(r - r_4) \right], \quad r_4 \leq r \leq \infty. \quad (5)
\]

In these expressions \( k \) is the wave number of the free waves and

\[
K_i(k) = \left( U_i - k^2 \right)^{1/2}, \quad i = 2, 4, \quad (6)
\]

\[
K_3(k) = \left( k^2 - U_3 \right)^{1/2} . \quad (7)
\]

The functions \( \alpha_i(k,d) \) are the logarithmic derivatives of the regular solution at the discontinuities of the potential,

\[
\alpha_i(k,d) = \frac{1}{K_{i+1}(k)} \frac{d}{dr} \ln \phi_i(k,r) |_{r=r_i}, \quad i = 1, 2, 3, 4, \quad (8)
\]

with \( K_3 = k \).

The logarithmic derivatives of \( \phi(k,r) \) at the consecutive discontinuities \( r_i \) and \( r_{i+1} \) are related by the matching conditions at \( r_{i+1} \),

\[
\alpha_1(k) = \frac{k}{K_2(k)} \cot kr_1, \quad (9)
\]

\[
\alpha_2(k,d) = \frac{K_2(k) \alpha_1(k) + \tanh(K_2(k)d)}{K_3(k) \frac{K_1}{1 + \alpha_1(k) \tanh(K_2(k)d)}}, \quad (10)
\]

\[ \alpha_3(k; d, V_3) = \frac{K_3(k; V_3) \alpha_2(k; d) - \tan(K_3(k)(r_3 - r_2))}{K_4(k) \cdot 1 + \alpha_2(k; d) \tan(K_3(k)(r_3 - r_2))}, \tag{11} \]
and
\[ \alpha_4(k; d, V_3) = \frac{K_4(k) \alpha_3(k; d, V_3) + \tanh(K_4(k)(r_4 - r_3))}{k \cdot 1 + \alpha_3(k; d, V_3) \tanh(K_4(k)(r_4 - r_3))}. \tag{12} \]

Since the first logarithmic derivative, \( \alpha_1(k) \), is explicitly known, an explicit solution for \( \alpha_2(k; d) \) is obtained by substitution of the expression (9) for \( \alpha_1(k) \) in Eq.(10). From the knowledge of \( \alpha_2(k; d) \) and Eq.(11) we solve for \( \alpha_3(k; d, V_3) \) which, combined with Eq.(12) gives an explicit solution for \( \alpha_4(k; d, V_3) \). Once the logarithmic derivatives \( \alpha_i \) are explicitly known as functions of the control parameters, an explicit expression for the regular solution \( \phi(k, r) \) is obtained from Eqs.(2-5).

The Jost function, \( f(-k) \), may now be readily obtained from the regular solution in the outer region and the knowledge of \( \alpha_4(k; d, V_3) \). When the regular solution in the outer region, given in Eq.(5), is written as a combination of an outgoing wave \( \exp(ikr) \) and an incoming wave \( \exp(-ikr) \)

\[ \phi_\text{in}(k, r) = \phi_\text{out}(k, r) \frac{1}{2} \left[ (1 - i \alpha_4(k; V_3, d)) \exp(ikr) - (1 + i \alpha_4(k; V_3, d)) \exp(-ikr) \right], r_4 \leq r \leq \infty, \tag{13} \]

the coefficient of the incoming wave is the Jost function. Making use of Eqs.(4), (12) and (13) we get

\[ f(-k) = \sin K_1 r_1 \left[ \cosh(K_2(k)d) + \alpha_1(k) \sinh(K_2d) \right] \left[ \cos \left( K_3(k)(r_3 - r_2) \right) + \alpha_2(k, d) \sin \left( K_3(k)(r_3 - r_2) \right) \right] \]
\[ \times \left\{ \frac{K_4(k)}{k} \left[ \sinh \left( K_4(k)(r_4 - r_3) \right) + \alpha_3(k; d, V_3) \cosh \left( K_4(k)(r_4 - r_3) \right) \right] - i \left[ \cosh(K_4(k)(r_4 - r_3)) \right. \right. \]
\[ + \alpha_3(k; d, V_3) \sinh \left( K_4(k)(r_4 - r_3) \right) \left. \right\} \exp ik r_4. \tag{14} \]

\[ \phi_\text{in}(k, r) = \phi_\text{out}(k, r) \frac{1}{2} \left[ (1 - i \alpha_4(k; V_3, d)) \exp(ikr) - (1 + i \alpha_4(k; V_3, d)) \exp(-ikr) \right], r_4 \leq r \leq \infty. \tag{13} \]

The zeros of the Jost function give resonance poles in the scattering wave function \( \psi(k, r) \), and in the matrix \( S(k) \). From (1) and (2-5), we also verify that all roots (zeros) of the Jost function are associated with energy eigenfunctions of the radial Schrödinger equation.

Unbound state eigenfunctions also called resonant-state or Gamow eigenfunctions are the solutions of Eq.(1) that vanish at the origin,

\[ u_n(k_n, 0) = 0, \tag{20} \]

and at infinity satisfy the outgoing wave boundary condition,

\[ \lim_{r \to \infty} \left[ \frac{1}{u_n(k_n, r)} \frac{du_n(k_n, r)}{dr} - ik_n \right] = 0, \tag{21} \]

where \( k_n \) is a zero of the Jost function,

\[ f(-k_n) = 0, \tag{22} \]

with \( k_n \) located in the fourth quadrant of the complex \( k \)-plane.

Hence, the resonant-state eigenfunctions are related to the regular solution by

\[ u_n(k_n, r) = N_n^{-1} \phi(k_n, r), \tag{23} \]

where \( N_n \) is a normalization constant. From (2-5) we verify that due to the vanishing of \( f(-k_n) \), \( \phi(k_n, r) \) is now proportional to the outgoing wave solution of (1), \( \exp(ik_n r) \) for \( r \) larger than the range of the potential.

