Correlation functions and long-range order for a nematic in nonequilibrium stationary states

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A thermotropic nematic thin film under the action of an external thermal gradient is studied theoretically. We use a fluctuating hydrodynamic approach and a time-scale perturbation formalism to calculate the orientation, temperature and velocity autocorrelation functions as well as the temperature-velocity fluctuations cross correlation function of the liquid crystal in the nonequilibrium state induced by the stationary heat flux. This method allows us to find, on slow time-scales, a contracted description in terms of the slow variables only, with a reduced dynamic matrix which can be constructed by the perturbation procedure. The wave number and frequency dependence of these correlation functions is evaluated analytically and their explicit functional form in the configuration space is also calculated for both equilibrium and steady states of the fluid. We show that, out of equilibrium, all these correlations are long-ranged. We calculate the effects of this long-range behavior on the light scattering structure factor. Our results also show that the temperature and velocity autocorrelations contain two contributions, namely, a local equilibrium and a mode coupling contribution. From our quantitative estimations of these correlations, it can be established that the contribution due to the mode coupling mechanism is dominant over that based on spatial inhomogeneities in the fluctuation-dissipation relation.

Keywords: Liquid crystals; fluctuations; nonequilibrium; correlation functions.

Se estudia teóricamente una película nemática termotrópica sometida a la acción de un gradiente térmico externo. Se usa una descripción hidrodinámica fluctuante y un formalismo perturbativo de escalas temporales para calcular las funciones de autocorrelación de orientación, temperatura y velocidad, así como la correlación cruzada temperatura-velocidad del cristal líquido en el estado fuera de equilibrio inducido por el flujo de calor estacionario. Este método permite encontrar en la escala de tiempos lenta, una descripción contraída en términos de las variables lentas solamente, con una matriz dinámica reducida que puede construirse mediante el método perturbativo. La dependencia con el número de onda y la frecuencia de estas correlaciones se calcula analíticamente y también su forma funcional explícita en el espacio de configuración, tanto para los estados de equilibrio como para los estados estacionarios del fluido. Se muestra que fuera de equilibrio todas las correlaciones anteriores son de largo alcance. También se calcula el efecto de largo alcance sobre el factor de estructura del sistema. Los resultados obtenidos muestran que las funciones de autocorrelación de temperatura y velocidad contienen dos contribuciones, a saber, de equilibrio local y de acoplamiento de modos. De las estimaciones cuantitativas obtenidas de estas correlaciones, se establece que la contribución debida al mecanismo de acoplamiento de modos domina sobre el mecanismo basado en las inhomogeneidades espaciales en la relación de fluctuación-dispersión.

Descriptores: Cristales líquidos; fluctuaciones; fuera de equilibrio; funciones de correlación.

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1. Introduction

In spite of the fact that the theory of fluctuations in nonequilibrium fluids was initiated in the late 70’s and pursued by many authors [1–16], nowadays several questions concerning the nature of hydrodynamic fluctuations in stationary nonequilibrium states are still of current active interest. One of these issues is the long-range character of these fluctuations far from instability points [17]. It is well known that thermal fluctuations in a simple fluid in equilibrium always give rise to short-range equal time correlation functions, except when close to a critical point. But when the fluid is driven to nonequilibrium steady states (NESS) by the action of externally applied gradients, the equal-time correlation functions may develop long-range contributions, whose nature is very different from those in equilibrium. For simple fluids in NESS, it has been shown theoretically that the existence of the so called generic scale invariance is the origin of the long-range observed in its correlation functions, [18–20], although these issues have been mostly investigated for simple fluids, some studies for equilibrium and nonequilibrium stationary states in complex fluids also exist, including the enhancement of concentration fluctuations in polymer solutions under external hydrodynamic and electric fields [21], or polymer solutions subjected to a stationary temperature gradient in the absence of any flow have been investigated [22]. Other applications have also been developed [23–26]. Also, the behavior of fluctuations about some stationary nonequilibrium states have been analyzed in the case of thermotropic nematic liquid crystals. Specific examples are the nonequilibrium states generated by a static temperature gradient [27], a stationary shear flow [28] generated by an externally imposed constant pressure gradient [29, 30] or by means of a concentration gradient of impurities in a dilute suspension [31]. In the first two cases, it was found that the nonequilibrium contributions to the corresponding light scattering spectrum were small, but in the last
two cases the nonequilibrium effects on the structure factor may be quite large and perhaps measurable. To our knowledge, however, at present there is no experimental confirmation of these effects, in spite of the fact that for nematics the scattered light intensity is several orders of magnitude greater than for ordinary simple fluids.

In this work we first present a model calculation based on a fluctuating hydrodynamic description which uses a time-scale perturbation formalism constructed upon the fact that for a thermotropic nematic the modes associated with the director relaxation are much slower than the visco-heat and sound modes [32, 33]. As a result, in the nonequilibrium stationary states the hydrodynamic fluctuations evolve on three widely separated times scales. By using this time-scale perturbation procedure, in this work we calculate analytically all the equal-time correlation functions of the transverse and longitudinal state variables for a thermotropic nematic liquid crystal in equilibrium and out of equilibrium states. We derive the explicit $k$ and $T$ dependence of these quantities and show that they exhibit long-range order not only in equilibrium, as could have been expected at least for the orientational correlation owing to the nematic nature of the fluid, but also in the NESS due to the heat flux induced by a stationary thermal gradient. We also evaluate theoretically the effect of the thermal gradient on the dynamic structure factor of the fluid. However, since to our knowledge there are no experimental results available in the literature for light scattering from a nematic in NESS to estimate quantitatively the correlation functions predicted by our model calculations, we used experimental parameter values similar to those used in light scattering experiments for an isotropic simple fluid, since to our knowledge there are no reported light scattering experiments with liquid crystals in NESS. Our results show that all these correlation functions exhibit long-range order. Furthermore, the temperature-temperature and velocity-velocity correlation functions contain two types of contributions, namely, a local equilibrium and a mode coupling contribution. From our quantitative estimate we also verified that the contribution due to the mode coupling mechanism is dominant over that based on spatial inhomogeneities in the fluctuation-dissipation relation. In this way our analysis and model calculations confirm the existence of the so-called generic scale invariance in a liquid crystal [17].

2. Basic Equations

2.1. Model

Consider a thermotropic nematic liquid crystal layer of thickness $d$ confined between two parallel plates in the presence of the gravitational field and of a stationary heat flux, as depicted in Fig. 1. The transverse dimensions of the cell along the $x$ and $y$ directions are large compared to $d$. The long axis of the nematic molecules are oriented normal to the lower and upper plates located at $z = −d/2, z = d/2$ and maintained at the uniform temperatures $T_1$ and $T_2$, respectively. The hydrodynamic state of the fluid is specified by the velocity field, $\vec{v}(\vec{r}, t)$, the unit vector, $\hat{n}(\vec{r}, t)$, defining the local symmetry axis (director field), and the pressure $p(\vec{r}, t)$ and the temperature $T(\vec{r}, t)$ fields.

We assume that the nematic has already reached a stationary state (NESS) where its state variables take on the values $T_{ss}, p_{ss}, \bar{v}_{ss}$ and $\bar{n}_{ss}$. We shall restrict ourselves to steady states without convection so that $\bar{v}_{ss} = 0$ and assume an initial homeotropic configuration for the director field, $\bar{n}_{ss} = \hat{e}_z$. By symmetry, $p_{ss}$ and $T_{ss}$ can only depend on the $z$ coordinate and, if the gravitational field is $\bar{g} = −g\hat{e}_z$, the stationary state is defined by the solution of the equations

$$d \quad T_2 \quad \bar{n}_{ss} \quad \nabla T \quad \hat{p}_1 \quad \mathcal{\hat{q}}_i \quad \mathcal{\hat{q}}_j \quad \hat{e}_z \quad \hat{e}_x \quad \hat{e}_y \quad \mathcal{\hat{p}}_s \quad \theta \quad \mathcal{\bar{q}}$$