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3. Degeneracy of unbound states

A degeneracy of unbound states, that is the equality of two (or more) complex resonance energy eigenvalues of the radial Schrödinger equation, results from the exact coincidence of two (or more) simple resonance zeros of the Jost function, which merge into one double (or higher rank) zero lying in the fourth quadrant of the complex \( k \)–plane. Hence, the condition for the occurrence of a degeneracy of two unbound states at some \( k = \tilde{k} \) is that both, the Jost function and its first derivative vanish at \( \tilde{k} \),

\[
f(\tilde{k}) = 0, \\
\left( \frac{df(-k)}{dk} \right)_{k=\tilde{k}} = 0,
\]

where \( f(\tilde{k}) \) is given in (14).

Therefore, to locate a degeneracy of unbound states, we have to solve this system of two coupled equations with two real, independent parameters, \( d \) and \( V \), whose values should be adjusted to satisfy (24) and (25).

The coupled equations (24) and (25) were solved numerically. The zeros of the Jost function are found by an algebraic computer package that searches for the minima of \( |f(-k)| \) in the complex \( k \)–plane. Starting with the values \( V_2 = V_4 = 2, r_1 = 1, r_3 - r_2 = 1, r_4 - r_3 = 0.304892 \) and \( d = 2, V_3 = 1.04 \), we find the first doublet of resonances at

\[
k_1 = 2.2101546 - i 0.1366887 \\
k_2 = 2.2321776 - i 0.0017984.
\]

We kept the five parameters \( (V_2, V_4, r_1, r_3 - r_2, r_4 - r_3) \) fixed, and we adjusted the control parameters \( d \) and \( V_3 \) until \( k_1 \) and \( k_2 \) became equal to some common value \( \tilde{k} \). Then, we computed numerically \( |df(-k)|/dk \) at \( k = \tilde{k} \) to verify that the second equation is also satisfied to some previously prescribed accuracy. Proceeding in this way, we found that by fine tuning the control parameters to the values \( d^* = 1.1314661145 \) and \( V_3^* = 1.038235081 \), the first doublet of resonances becomes degenerate, with a precision better than one part in \( 10^8 \), at

\[
\tilde{k} = 2.22697606 - i 0.07220139.
\]

Now let us turn our attention to the generalized Gamow eigenfunctions associated with a degeneracy of resonances. In the case of a one-channel problem with a short ranged, local potential and fixed angular momentum as the example we are considering here, the solution of the radial Schrödinger equation (1), which vanishes at the origin and behaves as a purely outgoing wave at distances larger than the range of the potential, is unique up to a multiplying constant. In other words, for each set of values of the external parameters \( (d, V) \) there is one and only one Gamow normalized eigenfunction \( u_n(k_n, r) \) associated with each complex zero of the Jost function, \( k_n \), lying in the fourth quadrant of the complex \( k \)–plane [25]. When we move in parameter space from the point \( (d, V) \) where all complex energy eigenvalues (complex zeros of the Jost function) are different to a point \( (d^*, V_3^*) \) where two complex energy eigenvalues, say \( k_1^2 \) and \( k_2^2 \), are equal, the corresponding Gamow eigenfunction \( u_1(k_1, r) \) and \( u_2(k_2, r) \) go to a common limit \( u_1(\tilde{k}, r) \). Hence, at degeneracy there is only one normal mode, the Gamow normalized eigenfunction \( u_1(\tilde{k}, r) \) associated with the repeated (degenerate) energy eigenvalue \( \tilde{k}^2 \). However, another, linearly independent, generalized eigenfunction or abnormal mode is provided by the same limiting process that gives rise to the degeneracy. As we move in parameter space from the point \( (d, V) \) to the degeneracy point \( (d^*, V_3^*) \), the difference of the two eigenvalues that become degenerate vanish, and the difference of the corresponding Gamow eigenfunctions also vanish. Then, by continuity of \( k_1(d, V) \) and \( k_2(d, V) \) at the common limit \( \tilde{k}(d, V_3^*) \), the derivative of the Gamow eigenfunction with respect to the complex energy eigenvalue exists:

\[
\tilde{u}_1(\tilde{k}, r) = \frac{du_1(\tilde{k}, r)}{d\tilde{\xi}} + c(\tilde{k})u_1(\tilde{k}, r)
\]

where \( \tilde{\xi} = h^2\tilde{k}^2/2m \), and \( c(\tilde{k}) \) is a function of \( \tilde{k} \) but is independent of \( r \),

\[
c(\tilde{k}) = \frac{2m}{h^2} \frac{1}{2k} \left[ \frac{1}{4k} \left( \frac{df(\tilde{k})}{dk} \right) - \frac{1}{4} \left( \frac{df(\tilde{k})}{dk} \right)^2 \right] \]

Therefore, when the Jost function has a double-resonance zero at \( k = \tilde{k} \), there is a chain of Gamow-Jordan generalized eigenfunctions of length two, \( \{u_1(\tilde{k}, r), \tilde{u}_1(\tilde{k}, r)\} \), which are solutions of the Jordan chain of differential equations

\[
- \frac{h^2}{2m} \frac{d^2u_1(\tilde{k}, r)}{dr^2} + V(r)u_1(\tilde{k}, r) = \tilde{\xi}u_1(\tilde{k}, r)
\]

and

\[
- \frac{h^2}{2m} \frac{d^2\tilde{u}_1(\tilde{k}, r)}{dr^2} + V(r)\tilde{u}_1(\tilde{k}, r) = \tilde{\xi}\tilde{u}_1(\tilde{k}, r) + u_1(\tilde{k}, r)
\]

and satisfy the same boundary conditions, namely, they vanish at the origin and at infinity they behave as outgoing waves,
In the particular case of a double barrier potential we are considering here

$$\tilde{u}_i(k, r) = N_i^{-1} \left[ \frac{\partial \phi(k, r)}{dk} \right]_k + c(\tilde{k}) \phi(k, r)$$

(34)

where $N_i$ is a normalization constant and $\phi(k, r)$ is given in Eqs. (2-13) and $\tilde{k}$ is given in (28).