**Figure 1.** Schematic representation of a plane homeotropic nematic cell. A stationary thermal gradient is applied in the direction $\hat{e}_z$. The light scattering geometry which will be used later is also shown. $\mathcal{\hat{q}}_i$ and $\mathcal{\hat{q}}_j$ denote the incident and scattered wave vectors, $\hat{p}_1$ and $\mathcal{\hat{p}}_s$ the incident and scattered polarizations, $\theta$ the scattering angle and $\mathcal{\bar{q}}$ the scattering vector.
where the explicit form of the elements of the hydrodynamic matrix $M_{ij}$ is given by

$$F_i \left( \vec{r}, t \right) = \frac{\partial}{\partial t} a_i \left( \vec{r}, t \right) - F_i \left( \vec{r}, t \right),$$

(4)

where the quantities $\alpha$, $c^2$, $c_p$, and $c_v$ have been defined in the Appendix. $Q_i$, $\Sigma_{ij}$, and $Y_i$ represent the stochastic components of the heat flux, the stress tensor, and the so-called director’s quasi-current respectively. $F_i$ has the following stochastic properties:

$$\langle F_i \left( \vec{r}, t \right) \rangle_{ss} = 0,$$

(6)

$$\langle F_i \left( \vec{r}_1, t_1 \right) F_j \left( \vec{r}_2, t_2 \right) \rangle_{ss} = Q_{ij} \langle \delta t - t_1 \rangle,$$

(7)

where $Q_{ij} \left( \vec{r}_1, \vec{r}_2 \right)$ is the covariance matrix and its explicit form in the stationary state is obtained by replacing, in its equilibrium expression, $Q_{ij}^{eq}$, all the equilibrium quantities appearing there as parameters, by their position-dependent steady state values. In Ref. 33 we have calculated $Q_{ij}^{eq}$ for a nematic liquid crystal for the same homeotropic geometry we are considering in this work.

It will be convenient to write the time evolution equations for the fluctuations in Fourier space by defining the space-time Fourier transform of an arbitrary field $f \left( \vec{r}, t \right)$ by

$$\hat{f} \left( \vec{k}, \omega \right) = \int dt e^{i \omega t} \hat{f} \left( \vec{k}, t \right),$$

(8)

where $\hat{f} \left( \vec{k}, t \right)$ is the space Fourier transform of $f$ defined as

$$\hat{f} \left( \vec{k}, t \right) = \int d\vec{r} e^{-i \vec{k} \cdot \vec{r}} f \left( \vec{r}, t \right).$$

(9)

In this way Eq. (4) becomes

$$\frac{\partial}{\partial t} \hat{a}_i \left( \vec{k}, t \right) = -H_{ij} \hat{a}_j \left( \vec{k}, t \right) - \hat{F}_i \left( \vec{k}, t \right),$$

(10)

where $H_{ij} \left( \vec{k} \right)$ is the linear hydrodynamic matrix in $\hat{k}$-space. Since the matrix elements $M_{ij}$ may depend on $z$, in arriving at Eq. (10) we restricted ourselves to considering only those fluctuations that can be described as a superposition of plane waves within a horizontal nematic layer of thickness $l_0$, so that the spatial variation of the thermodynamic quantities inside the layer can be neglected. In this way the elements $M_{ij}$ are considered to be constants. Furthermore, as a first approximation to a more general description of nematic fluctuations where boundary conditions on the nematic sample should be taken into account [16, 34], we shall calculate correlation functions for points $\vec{r}$ and $\vec{r}'$, with $z$ and $z'$ such that $|z - z'| < l_0$, which correspond also to positions far away from the boundaries, that is, $|z - z'| \ll d$ and $|z + z'| \ll d$.

### 2.2. Transverse and longitudinal variables

Since the cartesian components of the director and velocity fields are strongly coupled, it will be convenient to work with a different representation where this coupling is minimized. This may be accomplished by defining the transverse components of the director and velocity fluctuating fields as their projections along the perpendicular direction to the $\hat{k} - n_{ss}$ plane, that is,

$$\delta \hat{n}_1 = \frac{1}{k_\perp} \hat{n}_{ss} \cdot \left( \hat{k} \times \delta \hat{n} \right),$$

(11)

and

$$\delta \hat{v}_1 = \frac{1}{k_\perp} \hat{n}_{ss} \cdot \left( \hat{k} \times \delta \hat{v} \right),$$

(12)

where $k_\perp = \left( k_x^2 + k_y^2 \right)^{1/2}$. Similarly, the longitudinal components of these fields are defined as

$$\delta \hat{n}_3 = \frac{1}{k} \hat{k} \cdot \delta \hat{n},$$

(13)

$$\delta \hat{v}_3 = \frac{1}{k} \hat{k} \cdot \delta \hat{v},$$

(14)

$$\delta \hat{v}_2 = \frac{1}{k k_\perp} \hat{k} \times \left[ \hat{k} \times \hat{n}_{ss} \right] \cdot \delta \hat{v},$$

(15)
which correspond to independent projections of the fluctuating fields in the $\vec{k} - n_\perp$ plane. It follows directly from definitions (11)-(15) and the explicit form of Eq. (10) that the longitudinal and transverse components evolve independently. It is therefore convenient to split the state vector $\vec{a}$ into longitudinal and transverse parts, namely, $a_i^{(l)} = (\delta \tilde{n}_3, \delta \tilde{v}_2, \delta T, \delta \tilde{v}_3, \delta p)^T$ and $a_i^{(t)} = (\delta \tilde{n}_1, \delta \tilde{v}_1)^T$, which satisfy the equations

$$\frac{\partial}{\partial t} a_i^{(l)} = -H_j^{(l)}(\vec{k}) a_j^{(l)} - \tilde{F}_i^{(l)},$$

$$\frac{\partial}{\partial t} a_i^{(t)} = -H_j^{(t)}(\vec{k}) a_j^{(t)} - \tilde{F}_i^{(t)},$$

where $H_j^{(l)}(\vec{k})$ and $H_j^{(t)}(\vec{k})$ are the reduced longitudinal and transverse hydrodynamic matrices. Their explicit forms may be obtained from $M_{ij}$ and will be given explicitly in the next section for a set of variables defined below, which have the same dimensions and are more suitable for implementing the time scaling perturbation theory later on. In Eqs. (16), (17), $\tilde{F}_i^{(l)}$ and $\tilde{F}_i^{(t)}$ are the corresponding random forces given in terms of the stochastic currents $\tilde{\Sigma}_{ij}$, $\tilde{Q}_i$, $\tilde{Y}_i$

$$\tilde{F}_i^{(l)}(\vec{k}, t) = \left( \begin{array}{c} \frac{1}{\rho_{ss}} \left( k_x \tilde{Y}_x + k_y \tilde{Y}_y \right) \\
\frac{i}{\rho_{ss} c_n} k_j \tilde{Q}_j \\
\frac{i}{\rho_{ss} c_n} k_j \tilde{Q}_j \\
\frac{i}{\rho_{ss} c_n} k_j \tilde{Q}_j \\
\frac{i}{\rho_{ss} c_n} k_j \tilde{Q}_j \end{array} \right),$$

$$\tilde{F}_i^{(t)}(\vec{k}, t) = \left( \begin{array}{c} \frac{1}{\rho_{ss}} \left( k_x \tilde{Y}_y - k_y \tilde{Y}_x \right) \\
\frac{i}{\rho_{ss} c_n} k_j \tilde{Q}_j \\
\frac{i}{\rho_{ss} c_n} k_j \tilde{Q}_j \\
\frac{i}{\rho_{ss} c_n} k_j \tilde{Q}_j \\
\frac{i}{\rho_{ss} c_n} k_j \tilde{Q}_j \end{array} \right).$$

In the next section we shall solve Eqs. (16) and (17) by taking into account that the nematodynamic variables evolve on three widely separated time-scales and using a time scale perturbation theory which allows us to describe the dynamics of these variables separately.

### 3. Time Scaling Perturbation Theory

In previous works [31, 33], we have shown that, for a typical thermotropic nematic, the modes associated with the director relaxation are much slower than those of the velocity field, i.e. $\tau_{\text{orientation}}/\tau_{\text{velocity}} \approx 10^5$. Therefore, transverse (longitudinal) director fluctuations relax to equilibrium much more slowly than transverse (longitudinal) velocity fluctuations. Actually, this situation is common to a variety of physical systems where different variables evolve on different time-scales. The existence of widely separated time-scales may be exploited to eliminate the fast variables from the general dynamical equations, thus obtaining a reduced description in which only the slow variables are involved. A time-scale separation procedure may be implemented for our model in order to diminish the remaining couplings between nematodynamic fluctuations by using the time scaling perturbation method introduced by Geigenmüller et al. [37-36]. This method will allow us to find, on the slow time-scales, a contracted description in terms of the slow variables only, with a reduced dynamical matrix which can be constructed by a perturbation procedure. Moreover, a contracted description on the fast time-scales can be found in terms of the fast fluctuations only, and a corresponding reduced matrix given also by a perturbation expansion.

In order to compare the relative magnitudes of the various elements of the hydrodynamic matrices later on, we rescale the variables in such a way that they all have the same dimensions. Following the same procedure as for an isotropic fluid [40], we define the rescaled pressure, temperature, velocity and director fluctuations as

$$\delta \tilde{p}(\vec{k}, t) = \left( \frac{\chi_T}{\gamma} \right)^{1/2} \delta \tilde{p},$$

$$\delta \tilde{T}(\vec{k}, t) = \left( \frac{p_{ss} c_n}{T_{ss}} \right)^{1/2} \delta \tilde{T},$$

$$\delta \tilde{v}_{\mu}(\vec{k}, t) = \rho_{ss}^{1/2} \delta \tilde{v}_{\mu}, \mu = 1, 2, 3,$n

$$\delta \tilde{n}_{\mu}(\vec{k}, t) = \gamma_1 k_z^2 \delta n_{\mu}, \mu = 1, 3,$$

where $\chi_T$ denotes the isothermal compressibility coefficient and $\gamma_1$ is the orientational viscosity coefficient of the nematic.

### 3.1. Transverse variables

The dynamics of the transverse variables $(\delta \tilde{n}_1, \delta \tilde{v}_1)$ is given by the linear stochastic equation

$$\frac{\partial}{\partial t} \begin{pmatrix} \delta \tilde{n}_1 \\ \delta \tilde{v}_1 \end{pmatrix} = - \begin{pmatrix} \tilde{H}_{11}^{(t)} & \tilde{H}_{12}^{(t)} \\ \tilde{H}_{21}^{(t)} & \tilde{H}_{22}^{(t)} \end{pmatrix} \begin{pmatrix} \delta \tilde{n}_1 \\ \delta \tilde{v}_1 \end{pmatrix} - \begin{pmatrix} \tilde{F}_1^{(t)} \\ \tilde{F}_2^{(t)} \end{pmatrix},$$

where $\tilde{H}_{ij}^{(t)}$ and $\tilde{F}_i^{(t)}$ represent the normalized transverse hydrodynamic matrix and stochastic force elements. It follows from Eqs. (17), (22), (23), that

$$\tilde{H}^{(t)} = \begin{pmatrix} \frac{1}{\rho_{ss} c_n} \left( K_{22} k_z^2 + K_{33} k_z^2 \right) & -i \frac{(1+\lambda)\gamma_1}{2\rho_{ss}} k_z^2 \\ -i \frac{(1+\lambda)\gamma_1}{2\rho_{ss}} k_z^2 & \frac{1}{\rho_{ss} c_n} \left( \nu_2 k_z^2 + \nu_3 k_z^2 \right) \end{pmatrix}$$

$$\tilde{H}_{ij}^{(t)} = \begin{pmatrix} \tilde{H}_{11}^{(t)} & \tilde{H}_{12}^{(t)} \\ \tilde{H}_{21}^{(t)} & \tilde{H}_{22}^{(t)} \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{H}_{11}^{(t)} & \tilde{H}_{12}^{(t)} \\ \tilde{H}_{21}^{(t)} & \tilde{H}_{22}^{(t)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho_{ss} c_n} \left( K_{22} k_z^2 + K_{33} k_z^2 \right) & -i \frac{(1+\lambda)\gamma_1}{2\rho_{ss}} k_z^2 \\ -i \frac{(1+\lambda)\gamma_1}{2\rho_{ss}} k_z^2 & \frac{1}{\rho_{ss} c_n} \left( \nu_2 k_z^2 + \nu_3 k_z^2 \right) \end{pmatrix}$$

with $\bar{F}^{(t)}_1 = \gamma_1 k_2 \rho_{ss}^{-1/2} F^{(t)}_1$, $F^{(t)}_2 = \rho_{ss}^{1/2} F^{(t)}_2$. Here $K_2$ and $K_3$ represent the twist and bend elastic constants, respectively, while $\nu_2$ and $\nu_3$ denote two shear viscosity coefficients of the nematic and $\lambda$ is a non-dissipative coefficient associated with director relaxation.

We now classify these transverse variables as slow or fast by estimating the magnitude of the matrix elements $H^{(t)}_{ij}$ according to the method introduced in Refs. 37 to 39. To this end, note that $H^{(t)}_{11} \sim H^{(t)}_{21} \sim \nu k^2 / \nu$ and $H^{(t)}_{12} \sim H^{(t)}_{22} \sim \nu k^2 / \rho_{ss}$, where $\nu$ and $K$ denote any one of the viscosity coefficients and elastic constants of the nematic, respectively. For a typical thermotropic nematic $K \sim 10^{-7}$ dyn, $\nu \sim 10^{-1}$ poise, $\rho_{ss} \sim 1$ g cm$^{-3}$, the inequality $K/\nu \ll \nu/\rho_{ss}$ holds, $H^{(t)}_{11}$ and $H^{(t)}_{21}$ are small compared with $H^{(t)}_{12}$, $H^{(t)}_{22}$. If we follow the dynamics of $\delta \bar{n}_1$ and $\delta \bar{v}_1$ for a time of order $\tau_{sf} = 1 / H^{(t)}_{22}$ the contribution of $H^{(t)}_{22}$ is order unity, while that of $H^{(t)}_{11}$ is smaller, say of order $\varepsilon_1 \ll 1$. More precisely, $\varepsilon_1$ is the largest of the ratios of $H^{(t)}_{11} \sim H^{(t)}_{21}$ with respect to $H^{(t)}_{22}$, that is, $\varepsilon_1 \sim K \rho_{ss} / \nu^2$. This observation allows us to identify $\delta \bar{n}_1$ as a slow variable and $\delta \bar{v}_1$ as a fast variable. There remains, however, a strong dynamical coupling between these variables because $H^{(t)}_{12}$ is also of order unity. Physically this means that the slow variable is forced to participate in the fast motion, although it hardly influences the motion of the fast one. It is possible to decrease this coupling by using a transformation, $T^{(t)}$, which transforms the state vector $(\delta \bar{n}_1, \delta \bar{v}_1)$ and random force vector $(\bar{F}_{1}^{(t)}, \bar{F}_{2}^{(t)})$ according to [37]-[39]

$$
\begin{pmatrix}
\delta \bar{n}_1' \\
\delta \bar{v}_1'
\end{pmatrix} = T^{(t)}
\begin{pmatrix}
\delta \bar{n}_1 \\
\delta \bar{v}_1
\end{pmatrix},
$$

$$
\begin{pmatrix}
\bar{F}^{(t)}_{1}' \\
\bar{F}^{(t)}_{2}'
\end{pmatrix} = T^{(t)}
\begin{pmatrix}
\bar{F}^{(t)}_{1} \\
\bar{F}^{(t)}_{2}
\end{pmatrix},
$$

(26)

so that the new dynamical matrix

$$
[H^{(t)}]' = T^{(t)} H^{(t)} T^{(t)^{-1}}
$$

has the form

$$
\begin{pmatrix}
0 & 0 \\
0 & F^{(t)}
\end{pmatrix} + \varepsilon_1
\begin{pmatrix}
A^{(t)} & B^{(t)} \\
C^{(t)} & D^{(t)}
\end{pmatrix},
$$

(27)

where the quantities $F^{(t)}$, $A^{(t)}$, $B^{(t)}$, $C^{(t)}$ and $D^{(t)}$ are all of order $\nu k^2 / \rho_{ss}$. Following Geigenmüller et al., $T^{(t)}$ is chosen as

$$
T^{(t)} =
\begin{pmatrix}
1 + \frac{\rho_{ss}^{1/2} H^{(t)}_{22}}{\nu k^2} \\
0
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1 + \frac{\gamma_1 k_2^2}{2 \nu_2 k^2_1 + \nu_3 k^2_2} \\
0
\end{pmatrix}.
$$

(28)