The general theory of the Gamow-Jordan eigenfunctions associated with a degeneracy of unbound states is given by Hernández, Jáuregui and Mondragón [25].

4. Degeneracy of unbound states in parameter space

The radial Hamiltonian $H_r$ of a particle in a double barrier potential introduced in (1) and (32) and (33), is a smooth function of the parameters of the potential barriers; five of these parameters were kept fixed, but the width of the inner barrier, $d$, and the depth of the second well, $V_3$, were allowed to vary. Therefore, we may consider the Hamiltonian $H_r$ embedded in a population of Hamiltonians $H_r(d, V_3)$ smoothly parametrized by the two control parameters, $d$ and $V_3$, which take values in some domain $D$ of a manifold or parameter space. Each point in $D$ represents a Hamiltonian $H_r$. In this section, we will be concerned with the topology of the surfaces representing the complex energy eigenvalues as function of $(d, V_3)$ at a crossing of unbound states.

As explained in Sec. 2.2, the energy eigenvalues of the radial Schrödinger equation are determined by the zeros of the Jost function, Eqs.(14) and (22). Hence, when

$$f(-k_n) = \sin K_1(k_n) r_1 + \sinh K_2(k_n) d + \alpha_1(k_n)$$

$$\times \cosh K_3(k_n) (r_3 - r_2)$$

$$\times \sinh K_4(k_n) (r_4 - r_3) + \alpha_2(k_n, d)$$

$$\times \sinh K_5(k_n) (r_5 - r_4)$$

$$\times \cosh K_6(k_n, d) (r_6 - r_5)$$

$$- i \left[ \cosh K_7(k_n, d) (r_7 - r_6) \right]$$

$$+ \alpha_3(k_n, d)$$

$$\times \exp ik_n r_4 = 0.$$  

(35)

is satisfied by $k_n$, with $k_n$ lying in the fourth quadrant of the complex $k$-plane, the complex resonance energy eigenvalue is $E_n = k_n^2 k_n^2 / 2m$.

4.1. Energy surfaces

The Jost function, $f(-k_n; d, V_3)$, occurring in (35) is a function of many variables. As a function of $k$, it is an analytic function of $k$—complex, but it is also explicitly dependent on the real valued control parameters $(d, V_3)$. Therefore, the condition (35), implicitly defines the inverse functions

$$k_n(d, V_3) = f^{-1}(-k_n; d, V_3), \quad n = 1, 2, \ldots$$

(36)

as branches of a smooth, multivalued function of the parameters $(d, V_3)$. Then, not only the radial Hamiltonian $H_r$, but also its complex energy eigenvalues $E_n$ are smooth functions of the control parameters $(d, V_3)$ in the domain $D$.

We solved numerically the implicit equation (36) for $k_1(d, V_3)$ and $k_2(d, V_3)$ in the neighborhood of, and at the crossing of unbound states. The results of the numerical computation of $k_1(d, V_3)$ and $k_2(d, V_3)$ were represented as a two-sheeted hypersurface in a Euclidean space with Cartesian coordinates $(Re k, Im k, d, V_3)$.

When the control parameters take the critical values $d^* = 1.3114661145$ and $V_3^* = 1.038235081$, the two resonance zeros of the Jost function, $k_1$ and $k_2$, coalesce in one double zero at $k = 2.22697606 - i0.0720139$, all other zeros remaining simple. At this point, the surfaces representing $k_1(d, V_3)$ and $k_2(d, V_3)$ touch each other and the corresponding complex energy eigenvalues, $E_1$ and $E_2$, become degenerate.

In Figs. 2 and 3, the real function $\text{Re} k_{1,2}(d, V_3)$ is shown as a surface $S_R$ in the three-dimensional subspace with cartesian coordinates $(\text{Re} k, d, V_3)$ from two different points of view. Similarly, in Figs. 4 and 5, the real function $\text{Im} k_{1,2}(d, V_3)$ is shown as a surface, $S_I$ in the three-dimensional subspace with cartesian coordinates $(\text{Im} k, d, V_3)$ in two perspectives.

![Figure 2](image-url)

**Figure 2.** The two-sheeted surface $S_R$ that represents the real part of the eigenwave numbers $k_1$ and $k_2$ as functions of the control parameters $(d, V_3)$ in the neighborhood of a degeneracy of unbound states with complex resonance energies $E = k^2 k_n^2 / 2m$, $i = 1, 2$. Exact degeneracy, $k_1 = k_2$, occurs only at the critical point with coordinates $d^* = 1.3114661145$ and $V_3^* = 1.038235081$. 

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From Figs. 2 and 3, it can be seen that, close to the critical point \((d^*, V_3^*)\), where the two unbound states become degenerate, the function \(\text{Re} k_{1,2}(d, V_3)\) has two branches and the surface \(S_R\) representing this function has two sheets which are glued together from two copies of the plane \((d, V_3)\) which are cut and joined smoothly along a line \(L_R\). The projection of \(L_R\) on the plane \((d, V_3)\) is a line \(L'\) see Fig. 6. The cut starts at the critical point on \(L_R\) with coordinates \((d^*, V_3^*)\) and runs from this point to values of \(d\) larger than \(d^*\) and values of \(V_3\) larger than \(V_3^*\). The function \(\text{Im} k_{1,2}(d, V_3)\) also has two branches and the surface \(S_I\) representing this function is also glued from two copies of the plane \((d, V_3)\) which are cut and joined smoothly along a line \(L_I\), as shown in Figs. 4 and 5. The projection of \(L_I\) on the plane \((d, V_3)\) is also the line \(L'\) see Fig. 7. As in the case of \(\text{Re} k_{1,2}(d, V_3)\), the cut starts at the critical point on \(L_I\), but, in this case, the cut runs from the point \((d^*, V_3^*)\) to values of \(d\) smaller than \(d^*\) and values of \(V_3\) smaller than \(V_3^*\).

The lines \(L_R\) and \(L_I\) are orthogonal to each other - they are in orthogonal subspaces - but have one point in common, the critical point with coordinates \((d^*, V_3^*)\).

The projection of the lines \(L_R\) and \(L_I\) on the plane \((d, V_3)\) are the two halves of the line \(L'\), as shown in Figs 6 and 7. Close to the critical point the line \(L'\) may be approximated by its tangent at the critical point

\[
V_3 - V_3^* \approx m(d - d^*)
\]

with \(m \approx 0.19\)

![Figure 3](Image 330x611 to 562x782) ![Figure 4](Image 330x315 to 562x473) ![Figure 5](Image 70x616 to 302x782)

**Figure 3.** The surface \(S_R\) that represents the real part of the eigenwave numbers \(k_1\) and \(k_2\) as functions of the control parameters \((d, V_3)\) in the vicinity of a degeneracy of unbound states. This surface has two sheets which are copies of the plane \((d, V_3)\) cut and joined smoothly along a line \(L_R\) that starts at the degeneracy or critical point \((d^*, V_3^*)\) and runs to points such that \(d > d^*\) and \(V_3 > V_3^*\). Along \(L_R\), \(\text{Re} k_1 = \text{Re} k_2\), but exact degeneracy of unbound states, \(k_1 = k_2\), occurs only at the critical point with coordinates \((d^*, V_3^*)\).

**Figure 4.** The two-sheeted surface \(S_I\) that represents the imaginary part of eigenwave numbers \(k_1\) and \(k_2\) as functions of the control parameters \((d, V_3)\) in the neighborhood of a degeneracy of unbound states with complex resonance energies \(\varepsilon_i = k^2 k_i^2 / 2m\), \(i = 1, 2\). This surface has two sheets which are copies of the plane \((d, V_3)\) cut and joined smoothly along a line \(L_I\) that starts at the degeneracy (critical) point \((d^*, V_3^*)\) and runs to points such that \(d < d^*\) and \(V_3 < V_3^*\). Along \(L_I\), \(\text{Im} k_1 = \text{Im} k_2\) but exact degeneracy, \(k_1 = k_2\), occurs only at the critical point \((d^*, V_3^*)\).

**Figure 5.** Another view of the surface \(S_I\) that represents the imaginary parts of the resonance eigenwave numbers \(k_1\) and \(k_2\) as functions of the control parameters \((d, V_3)\) in the vicinity of a degeneracy of unbound states. The two sheets of this surface are copies of the plane \((d, V_3)\) cut and joined smoothly along a line \(L_I\), that starts at the degeneracy (critical) point \((d^*, V_3^*)\) and runs to points such that \(d < d^*\) and \(V_3 < V_3^*\). Along \(L_I\), \(\text{Im} k_1 = \text{Im} k_2\) but exact degeneracy, \(k_1 = k_2\), occurs only at the critical point \((d^*, V_3^*)\).

Let us call \(\text{Re} k_U(d, V_3)\) the function represented by points on the upper sheet of the surface \(S_R\), and \(\text{Re} k_L(d, V_3)\) the function represented by points on the lower sheet of the surface \(S_R\). Likewise, let us call \(\text{Im} k_U(d, V_3)\) the function represented by points on the upper sheet of the surface \(S_I\) and \(\text{Im} k_L(d, V_3)\) the function represented by points on the lower sheet of the surface \(S_I\).
Similarly, at the points on the line $S_R$ that represents the two branched function $Re k_{1,2}(d, V_3)$. The line $L'$ is the projection of $L_R$ in parameter space. The dot marks the critical point with coordinates $(d^*, V_3^*)$.

Then, at the points on the line $L_R$ such that $d' > d^*$ and $V_3' > V_3^*$,

$$Re k_U(d', V_3') = Re k_L(d', V_3'),$$

but

$$Im k_U(d', V_3') = Im k_L(d', V_3').$$

Similarly, at the points on the line $L_I$, such that $d'' < d^*$ and $V_3'' < V_3^*$,

$$Im k_U(d'', V_3'') = Im k_L(d'', V_3'').$$

At the critical point with coordinates $(d^*, V_3^*)$, and only at that point, both, the real and imaginary parts of $k_1$ and $k_2$ are equal

$$k_1(d^*, V_3^*) = k_2(d^*, V_3^*).$$

Therefore, in the complex $k$–plane, the crossing point of the two simple zeros of the Jost function is an isolated point, where the Jost function has one double zero.

### 4.2. Sections of the energy hypersurface

Let us consider a point $(d, V_3)$ in parameter space away from the critical point. That is, a point in the plane of the control parameters $d$ and $V_3$ with cartesian coordinates $(d, V_3) \neq (d^*, V_3^*)$. To this point corresponds a pair of non-degenerate eigenwave numbers

$$k_1(d, V_3) \neq k_2(d, V_3) \text{ if } (d, V_3) \neq (d^*, V_3^*),$$

these two eigenwave numbers are represented by two points on the $k_{1,2}(d, V_3)$ hypersurface.

When the point $(d, V_3)$ traces a path $\pi$ in parameter space, the corresponding points $k_1(d, V_3)$ and $k_2(d, V_3)$ trace two curving trajectories, $C_1(\pi)$ and $C_2(\pi)$, on the $k_{1,2}(d, V_3)$ hypersurface. The topological structure of the hypersurface $k_{1,2}(d, V_3)$ will be most clearly evident in the shape and properties of the trajectories $C_1(\pi)$ and $C_2(\pi)$ for a path $\pi$ that crosses the line $L'$ at a point close to the critical point.

We define three straight line paths in parameter space, $\pi_1, \pi_2$ and $\pi_3$, by keeping the parameter $V_3$ fixed at some value $V_3^{(i)}$, $i = 1, 2, 3$, and letting the parameter $d$ vary. The values of $V_3^{(i)}$ were chosen in such a way that the paths $\pi_1, \pi_2$ and $\pi_3$, cross the line $L'$ at points located just before, at, and just after the critical point.