As a consequence, $\delta \bar{n}_1' = \delta \bar{n}_1$, $\bar{F}^{(t)}_{1}' = \bar{F}^{(t)}_{1}$,

$$
\delta \bar{n}_1' = \delta \bar{n}_1 + i \frac{1 + \lambda}{2 \nu_2 k^2_1 + \nu_3 k^2_2} \delta \bar{v}_1
$$

(29)

and

$$
\bar{F}^{(t)}_{1}' = \bar{F}^{(t)}_{1} + i \frac{1 + \lambda}{2 \nu_2 k^2_1 + \nu_3 k^2_2} \bar{F}^{(t)}_{2}.
$$

(30)

From (28) it can be shown that $[H^{(t)}]'$ has the desired form (27). Thus after the transformation $T^{(t)}$, Eq. (26) takes the form

$$
\frac{\partial}{\partial t} \begin{pmatrix}
\delta \bar{n}_1' \\
\delta \bar{v}_1'
\end{pmatrix} =
\begin{pmatrix}
0 & 0 \\
0 & F^{(t)}
\end{pmatrix} + \varepsilon_1
\begin{pmatrix}
A^{(t)} & B^{(t)} \\
C^{(t)} & D^{(t)}
\end{pmatrix}
\begin{pmatrix}
\delta \bar{n}_1' \\
\delta \bar{v}_1'
\end{pmatrix} - \begin{pmatrix}
\bar{F}^{(t)}_{1}' \\
\bar{F}^{(t)}_{2}'
\end{pmatrix}.
$$

(31)

This equation may be rewritten as a set of independent equations for $\delta \bar{n}_1'$ and $\delta \bar{v}_1$ by applying the time scaling perturbation theory presented in Sec. 5 of Ref. 39. The dynamics of $\delta \bar{n}_1'$ and $\delta \bar{v}_1$ is described by two reduced hydrodynamic coefficients $\omega^{red}_{n_1}(\bar{k})$ and $\omega^{red}_{v_1}(\bar{k})$, respectively, such that

$$
\frac{\partial}{\partial t} \delta \bar{n}_1' = -\omega^{red}_{n_1}(\bar{k}) \delta \bar{v}_1 - \bar{F}^{(t)}_{1}',
$$

(32)

and

$$
\frac{\partial}{\partial t} \delta \bar{v}_1 = -\omega^{red}_{v_1}(\bar{k}) \delta \bar{n}_1' - \bar{F}^{(t)}_{2}'.
$$

(33)

It is important to emphasize that these equations are valid for different time scales. Eq. (32) describes the dynamics of $\delta \bar{n}_1'$ in the slow-time scale at times much larger than $\tau_{sf}$, while (33) describes the evolution of $\delta \bar{v}_1$ in the fast time scale, that is, for times of order $\tau_{sf}$. Up to the first order in $\varepsilon_1$, we have $\omega^{red}_{n_1}(\bar{k}) = \varepsilon_1 A^{(t)}$ and $\omega^{red}_{v_1}(\bar{k}) = F^{(t)} + \varepsilon_1 D^{(t)}$. The explicit expressions for these first order corrections are then

$$
\omega^{red}_{n_1}(\bar{k}) = \frac{1}{\gamma_1} \left( K_2 k^2_1 + K_3 k^2_2 \right)
\times \left[ 1 + \frac{\gamma_1 (1 + \lambda)^2 k^2_2}{4 \nu_2 k^2_1 + \nu_3 k^2_2} \right],
$$

(34)

$$
\omega^{red}_{v_1}(\bar{k}) = \frac{1}{\rho_{ss}} \left( \nu_2 k^2_1 + \nu_3 k^2_2 \right)
\times \left[ 1 - \left( \frac{1 + \lambda}{2} \right) \frac{\rho_{ss} \left( K_2 k^2_1 + K_3 k^2_2 \right) k^2_2}{(\nu_2 k^2_1 + \nu_3 k^2_2)^2} \right].
$$

(35)
3.2. Longitudinal variables

Let us now consider the longitudinal variables $(\delta n_3, \delta v_2, \delta T, \delta v_3, \delta \bar{p})$ and use the same procedure as that followed for the transverse variables in the previous subsection. However, we notice that in this case the time scaling perturbation method may be implemented twice, because Eq. (16) describes director relaxation as well as visco-heat and propagating modes which evolve on three widely different time scales [33]. Thus, we shall apply the time scaling procedure in two steps in order to systematically decouple the fast variables. For this purpose it will be convenient to start by grouping the longitudinal normalized variables in the following vectors:

$$\vec{w}(\vec{k}, t) \equiv \begin{pmatrix} \delta n_3 \\ \delta v_2 \\ \delta T \\ \delta v_3 \\ \delta \bar{p} \end{pmatrix}$$

and

$$\vec{z}(\vec{k}, t) \equiv \begin{pmatrix} \delta v_3 \\ \delta \bar{p} \end{pmatrix}.$$ 

From Eqs. (16) and (20) - (23), it follows that they satisfy the stochastic equation

$$\frac{\partial}{\partial t} \begin{pmatrix} \vec{w} \\ \vec{z} \end{pmatrix} = - \begin{pmatrix} \vec{H}_{ww} & \vec{H}_{wz} \\ \vec{H}_{zw} & \vec{H}_{zz} \end{pmatrix} \begin{pmatrix} \vec{w} \\ \vec{z} \end{pmatrix} - \begin{pmatrix} \vec{F}_w \\ \vec{F}_z \end{pmatrix}$$

with

$$\vec{H}_{ww} = \begin{pmatrix} \omega_{d3}(\vec{k}) & \frac{\gamma_1 k_i k_j}{\rho_{xx}} \tilde{\lambda}(\vec{k}) & 0 \\ \frac{k^2}{k_i k_j} \tilde{\lambda}(\vec{k}) \omega_{d3}(\vec{k}) & \omega_{v2}(\vec{k}) & \left(\frac{\gamma - 1}{\gamma}\right)^{1/2} \frac{\kappa k}{c} \omega_T(\vec{k}) \\ -i \frac{\rho_{xx} D^2 k_i k_j}{\gamma_1} k \tilde{\omega}_T & \frac{k_i k_j}{k} \omega_{v2} \tilde{\omega}_T & \omega_T(\vec{k}) \end{pmatrix},$$

$$\vec{H}_{wz} = \begin{pmatrix} \frac{\left(\frac{\gamma - 1}{\gamma}\right)^{1/2} \kappa k}{c} \omega_T & 0 \\ -i \frac{\kappa}{c} \frac{k^2}{k_i k_j} \tilde{\omega}_v & \omega_v(\vec{k}) & \left(\frac{\gamma - 1}{\gamma}\right)^{1/2} \frac{\kappa k}{c} \omega_T \tilde{\omega}_v \\ i \left(\frac{\gamma - 1}{\gamma}\right)^{1/2} \kappa k & \frac{k_i k_j}{k} \omega_{v2} \tilde{\omega}_v & 0 \end{pmatrix},$$

$$\vec{H}_{zw} = \begin{pmatrix} \omega_{v3}(\vec{k}) & 0 & \left(\frac{\gamma - 1}{\gamma}\right)^{1/2} \frac{k_i k_j}{k} \omega_T \tilde{\omega}_v \\ -i \frac{\kappa}{c} \frac{k^2}{k_i k_j} \tilde{\omega}_v & \omega_v(\vec{k}) & \left(\frac{\gamma - 1}{\gamma}\right)^{1/2} \frac{\kappa k}{c} \omega_T \tilde{\omega}_v \\ ic k - \frac{\gamma k}{c} & \frac{k_i k_j}{k} \omega_{v2} \tilde{\omega}_v & 0 \end{pmatrix},$$

$$\vec{H}_{zz} = \begin{pmatrix} \omega_v(\vec{k}) & 0 \\ ic k - \frac{\gamma k}{c} & \frac{k_i k_j}{k} \omega_{v2} \tilde{\omega}_v \\ \frac{\gamma k}{c} & \omega_T(\vec{k}) \end{pmatrix},$$

where $c$ denotes the isentropic sound speed of the nematic as presented in the Appendix, and we have introduced the following
described hydrodynamically. On the other hand, typically spatial distances of the order $10 \, \text{Å}$ are small in comparison with the correlation length $\lambda(\vec{k})$, with magnitudes such that $\nu \kappa / \rho \ll 1$. This estimate shows that the gradient terms in Eqs. (38) are much smaller than $\nu \kappa / \rho$, which, again, can be chosen so that the new dynamic matrix $H_{zz}$ is of order unity. Additionally, $H_{ww}$ and $H_{zz}$ are much larger than all the other elements.