As a point moves in parameter space along the straight line path $\pi_i$ from the starting point $(d_1, V_3^{(i)})$ to the end point $(d_2, V_3^{(i)})$, the points representing $k_1(d_1, V_3^{(i)})$ and $k_2(d_2, V_3^{(i)})$ move along the curving trajectories $C_1(\pi_i)$ and $C_2(\pi_i)$ on the hypersurface $k_{1,2}(d, V_3)$. The trajectories $C_1(\pi_i)$ and $C_2(\pi_i)$ are the intersection of the hypersurface $k_{1,2}(d, V_3)$ and the hyperplane $V_3 = V_3^{(i)}$ in the space with cartesian coordinates $(Re k_U, Im k_U, d)$. Since $V_3$ is kept constant at the fixed value $V_3^{(i)}$, the trajectories $C_1(\pi_i)$ and $C_2(\pi_i)$ may be represented as three dimensional curves in the space with cartesian coordinates $(Re k, Im k, d)$.

The trajectories $C_1(\pi_i)$ and $C_2(\pi_i)$, for each path $\pi_i$, were computed numerically. The results are shown as three-dimensional graphs in Figs. 8, 9 and 10. To each point on the trajectories $C_1(\pi_i)$ and $C_2(\pi_i)$ corresponds a triple of numbers $(Re k_1, Im k_1, d)$ and $(Re k_2, Im k_2, d)$ respectively, the numbers $(Re k, Im k, d)$ are shown in the figures as cartesian
coordinates. We show the trajectories $C_1(\pi_i)$ and $C_2(\pi_i)$ in perspective view and the three projections of these curves on the planes $(Rek, d)$, $(Imk, d)$ and on the complex $k-$plane $(Rek, Imk)$. Notice that, in each one of these figures, the path $\pi_i$ is shown as the vertical axis $Od$, while in Figs. 2, 3, 4 and 5, $Od$ is shown as a horizontal axis.

The properties of the trajectories $C_1(\pi_i)$ and $C_2(\pi_i)$ may now be readily understood in terms of the properties of the two sheeted surfaces $S_R$ and $S_I$ that represent the functions $Rek_{1,2}(d, V_3)$ and $Imk_{1,2}(d, V_3)$.

Let us consider first Fig. 8, which shows the trajectories $C_1(\pi_1)$ and $C_2(\pi_1)$ traced by the points $k_1(d, V_3)$ and $k_2(d, V_3)$ on the hypersurface $k_{1,2}(d, V_3)$ when the point $(d, V_3)$ moves along the straight line path $\pi_1$ in parameter space. The path $\pi_1$ is defined by the conditions

$$1.05 \leq d \leq 1.20 \quad \text{and} \quad V_3 = 1.0381,$$

it crosses the line $L'$ to the left of the critical point at a point $d^{(1)} = 1.1308$. Corresponding to this point, there is a point on the line $L_I$ where the two sheets of $S_I$ cross, but there is no corresponding point on the line $L_R$ where the two sheets of $S_R$ cross. Therefore, as the point $d$ moves up on $\pi_1$, from the starting point at $d = 1.05$, the points $Imk_{1}(d, V_3^{(1)})$ and $Imk_{2}(d, V_3^{(1)})$ also move up and approaching each other on the upper and lower sheets of $S_I$ respectively, until they meet when $d = d^{(1)}$ and the two projections of $C_1(\pi_1)$ and $C_2(\pi_1)$ on the plane $(Imk, d)$ cross at a point on the line $L_I$ where the two sheets of $S_I$ cross. After crossing, as $d$ moves from $d^{(1)} = 1.1308$ further up, the points $Imk_{1}(d, V_3^{(1)})$ and $Imk_{2}(d, V_3^{(1)})$ move now on the lower and upper sheets of $S_I$ respectively further up and away from each other, as shown in the projections on the plane $(Imk, d)$ in Fig. 8.

The points $Rek_{1}(d, V_3^{(1)})$ and $Rek_{2}(d, V_3^{(1)})$ can not cross, hence, as the point $d$ moves up on $\pi_1$ from the starting point at $d = 1.05$, the points $Rek_{1}(d, V_3^{(1)})$ and $Rek_{2}(d, V_3^{(1)})$ also move up and approaching each other on the lower and upper sheets of $S_R$ respectively, they come close together when $d = d^{(1)}$, but since they cannot cross, when $d$ moves further up, they also move further up and away from each other staying on the same lower and upper sheets of $S_R$ they were at the initial value $d_i = 1.05$, as shown in the projections on the plane $(Rek, d)$ in Fig. 8. We may now try to understand why is that, as $d$ increases from the starting point at $d = 1.05$, the points $k_1(d, V_3^{(1)})$ and $k_2(d, V_3^{(1)})$ moving up on the trajectories $C_1(\pi_1)$ and $C_2(\pi_1)$ approach each other until they come close together when $d = d^{(1)}$, but then, their trajectories make a sudden turn in opposite directions: $C_1(\pi_1)$ turns almost $90^\circ$ towards smaller values of $Imk$ and $C_2(\pi_1)$ turns almost $90^\circ$ towards larger values of $Imk$. As $d$ moves further up on $\pi_1$, the points $k_1(d, V_3^{(1)})$ and $k_2(d, V_3^{(1)})$ move on $C_1(\pi_1)$ and $C_2(\pi_1)$ further up and away from each other. From the Figs. 2, 3, 4 and 5, the sudden turn in the trajectories seem to be produced by the crossing of $Imk_{1}(d, V_3^{(1)})$ and $Imk_{2}(d, V_3^{(1)})$ while $Rek_{1}(d, V_3^{(1)})$ and $Rek_{2}(d, V_3^{(1)})$ cannot cross, which means that $Rek_{1} < Rek_{2}$ for all values of $d \in \pi_1$, while $Imk_{1}$ and $Imk_{2}$ can move freely from values $Imk_{1} > Imk_{2}$ to values $Imk_{1} < Imk_{2}$ when $d$ moves along $\pi_1$ past the point $d^{(1)}$. That this explanation is not fully convincing will be evident when we examine the results shown in Fig. 9.

Figure 9 shows the trajectories $C_1(\pi_2)$ and $C_2(\pi_2)$ traced by the points $k_1(d, V_3)$ and $k_2(d, V_3)$ on the hypersurface $k_{1,2}(d, V_3)$ when the point $(d, V_3)$ moves along the straight line path $\pi_2$ in parameter space. The path $\pi_2$ is defined by the conditions

$$1.05 \leq d \leq 1.20 \quad \text{and} \quad V_3^{(1)} = V_3^{(2)} = 1.038235081,$$

This path crosses the line $L'$ precisely at the critical point. As explained above, at the critical point and only at that point, $k_1 = k_2$, the two lines $L_I$ and $L_R$ meet and the two sheets of each of the two surfaces $S_R$ and $S_I$ cross. Therefore, as the point $(d, V_3^{(1)})$ moves on $\pi_2$ from the starting point at $d = 1.05$ up to $d = d^*$, the points $k_1(d, V_3^{(1)})$ and $k_2(d, V_3^{(1)})$ move up on the trajectories $C_1(\pi_2)$ and $C_2(\pi_2)$, they approach each other until they meet when $d = d^*$. At this point, the trajectories make a sudden $90^\circ$ turn in the same direction. Since both $Rek_{1,2}$ and $Imk_{1,2}$ can cross at this point, the sudden turn in the trajectories must be understood in terms of some property other than the impossibility of crossing from upper to lower sheets or viceversa.
For values of \( (d, V_3) \), where \( d = \eta + d^* \) and \( \eta > 0 \),
\[
\epsilon(\pi_2) \approx \left[ \epsilon_1(d^* - d) \right]^p = \left[ \epsilon_1 \eta \right]^p.
\]
(43)

For values of \( (d, V_3) \) in the path \( \pi_2 \), just after the crossing, with \( d = d^* + \eta \) and \( \eta > 0 \),
\[
\epsilon(\pi_2) \approx (-1)^p \left[ \epsilon_1 \eta \right]^p
\]
(44)

Therefore, just before the crossing, we may write
\[
k_1(d^* - \eta, V_3^*) \approx K(d^*, V_3^*) + [\epsilon\eta]^p, \quad d = d^* - \eta
\]
(45)

\[
k_2(d^* + \eta, V_3^*) \approx K(d^*, V_3^*) - [\epsilon\eta]^p
\]
(46)

and, just after the crossing, we get
\[
k_1(d^* + \eta, V_3^*) \approx K(k^*, V_3^*) + \exp(ip\pi)[\epsilon\eta]^p
\]
(47)

\[
k_2(d^* + \eta, V_3^*) \approx K(k^*, V_3^*) - \exp(ip\pi)[\epsilon\eta]^p
\]
(48)

for \( d - d^* = \eta \).

Then, the sudden 90° turn at the critical point is telling us that
\[
p = \frac{1}{2},
\]
(49)

and close to the critical point, \( \epsilon(d, V_3) \) behaves as
\[
\epsilon \approx \left[ \epsilon_1(d^* - d) + \epsilon_2(V_3^* - V_3) \right]^{1/2}
\]
(50)

Now, since the critical point is the starting point of the cuts in the surfaces \( S_2 \) and \( S_1 \) representing \( Re k_{1,2}(d, V_3) \) and \( Im k_{1,2}(d, V_3) \) as functions of \( (d, V_3) \) the conditions (38), (39) and (40) characterize the singularity of the complex energy hypersurface \( k_{1,2}(d, V_3) \) at the crossing of unbound states.

Close to the critical point, \( k_1 \) and \( k_2 \) may be written as
\[
k_1(d, V_3) \approx K + \sqrt{\epsilon_1(d^* - d) + \epsilon_2(V_3^* - V_3)}
\]
(51)

and
\[
k_2(d, V_3) \approx K - \sqrt{\epsilon_1(d^* - d) + \epsilon_2(V_3^* - V_3)}.
\]
(52)

But, since we had previously identified \( k_1 \) and \( k_2 \) with two branches of the multivalued function \( f^{-1}(-k_n; d, V_3) \) defined in Eq. (36), we are justified in writing
\[
k_{1,2}(d, V_3) \approx K(d, V_3) \pm \sqrt{\epsilon_1(d^* - d) + \epsilon_2(V_3^* - V_3)}.
\]
(53)

This result gives us the clue to the nature of the topological structure of the Riemann surfaces of the function \( k_n = f^{-1}(-k_n; d, V_3) \) in the neighborhood of a crossing of unbound states which is that of the square root function on the right hand side of Eq. (53).

An equivalent result may be derived from a phenomenological representation of the scattering matrix written in terms of an effective \( 2 \times 2 \) matrix propagator or resolvent of the Schrödinger equation [7, 15, 22].

If the system has an isolated doublet of interfering resonances with eigenwave numbers \( k_1 \) and \( k_2 \) far from all other...
resonances, the corresponding zeros of the lost function may be factored out and the scattering matrix takes the form

\[ S(k) = S^{(\text{res})}(k) \exp i2\delta_B(k) \]  

(54)

where \( \delta_B(k) \) is the background phase shift due to the hard sphere scattering and the contribution of the far away resonances. \( S^{(\text{res})}(k) \) is the resonating part of \( S(k) \) due to the doublet of interfering resonances

\[ S^{(\text{res})}(k) = \frac{(k - k_1^*)(k - k_2^*)}{(k - k_1)(k - k_2)} \]  

(55)

where \( k_1 \) and \( k_2 \) are the position of the two poles of the \( S \)–matrix corresponding to the doublet of resonances.