In Figs. (33)-(39), $K_1$ represents the splay elastic constant, $\nu_1$, $\nu_2$, ... , $\nu_5$ are viscosity coefficients, and $D_{ss}$ and $D_{\perp s}$ denote the thermal diffusivities of the nematic along $\vec{n}_{ss}$ and along the direction perpendicular to it, respectively; they are related to the thermal diffusion coefficients by the usual expression

$$D_{ss} = \frac{\kappa_{ss}}{\rho_{ss} c_p}.$$  

The stochastic force vectors $\vec{F}_w$ and $\vec{F}_z$ are given by

$$\vec{F}_w = \left( \begin{array}{c} F_{1w}^{(l)} \\ F_{2w}^{(l)} \\ F_{3w}^{(l)} \end{array} \right) = \left( \begin{array}{c} \frac{\gamma_1 k_1}{\rho_{ss} c_p} F_1^{(l)} \\ \frac{\nu_1}{\rho_{ss}} F_2^{(l)} \\ \frac{\nu_2}{\rho_{ss}} F_3^{(l)} \end{array} \right),$$  

$$\vec{F}_z = \left( \begin{array}{c} F_{4z}^{(l)} \\ F_{5z}^{(l)} \end{array} \right) = \left( \begin{array}{c} \frac{1}{\rho_{ss} c_p} F_4^{(l)} \\ \frac{\nu_4}{\rho_{ss}} F_5^{(l)} \end{array} \right).$$

As for the transverse variables, $\vec{w}$ and $\vec{z}$ may be classified as slow and fast variables by estimating the magnitude of the elements $H_{ww}$, $H_{zw}$, $H_{zw}$ and $H_{zz}$. However, before carrying out this separation, we impose the condition of considering only wave vectors $\vec{k}$ with magnitudes such that

$$k_1 \ll k \ll k_2,$$  

where $k_1 = c^{-1} (c_0 / T_{ss})^{1/2} (d T_{ss} / dz)$ and $k_2 = c / D_T$. Here $D_T = \kappa / \rho_{ss} c_p$ and $\kappa \sim \kappa_{ss} \sim \kappa_{\perp s}$. The upper bound $k_2$ guarantees that the terms of order $D_T k_2^2$ are small compared to $ck$. Similarly, the lower bound $k_1$ ensures that all the gradient terms in Eqs. (38) are also much smaller than $ck$. However, notice that for typical parameter values of a thermotropic nematic, $D_T \sim 10^{-3} \, \text{cm}^2 \, \text{s}^{-1}$, and $k_2$ probes spatial distances of the order 10 Å, which are too small to be described hydrodynamically. On the other hand, typically

$$k_1 \sim l^{-1},$$  

where $l \gg l_0$ is the distance over which the variations of the gradient are significant. Since we are restricted to considering distances $|\vec{T}' - \vec{T}| \ll l_0 \ll l$, which implies $k \gg l^{-1}$, we notice that (53) is not a new restriction in our analysis, but is consistent with the hydrodynamic treatment and our previous assumptions. Thus, we note that the elements of $H_{ww}$, $\alpha, \beta = w, z$, proportional to $ck$ and which are contained only in $H_{ww}$ and $H_{zz}$, are much larger than all the other elements.

If we follow the dynamics of $\vec{w}$ and $\vec{z}$ for a time of order $\tau_{2f} = 1 / ck$, the contribution of $H_{zz}$ is order unity, while that of $H_{ww}$ is small, say of order $\varepsilon_2 \ll 1$. More precisely, $\varepsilon_2$ is the largest of the ratios of the elements in $H_{ww}$ and $H_{zw}$ with respect to $ck$. In our case, $\varepsilon_2 \sim \nu c / \rho_{ss}$. Furthermore, $H_{zw}$ is of order $\varepsilon_2$. This observation allows us to identify the two groups of variables, $\vec{w}$ and $\vec{z}$, as slow and fast, respectively. There remains, however, a strong dynamic coupling between $\vec{w}$ and $\vec{z}$ because $H_{zw}$ is of order unity. As before, to decrease this coupling we introduce a transformation $T$, such that

$$\left( \begin{array}{c} \vec{w}' \\ \vec{z}' \end{array} \right) = T \left( \begin{array}{c} \vec{w} \\ \vec{z} \end{array} \right),$$  

$$\left( \begin{array}{c} \vec{F}_w' \\ \vec{F}_z' \end{array} \right) = T \left( \begin{array}{c} \vec{F}_w \\ \vec{F}_z \end{array} \right),$$  

which, again, can be chosen so that the new dynamic matrix has the form proposed by Geigenmüller et al.,

$$H_{ww} \quad H_{ww}$$  

$$H_{zw} \quad H_{zz}$$  

$$= \left( \begin{array}{cc} 0 & 0 \\ 0 & F \end{array} \right) + \varepsilon_2 \left( \begin{array}{cc} A & B \\ C & D \end{array} \right),$$  

where $F$, $A$, $B$, $C$ and $D$ are all of order $ck$. For the present model, the explicit form of $T$ is

$$T = \left( \begin{array}{cc} 1 & c \\ 0 & 1 \end{array} \right),$$  

with
\[
e = - \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
(\frac{\gamma - 1}{\gamma})^{1/2} & 0 & 0 
\end{pmatrix}.
\] (57)

From (56) and (57) we obtain \( \bar{w}' = \bar{z}, \bar{F}'_z = \bar{F}_z \),
\[
\bar{w}' = \begin{pmatrix}
\delta \bar{n}_3 \\
\delta \bar{n}_2 \\
\delta \bar{T}'
\end{pmatrix},
\] (58)
with \( \delta \bar{T}' = \delta \bar{T} - [(\gamma - 1)/\gamma]^{1/2}\delta \bar{p}, \) and
\[
\bar{F}'_w = \begin{pmatrix}
\bar{F}'_1(l) \\
\bar{F}'_2(l) \\
\bar{F}'_3(l)
\end{pmatrix},
\] (59)
with \( \bar{F}'_3(l) = \bar{F}_3(l) - [(\gamma - 1)/\gamma]^{1/2}\bar{F}_5(l) \). Therefore,
\[
\begin{align*}
H^{red}_{ww} &= \begin{pmatrix}
\omega_{d3}(\bar{k}) & i\gamma_{kl} k_{l \perp} & -i\frac{2}{\gamma} \frac{\delta}{\nu_s} \\
i\gamma_{kl} k_{l \perp} & \omega_{d3}(\bar{k}) & 0 \\
i\frac{2}{\gamma} \frac{\delta}{\nu_s} & 0 & \frac{\nu_s}{\nu_c} \omega_T(\bar{k})
\end{pmatrix}, \\

H^{red}_{zz} &= \begin{pmatrix}
\omega_{z3}(\bar{k}) & i\gamma_{kl} k_{l \perp} & -i\frac{2}{\gamma} \frac{\delta}{\nu_s} \\
i\gamma_{kl} k_{l \perp} & \omega_{z3}(\bar{k}) & 0 \\
i\frac{2}{\gamma} \frac{\delta}{\nu_s} & 0 & \frac{\nu_s}{\nu_c} \omega_T(\bar{k})
\end{pmatrix}.
\end{align*}
\] (60)

Thus we have eliminated from the description of the slow variables \( \bar{w}' \), the coupling with the fast ones. At this point it is essential to point out that, if the elements of \( H^{red}_{ww} \) are evaluated for typical parameter values of a thermotropic, we find that some of them are much larger than others in the same sense as in the case of the transverse variables. The dominant elements are of order \( \nu k^2/\rho_{ss} \) and are contained in the matrix elements which describe normalized velocity fluctuations only. This result implies that some of the slow variables \( \bar{w}' \) evolve in time faster than others and suggests that, to describe the dynamics of the slower variables, it is convenient to apply the time-scaling perturbation theory again. Thus, following the same line of reasoning implemented in the previous cases, we group the components of \( \bar{w}' \) as
\[
\begin{pmatrix}
\delta \bar{n}_3 \\
\delta \bar{p} \\
\delta \bar{T}'
\end{pmatrix} = \begin{pmatrix}
\delta \bar{n}_3 \\
\delta \bar{p} \\
\delta \bar{T}'
\end{pmatrix} + \varepsilon_2 \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \begin{pmatrix}
\bar{w}' \\
\bar{z}
\end{pmatrix}.
\] (61)