The analytical properties of \( S^{(\text{res})}(k) \) as function of the control parameters \( (d, V_3) \) may be exhibited by rewriting \( S^{(\text{res})}(k) \) as

\[ S^{(\text{res})}(k) = 1 - iW \frac{1}{|k\mathbf{1}_{2 \times 2} - \mathbf{K}|} W^\dagger \]  

(56)

in this expression the \( 2 \times 2 \) matrix \( |k\mathbf{1}_{2 \times 2} - \mathbf{K}|^{-1} \) plays the role of an effective propagator. The matrix \( \mathbf{K} \) is

\[ \mathbf{K} = \mathcal{K} - i \frac{1}{2} W^\dagger W \]  

(57)

where

\[ \mathcal{K} = \frac{1}{2} \text{tr} \mathbf{1}_{2 \times 2} + \begin{pmatrix} z & x \\ x & -z \end{pmatrix} \]  

(58)

and

\[ W^\dagger W = \sqrt{2}\Gamma \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]  

(59)

The \( 2 \times 1 \) row matrix \( W \) is the matrix of the decay amplitudes which couples the elastic channel to the two resonant states in the doublet. The form of the anti-Hermitian part of \( \mathbf{K} \) ensures the unitarity of \( S^{(\text{res})}(k) \). The poles of \( S^{(\text{res})}(k) \) are the eigenvalues of \( \mathbf{K} \). The matrix elements \( k, x, z \) and \( \Gamma_z \) are, functions of the control parameters \( (d, V_3) \).

It is convenient to write \( \mathbf{K} \) in terms of the Pauli matrix valued vectors \( (\sigma_x, \sigma_y, \sigma_z) \) as

\[ \mathbf{K} = \mathbf{K} \mathbf{1}_{2 \times 2} + (\bar{R} - i \frac{1}{2} \bar{\Gamma}) \cdot \hat{\sigma} \]  

(60)

where \( \bar{R}(d, V_3) \) and \( \bar{\Gamma}(d, V_3) \) are real vectors with Cartesian components \( (x, 0, z) \) and \( (0, 0, \Gamma_z) \).

Then, the eigenvalues of \( \mathbf{K} \) are given by

\[ k_{1,2}(d, V_3) = K \pm \sqrt{(\bar{R} - i \frac{1}{2} \bar{\Gamma})^2} \]  

(61)

which is of the same form as Eq. (53).

Let us assume that \( \bar{R} \) and \( \bar{\Gamma} \) can be expanded in a Taylor series about the critical point, as

\[ \bar{R} = \bar{R}_0 + \bar{r}_{1d}(d - d^*) + \bar{r}_{1V_3}(V_3 - V_3^*) + \ldots \]  

(62)

and

\[ \bar{\Gamma} = \bar{\Gamma}_0 + \bar{\gamma}_{1d}(d - d^*) + \bar{\gamma}_{1V_3}(V_3 - V_3^*) + \ldots \]  

(63)

Then, keeping terms of the first order in \( (d - d^*) \) and \( (V_3 - V_3^*) \), the expression under the square root becomes

\[ (\bar{R} - i \frac{1}{2} \bar{\Gamma})^2 = \bar{R}_0^2 - \frac{1}{4}(\bar{R}_0^2 + \bar{\Gamma}_0^2) + \bar{c}_1(d - d^*)^2 + \bar{c}_2(V_3 - V_3^*)^2 + \ldots \]  

(64)

where

\[ \bar{c}_1 = (2\bar{R}_0 \cdot \bar{r}_{1d} + \frac{1}{2}\bar{\Gamma}_0^2 \cdot \bar{\gamma}_{1d}) - i(\bar{R}_0 \cdot \bar{\gamma}_{1d} + \bar{r}_{1d}) \]  

(65)

and

\[ \bar{c}_2 = (2\bar{R}_0 \cdot \bar{r}_{1V_3} + \frac{1}{2}\bar{\Gamma}_0^2 \cdot \bar{\gamma}_{1V_3}) - i(\bar{R}_0 \cdot \bar{\gamma}_{1V_3} + \bar{r}_{1V_3}) \]  

(66)

Close to the critical point, the eigenwave numbers are given by

\[ k_{1,2}(d, V_3) \approx K \pm \sqrt{R_0^2 - \frac{1}{4}\Gamma_0^2} \]  

(67)

The eigenvalues \( k_1 \) and \( k_2 \) coincide at the critical point \( (d^*, V_3^*) \), when the term under the square root vanishes. Since real and imaginary parts should vanish, the degeneracy condition is expressed by the pair of equations

\[ R_0^2 - \frac{1}{4}\Gamma_0^2 = 0 \quad \bar{R}_0 \cdot \bar{\Gamma}_0 = 0 \]  

(68)

Hence, when these two conditions are satisfied, close to the crossing point, we get

\[ k_{1,2}(d, V_3) = K \pm \sqrt{c_1(d - d^*)^2 + c_2(V_3 - V_3^*)^2} \]  

(69)

which is the same expression as in Eq. (53).

Finally, Fig. 10 shows the trajectories \( C_1(\pi_3) \) and \( C_2(\pi_3) \) traced by the points \( k_1(d, V_3) \) and \( k_2(d, V_3) \) on the hypersurface \( k_1(d, V_3) \) when the point \( (d, V_3) \) moves along the straight line path \( \pi_3 \) in parameter space. The path \( \pi_3 \) is defined by the condition

\[ 1.05 \leq d \leq 1.20 \quad \text{and} \quad \bar{V}_3^{(3)} = 1.0384 \]  

(70)

this path crosses the line \( L' \) to the right of the critical point at the point \( d^{(3)} = 1.132 \). Corresponding to the point \( d^{(3)} \), there is a point on the line \( L_R \) where the two sheets of \( S_R \) cross, but there is no corresponding point on the line \( L_I \) where the two sheets of the surface \( S_I \) cross. Hence, as the point \( (d, \bar{V}_3^{(3)}) \) moves up on \( \pi_3 \), the points \( \bar{R}k_1(d, \bar{V}_3^{(3)}) \) and \( \bar{R}k_2(d, \bar{V}_3^{(3)}) \) also move up and approach each other on the lower and upper sheets of \( S_R \) respectively, until they meet when \( d = d^{(3)} \) and the two projections of \( C_1(\pi_3) \) and \( C_2(\pi_3) \)
Making use the Eq.(37) for points on the line $L'$, $\epsilon(\pi_3)$ may be written as

$$
\epsilon(\pi_3) \approx \left[ c_1(d - d^*) + c_2(V^*_3 - V^*_3) \right]^{1/2}. \quad (72)
$$

since $c_1$ and $c_2$ are complex $\epsilon(\pi_3)$ never vanishes, but when the term $c_1(d - d^*)$ changes sign, $\epsilon(\pi_3)$ makes a sudden turn.