If we now apply the time-scaling perturbation theory of Geigenmüller et al., we may rewrite Eq. (60) as a set of independent equations for \( \bar{w}' \) and \( \bar{z} \), in terms of the reduced matrices \( H^{red}_{ww} \) and \( H^{red}_{zz} \), namely,
\[
\begin{align*}
\frac{\partial}{\partial \bar{t}} \bar{w}' &= -H^{red}_{ww} \bar{w}' - \bar{F}'_w, \\
\frac{\partial}{\partial \bar{t}} \bar{z} &= -H^{red}_{zz} \bar{z} - \bar{F}'_z.
\end{align*}
\] (62)

Up to the first order in \( \varepsilon_2 \) we have \( H^{red}_{ww} = \varepsilon_2 A \) and \( H^{red}_{zz} = F + \varepsilon_2 D \), whose explicit expressions read
\[
H^{red}_{ww} = \begin{pmatrix}
\omega_{d3}(\bar{k}) & i\gamma_{kl} k_{l \perp} & -i\frac{2}{\gamma} \frac{\delta}{\nu_s} \\
i\gamma_{kl} k_{l \perp} & \omega_{d3}(\bar{k}) & 0 \\
i\frac{2}{\gamma} \frac{\delta}{\nu_s} & 0 & \frac{\nu_s}{\nu_c} \omega_T(\bar{k})
\end{pmatrix},
\] (63)

\[
H^{red}_{zz} = \begin{pmatrix}
\omega_{z3}(\bar{k}) & i\gamma_{kl} k_{l \perp} & -i\frac{2}{\gamma} \frac{\delta}{\nu_s} \\
i\gamma_{kl} k_{l \perp} & \omega_{z3}(\bar{k}) & 0 \\
i\frac{2}{\gamma} \frac{\delta}{\nu_s} & 0 & \frac{\nu_s}{\nu_c} \omega_T(\bar{k})
\end{pmatrix}.
\] (64)

Thus we have eliminated from the description of the slow variables \( \bar{w}' \), the coupling with the fast ones. At this point it is essential to point out that, if the elements of \( H^{red}_{ww} \) are evaluated for typical parameter values of a thermotropic, we find that some of them are much larger than others in the same sense as in the case of the transverse variables. The dominant elements are of order \( \nu k^2/\rho_{ss} \) and are contained in the matrix elements which describe normalized velocity fluctuations only. This result implies that some of the slow variables \( \bar{w}' \) evolve in time faster than others and suggests that, to describe the dynamics of the slower variables, it is convenient to apply the time-scaling perturbation theory again. Thus, following the same line of reasoning implemented in the previous cases, we group the components of \( \bar{w}' \) as
and the semi-slow variable \( \bar{y} \), with the resulting equations:

\[
\frac{\partial}{\partial t} \delta n_3' = - \omega_{n3}^{\text{red}} \left( \frac{\bar{k}}{k} \right) \left( F_1^{(1)} \right) \delta n_3 - F_1^{(1)r}, \quad (67)
\]

with

\[
\omega_{n3}^{\text{red}} \left( \frac{\bar{k}}{k} \right) = \frac{1}{\gamma_1} \left( \frac{1}{K_{1} k_{\perp}^2 + K_{3} k_{\parallel}^2} \right) \left\{ 1 + \frac{1}{4} \frac{\gamma_1 \left( \left( 1 + \lambda \right) k_{\perp}^2 + \left( \frac{1 - \lambda}{\nu_3} k_{\parallel}^2 \right) \right)}{\nu_3 k_{\perp}^4 + 2 \left( \nu_1 + \nu_2 - \nu_3 \right) k_{\perp}^2 k_{\parallel}^2 + \nu_3 k_{\parallel}^4} \right\}, \quad (68)
\]

\[
F_1^{(1)r} = \bar{F}_1^{(1)} + \frac{i}{2} \frac{\gamma_1 k_{\perp} k_{\parallel}}{\nu_3 k_{\perp}^4 + 2 \left( \nu_1 + \nu_2 - \nu_3 \right) k_{\perp}^2 k_{\parallel}^2 + \nu_3 k_{\parallel}^4} \bar{F}_1^{(2)}, \quad (69)
\]

\[
\frac{\partial}{\partial t} \bar{y} = - \mathbf{H}_{yy}^{\text{red}} \bar{y} - \bar{F}_y, \quad (70)
\]

where

\[
\frac{\partial}{\partial t} \delta n_3' = - \omega_{n3}^{\text{red}} \left( \frac{\bar{k}}{k} \right) \delta n_3 - \bar{F}_3^{(1)r}, \quad (72)
\]

Equations (67) and (70) are correct in the slow and semi-slow time-scales up to the first order in \( \varepsilon_1 \). Therefore, we may then say that the time evolution of the nematic’s longitudinal variables at the hydrodynamic level is described by Eqs. (62), (67) and (70) for the slow variable \( \delta n_3' \), the semi-slow variable \( \bar{y} \), and the fast variable \( \bar{z} \), respectively.

### 4. Orientational Correlation Functions

#### 4.1. Long-Range Order

In previous work we have calculated the correlation functions of the director fluctuations \( \delta n_1 \) and \( \delta n_3 \) in both the equilibrium state and the stationary state induced by the thermal gradient [32]. Since those calculations were performed by using the approach discussed in the previous section, for the sake of completeness in this section we shall briefly review those results. The space-time Fourier transform of Eqs. (32) and (67) in the slow time-scale, may be written in the form

\[
\frac{\partial}{\partial t} \delta n_\mu = - \omega_{n\mu}^{\text{red}} \left( \frac{\bar{k}}{k} \right) \delta n_\mu - \bar{\sigma}_n\mu, \quad (73)
\]

where \( \omega_{n1}^{\text{red}} \) and \( \omega_{n3}^{\text{red}} \) are given by Eqs. (34) and (68). The fluctuating sources \( \bar{\sigma}_n\mu \) are obtained from Eqs. (30), (69) and are defined by

\[
\bar{\sigma}_n = \frac{1}{k_{\perp}} \left\{ k_{\perp} \left( k_{\perp}^2 - k_{\parallel}^2 \right) + \left( 1 + \lambda \right) k_{\perp}^2 \left( k_{\perp}^2 + k_{\parallel}^2 \right) - k_{\perp}^2 k_{\parallel} \left( k_{\perp}^2 + k_{\parallel}^2 \right) \right\}, \quad (74)
\]

As previously discussed, the stochastic components of the quasi-current of the director field and the stress tensor, \( \bar{\Sigma}_i \) and \( \Sigma_{ij} \), respectively, are zero averaged \( \langle \bar{\Sigma}_i \rangle = \langle \Sigma_{ij} \rangle = 0 \) and in equilibrium they satisfy the fluctuation dissipation relations [33]

\[
\langle \bar{\Sigma}_{ij} \bar{r}^\prime, t \rangle = k_B T \frac{1}{\gamma_1} \delta_{ij} \delta \left( \bar{r} - \bar{r}^\prime \right) \delta \left( t - t^\prime \right), \quad (76)
\]

where \( k_B \) is Boltzmann’s constant, \( T \) is the equilibrium temperature of the nematic, \( \delta_{ij} = \delta_{ij} - n_i n_j \) is a projection operator and \( \nu_{ijlm} \) is the viscous tensor as given in Ref. 33. Relations (76) and (77) describe stationary Gaussian Markov processes.
Now, we shall assume that in the non-equilibrium steady state the fluctuation dissipation relations for $\Upsilon_i$ and $\Sigma_{ij}$ can be obtained from (76) and (77) by replacing the equilibrium temperature $T$ by the local $z$-dependent stationary temperature $T_{ss}(z)$ (see Eqs. (A.8) - (A.10)). This assumption leads to

$$\langle \hat{Y}_i (\vec{k}^\prime, \omega) \hat{Y}_j (\vec{k}^\prime, \omega) \rangle = 2 (2\pi)^4 k_B T_0 \frac{1}{\gamma_1} \delta_{ij} \left( 1 + i \frac{\nu_1}{\gamma_1} \frac{\partial}{\partial \Omega_{\vec{k}}^\prime} \right) \delta (\vec{k} + \vec{k}^\prime) \delta (\omega + \omega'),$$