5. Conclusions

In this paper we discussed some aspects of the degeneracy of unbound states in the scattering of a beam of particles by a double barrier potential. It was shown that degeneracies of unbound states (resonances) and the concomitant double poles of the scattering matrix may easily be brought about by adjusting the values of only two real independent parameters in the Hamiltonian of the system, so as to satisfy the degeneracy conditions. In the example discussed here, the control parameters of the system are the width, $d$, of the inner barrier and the depth, $V_3$, of the external potential.

The resonance energy eigenvalues $E_n = \hbar k^2 / 2m$, and the corresponding complex eigenwave numbers $k_n$ are smooth functions of the control parameters. In the vicinity of a degeneracy of two unbound states with complex energy eigenvalues $E_1$ and $E_2$, the resonance conditions define the corresponding wave numbers, $k_1(d, V_3)$ and $k_2(d, V_3)$, as branches of a multivalued function $k_{1,2}(d, V_3)$.

With the purpose of exploring the geometrical and topological properties of the hypersurface representing $k_{1,2}(d, V_3)$ in parameter space, we solved numerically the implicit equation for $k_1(d, V_3)$ and $k_2(d, V_3)$ in the neighbourhood of, and at a degeneracy of unbound states.

We found that, close to the degeneracy of unbound states,

- The function $Re k_{1,2}(d, V_3)$ has two branches and is represented by a two sheeted surface $S_R$. The two sheets of $S_R$ are two copies of the plane $(d, V_3)$ which are cut and joined smoothly along a line $L_R$ starting at the critical point and extending to values $d \geq d^*$ and $V_3 \geq V^*_3$.

- The function $Im k_{1,2}(d, V_3)$ also has two branches and is represented by a two sheeted surface $S_I$. The two sheets of $S_I$ are two copies of the plane $(d, V_3)$ which are cut and pasted smoothly along a line $L_I$ starting from the critical point to values $d \leq d^*$ and $V_3 \leq V^*_3$.

- The projections of the lines $L_R$ and $L_I$ on the plane $(d, V_3)$ are the two halves of a line $L'$ that goes through the critical point $(d^*, V^*_3)$.

- At the critical point, and only at that point, both the real and imaginary parts of $k_1$ and $k_2$ are equal. Therefore, in the complex $k-$plane, at the crossing point, the two simple zeros of the Jost function merge in one double zero which is an isolated point (no branch cuts in the $k-$plane).
We also computed sections, $V_3 = V_3$, of the hypersurface representing $k_{1,2}(d, V_3)$. These sections were represented as trajectories $C_1(\pi)$ and $C_2(\pi)$ traced by the points $k_1(d, V_3)$ and $k_2(d, V_3)$ on the hypersurface $k_{1,2}(d, V_3)$ when the point $(d, V_3)$ traces a straight line path $\pi$ in parameter space; $\pi$ is defined by the conditions $V_3 = V_3$ and $1.05 \leq d \leq 1.2$ for various values of $V_3$.

A careful examination of the properties of the surfaces $S_R$ and $S_I$ and the trajectories $C_1(\pi)$ and $C_2(\pi)$ for various paths $\pi_i$ suggests that the topological structure of the hypersurface representing $k_{1,2}(d, V_3)$ is the same as the topological structure of the Riemann surface of the square root function in the right hand side of the expression

$$k_{1,2}(d, V_3) \approx K(d, V_3) \pm \sqrt{\epsilon_0(d^*, V_3^*) + \pi(d, V_3)} \quad (73)$$

where $\epsilon_0(d^*, V_3^*)$ and $\pi(d, V_3)$ are complex functions of the real arguments $(d^*, V_3^*)$ and $((d - d^*), (V_3 - V_3^*))$ respectively.

In our previous papers on the degeneracy and crossing of resonances [7, 15], see also [25] we had found essentially the same result written in a slightly different form, namely

$$k_{1,2}(d, V_3) = K(d, V_3) \pm \sqrt{\vec{R} - \frac{i}{2} \vec{\Gamma}}^2 \quad (74)$$

where $\vec{R}(d, V_3)$ and $\vec{\Gamma}(d, V_3)$ are real vectors with Cartesian components $(x, y, z)$ and $(\Gamma_1, 0, \Gamma_2)$. The components $x, y, \Gamma_1$ and $\Gamma_2$ are regular functions of the control parameters in a neighbourhood of the critical point. Expanding $\vec{R}$ and $\vec{\Gamma}$ in Taylor series about the critical point, and keeping only terms of the first order in $(d - d^*)$ and $(V_3 - V_3^*)$, eq.(74) takes the form

$$k_{1,2}(d, V_3) \approx K(d, V_3) \pm \sqrt{(\vec{R}_0 - i \frac{1}{2} \vec{\Gamma})^2 + c_1(d - d^*) + c_2(V_3 - V_3^*)} \quad (75)$$

which is essentially the same as in (73).

The degeneracy conditions are

$$R_0^2 - \frac{1}{4} \Gamma_0^2 = 0 \quad \text{and} \quad \vec{R}_0 \cdot \vec{\Gamma}_0$$

as in our previous papers [7, 15, 25].

In conclusion, some geometric and topological properties of a degeneracy of unbound or resonant states were explicitly exhibited in a simple model of the scattering of a beam of particles by a double barrier potential with two regions of trapping. We found that, in the vicinity of a degeneracy of unbound states, the surfaces that represent the complex resonance eigenwave numbers as functions of two real control parameters have the same topology as the Riemann surface of a square root of the difference of two complex regular functions of the same real control parameters.

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