(78)

$$\langle \hat{\Sigma}_{ij} (\vec{k}, \omega) \hat{\Sigma}_{lm} (\vec{k}, \omega) \rangle = 2 (2\pi)^4 k_B T_0 \nu_{1ijlm} \left( 1 + i \frac{\nu_1}{\gamma_1} \frac{\partial}{\partial \Omega_{\vec{k}}^\prime} \right) \delta (\vec{k} + \vec{k}^\prime) \delta (\omega + \omega').$$

(79)

Here $T_0 = (T_1 + T_2)/2$ is the average temperature in the cell and $\bar{T} \equiv (dT_{ss}/dz)/T_0$. The space-time Fourier transform of Eqs. (32) and (67) may be written in the form

$$\delta \hat{n}_\mu (\vec{k}, \omega) = \frac{\sigma_{n\mu} (\vec{k}, \omega)}{-i \omega + \omega_{n\mu} (\vec{k})},$$

(80)

where $\mu = 1, 3$, and for simplicity we have omitted the superscript red in the reduced frequencies $\omega_{n\mu}$. The director auto-correlation functions

$$\hat{X}_{ij} (\vec{k}, \vec{k}^\prime; \omega, \omega') = \left\langle \delta \hat{n}_i (\vec{k}, \omega) \delta \hat{n}_j (\vec{k}^\prime, \omega') \right\rangle_{ss}$$

(81)

can be constructed in terms of the corresponding fluctuation-dissipation relations for $\sigma_{n1}$ and $\sigma_{n3}$,

$$\hat{X}_{ij} = \frac{\left\langle \sigma_{n\mu} (\vec{k}, \omega) \sigma_{n\nu} (\vec{k}^\prime, \omega') \right\rangle_{ss}}{-i \omega + \omega_{n\mu} (\vec{k}) - i \omega' + \omega_{n\mu} (\vec{k}^\prime)},$$

(82)

where no summation over the repeated index $\mu$ is implied. The fluctuation-dissipation relations satisfied by $\sigma_{n1}$ and $\sigma_{n3}$ may be found from Eqs. (74), (75), (78) and (79). By inserting the resulting expressions into Eq. (82), we obtain explicit forms for $\hat{X}_{ij}$. First we shall examine the spatial limiting behavior of these correlations, which is given by

$$X_{ij} (\tau, \tau'; t, t') = \frac{1}{(2\pi)^8} \int d\vec{k} d\vec{k}' d\omega d\omega' e^{i (\vec{k} \cdot \vec{r} + \vec{k}' \cdot \vec{r}' - \omega t - \omega' t')}
\times \hat{X}_{ij} (\vec{k}, \vec{k}^\prime; \omega, \omega'),$$

(83)

when $(t' - t) \to 0$. Since the calculation of $X_{11}$ and $X_{33}$ is formally the same, we shall only describe the procedure for $X_{11}$. Calculating $\left\langle \sigma_{n1} (\vec{k}, \omega) \sigma_{n1} (\vec{k}^\prime, \omega') \right\rangle_{ss}$ with the help of Eqs. (74), (78) and (79), replacing the result into (82), and integrating over $\omega$, $\vec{k}$ and $\vec{k}'$, we find that

$$X_{11} (\tau, \tau'; t, t') = X_{11}^{(1)} (\tau, \tau'; t, t') + X_{11}^{(2)} (\tau, \tau'; t, t'),$$

(84)

$$\lim_{\xi \to 0} |X_{11}| = \frac{k_B T_{ss}(z)}{4\pi (K_1 K_3)^{1/2}} \times \frac{1}{\left( \frac{2K_3}{K_1} \right)^{1/2}} \times \left[ \frac{1 - \left( \frac{dT_{ss}}{dz} \right) \frac{z - z'}{2T_{ss}(z)} \left| \tau' - \tau'' \right|^2 + \left( \frac{K_3}{K_1} \right)^{1/2} \left| \tau - \tau' \right|^2 + \left( \frac{K_3}{K_1} \right)^{1/2} \left| \tau - \tau'' \right|^2 + \left( \frac{K_3}{K_1} \right)^{1/2} \left| \tau - \tau' \right|^2 + \left( \frac{K_3}{K_1} \right)^{1/2} \left| \tau - \tau'' \right|^2 + \left( \frac{K_3}{K_1} \right)^{1/2} \left| \tau - \tau' \right|^2 + \left( \frac{K_3}{K_1} \right)^{1/2} \left| \tau - \tau'' \right|^2 \right].$$

(88)

where $|\tau' - \tau''|^2 = (x - x')^2 + (y - y')^2$. Similarly, for the longitudinal director fluctuations we obtain
Note that, in equilibrium, $\mathbf{\mathcal{B}} = 0$, $T_{ss}(z) = T$, a constant, and the transverse orientational correlation function reduces to

$$\lim_{\xi \to 0} |X_{11}^{\xi}| = \frac{k_B T}{4\pi (K_2 K_3)^{1/2}} \frac{1}{\left| \mathbf{\tau} - \mathbf{\tau}\right|^2}.$$  \hspace{1cm} (90)

while the corresponding correlation function for the longitudinal director components is

$$\lim_{\xi \to 0} |X_{33}^{\xi}| = \frac{k_B T}{4\pi (K_1 - K_3)} \left\{ \frac{1}{\left| \mathbf{\tau} - \mathbf{\tau}\right|^2} - \frac{(K_3/K_1)^{1/2}}{\left[ \left| \mathbf{\tau} - \mathbf{\tau}\right|^2 + \frac{K_3}{K_1} (z - z')^2 \right]^{1/2}} \right\}.$$  \hspace{1cm} (91)

Note that both $X_{11}^{\xi}$ and $X_{33}^{\xi}$ are long range as could have been anticipated due to the well-known property of a nematic which spontaneously exhibits a macroscopic orientational order. They decay algebraically as $|\mathbf{\tau} - \mathbf{\tau}|^{-1}$. In Fig. 2, we plot the normalized static correlation in equilibrium $X_{11}^{0} \equiv 4\pi d (K_2 K_3)^{1/2} X_{11}^{\xi}/k_B T_0$ for $x - x' = y - y' = z = 0$, $T = T_0$ and as a function of normalized distance $z'/d$. In the stationary state, $\mathbf{\mathcal{B}} \neq 0$, the behavior of both correlations is modified by the presence of a term proportional to the temperature gradient which does not decay, in the direction of the temperature gradient. This is also depicted in Fig. 2, where we plot the normalized static correlation $X_{11} \equiv 4\pi d (K_2 K_3)^{1/2} X_{11}/k_B T_0$ for the same conditions as in the equilibrium case and a value of the normalized thermal gradient $\beta' = d\mathcal{B}/d\mathcal{T} = 0.5 > 0$. Note that, because of the presence of $dT_{ss}/dz$, $X_{11}$ becomes asymmetric. When compared to its equilibrium correlations, variations in the direction of lower temperatures and decrease in the opposite direction. Indeed, for a plate separation $d = 10^{-2}$ cm, the difference between both curves becomes significant for $z'/d \sim 10^{-1}$. This means that the wave vectors sensitive to this difference are of the order of $q \sim 10^3$ cm$^{-1}$. Since for light scattering the wave vector $k$ and the scattering angle $\theta$ are related by $q = 2q \sin \theta/2$, where $q_i \sim 10^3$ m$^{-1}$ is the incident wave number, this implies very low scattering angles $\theta \sim 0.1^\circ$. A quantitative evaluation of this nonequilibrium effect on a measurable property will be discussed in the next section for the structure factor of the fluid.

4.2. Light Scattering Spectrum

As an application of the above theory, we now calculate the light scattering spectrum of the nematic in the stationary state and for the scattering geometry defined in Fig. 1. For a nematic dielectric tensor, fluctuations come mainly from director fluctuations, and for the present model the spectral intensity of the scattered light is proportional to the dynamic structure factor

$$S(q', \omega) = \frac{\varepsilon_0^2 \cos^2 \theta}{V_s t_s} \times \text{Re} \left\{ \langle \delta n_1(q', \omega) \delta n_1(-q', -\omega) \rangle \right\},$$  \hspace{1cm} (92)

where $V_s$ and $t_s$ are the scattering volume and scattering time, respectively, $\mathbf{q}' = \mathbf{q} - \mathbf{q}$ is the scattering vector ($q_y = 0$), and $\omega = \omega_2 - \omega_1$ is the frequency shift. $\varepsilon_\parallel = \varepsilon_0 - \varepsilon_\perp$ denotes the dielectric constant anisotropy. Evaluating this expression with the help of Eqs. (78), (79), we obtain

$$S(q', \omega) = \frac{2\varepsilon_0^2 k_B T_0 \cos^2 \theta}{\gamma_1} \frac{\alpha(q')}{\omega^2 + \omega_n^2(q')} \times \left\{ 1 - \frac{1}{T_0} \left( \frac{dT_{ss}}{dz} \right) \frac{2\omega q \beta(q')}{\omega^2 + \omega_n^2(q')} \right\}.$$  \hspace{1cm} (93)

In this result we have considered the spectrum produced by a scattering volume located at the center of the cell. The nonequilibrium contribution has been written up to the smallest power of the wave number $q$, that is, up to the leading term in the hydrodynamic limit $q \to 0$; $\omega_n(q')$ is given by Eq. (34), and the functions $\alpha(q')$ and $\beta(q')$ contain angular information through

$$\alpha(q') = 1 + \left( \frac{1 + \lambda}{2} \right)^2 \frac{\gamma_1 q^2}{\nu_2 q^2 + \nu_3 q^2}$$  \hspace{1cm} (94)

and

$$\beta(q') = \frac{1}{\gamma_1} \left\{ K_3 \alpha(q') + K_2 q^2 + K_3 q^2 \right\},$$  \hspace{1cm} (95)

where $K_2 = 4.4 \times 10^{-3}$ dyne, $K_3 = 8.9 \times 10^{-3}$ dyne.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{(- - -) Decay of $X_{11}$ as a function of $z'/d$. (---) Decay of $X_{11}$ as a function of $z'/d$ for $\beta' = 0.5$. The elastic constant values are $K_2 = 4.4 \times 10^{-3}$ dyne, $K_3 = 8.9 \times 10^{-3}$ dyne.}
\end{figure}
since they do not depend on the magnitude of \( \mathbf{q} \) but only on its orientation. In equilibrium, (93) reduces to

\[
S^\text{eq} (\mathbf{q}, \omega) = \frac{2\pi^2 k_B T \cos^2 \theta}{\gamma_1} \frac{\alpha (\mathbf{q})}{\omega^2 + \omega_{n1} (\mathbf{q})^2},
\]

and exhibits a \( q^{-4} \) dependence which is responsible for the well-known long-range order spatial behavior of the orientational correlations exhibited spontaneously by a nematic in equilibrium. On the other hand, the nonequilibrium contribution is

\[
S^{\text{neq}} (\mathbf{q}, \omega) = -\frac{2\pi^2 k_B \sin^2 \theta}{\gamma_1} \frac{d\mathbf{T}_{ss}}{dz} \times \frac{2\omega q_z \alpha (\mathbf{q})}{|\omega^2 + \omega_{n1} (\mathbf{q})^2|^2},
\]

which also shows a long-range order decaying as \( q^{-5} \). Its odd dependence on \( \omega \) introduces an asymmetry in the shape of the structure factor, shifting the maximum towards the region of negative values of \( \omega \), as shown in Fig. 3. Note that close to equilibrium the size of the shift is indeed proportional to \( d\mathbf{T}_{ss}/dz \).

For a fixed scattering vector \( \mathbf{q} \), we define the dimensionless dynamic structure factor, \( S_0 (\omega_0) \), in terms of the normalized thermal gradient \( \beta' = d\mathbf{T} / d\mathbf{B} \) and the normalized frequency \( \omega_0 = \omega / \omega_{n1} (\mathbf{q}) \), by

\[
S_0 (\omega_0) = \frac{S (\mathbf{q}, \omega)}{S^\text{eq} (\mathbf{q}, 0)} = \frac{1}{1 + \omega_0^2} \left\{ 1 - \beta' \frac{2A\omega_0}{1 + \omega_0^2} \right\},
\]

where \( A = q_z \beta (\mathbf{q}) / \omega_{n1} (\mathbf{q}) d \). In Fig. 3, we compare \( S_0 (\omega_0) \) and \( S_0^{\text{eq}} (\omega_0) \) for low scattering angles, \( \theta \sim 0.1^\circ \), \( q_1 \sim 10^5 \text{ cm}^{-1} \), \( \beta' = 0.5 \) and typical values of the material parameters of a thermotropic nematic [41]. From this result it follows that the increase in the maximum is about 7% and that the decrease in the half width at half height is 10%. This shows that the nonequilibrium state may induce changes in the dynamic structure factor which could be detected experimentally.

5. Semi-slow variables correlation functions

In this section, we use the same method developed in the previous section to calculate the correlation functions of the semi-slow variables \( \delta v_1, \delta v_2 \) and \( \delta T \). In the time-scale where Eqs. (33) and (70) are valid, the fast variables \( \delta \tilde{p} \) and \( \delta \tilde{v}_3 \) have already decayed and average zero. Therefore, we can make the approximations

\[
y_1 = \delta \tilde{v}_2 = k_1^{-1} (k \delta \tilde{v}_z - k_z \delta \tilde{v}_3) \approx k_1^{-1} k \delta \tilde{v}_z \tag{99}
\]

and

\[
y_2 = \delta \tilde{T} - \gamma^{-1/2} (\gamma - 1)^{1/2} \delta \tilde{p} \approx \delta \tilde{T}, \tag{100}
\]

where \( \delta \tilde{v}_z = \rho^{1/2} \delta \tilde{v}_z \) is the normalized \( z \) component of the velocity field. On the other hand, in the same time-scale, the slow variables only slightly perturb the motion of the semi-slow ones through the different powers of the small quantity \( \varepsilon_1 \sim K \rho_{ss}/\nu^2 \). In Eqs. (33) and (70), we have explicitly written out the first of such perturbation corrections. However, for a typical thermotropic \( \varepsilon_1 \sim 10^{-5} \) and in a first analysis, these perturbation terms can be neglected.

Also notice that we can approximate \( (\mathbf{B}_{\mu \nu})_{21} \approx k_1 \omega_\gamma T / k \) in Eq. (71), since for a typical thermotropic nematic and typical experimental values of the thermal gradient the following relations hold:

\[
\frac{\lambda}{\nu} \left( \mathbf{B} \right) \frac{D_{\alpha} k^2}{\omega_{v1} \left( \mathbf{B} \right)} \sim \frac{\rho_{ss} D_{\alpha} \omega}{\nu} \sim 10^{-3} \ll 1 \tag{101}
\]

and

\[
\frac{\rho_{ss}}{c} \ll \omega_\gamma T. \tag{102}
\]

Under these conditions, the stochastic equations (33) and (70) in \( \mathbf{B} = \mathbf{B} - \omega \mathbf{v} \) in terms of the non-scaled variables \( \delta \tilde{v}_1, \delta \tilde{T} \) and \( \delta \tilde{v}_2 \), respectively, read

\[
\delta \tilde{v}_1 = \frac{1}{-i\omega + \omega_{v1} \left( \mathbf{B} \right)} \dot{\delta \tilde{v}_1}, \quad \delta \tilde{v}_2 = \frac{k_2}{\kappa} g \alpha \left( \mathbf{B} \right)
\]

\[
\left( \begin{array}{c}
-k_2 \frac{\omega_2}{\kappa} \left( \mathbf{B} \right)

\frac{d\omega_2}{dz} \quad -i\omega + \omega_T \left( \mathbf{B} \right)
\end{array} \right) \dot{\delta \tilde{v}_2}
\]

\[
\times \left( \begin{array}{c}
\delta \tilde{v}_3

\delta \tilde{T}
\end{array} \right) = - \left( \begin{array}{c}
\dot{\delta \tilde{v}_2}

\dot{\delta \tilde{T}}
\end{array} \right), \tag{104}
\]

where we have introduced the following abbreviations for the stochastic force terms:

\[
\dot{\delta \tilde{v}_1} = \frac{i}{\rho_{ss}} \left( k_2 \delta \tilde{v}_2 - k_3 \delta \tilde{v}_3 \right), \tag{105}
\]

\[
\dot{\delta \tilde{v}_2} = \frac{i}{\rho_{ss}} \left( k_3 \delta \tilde{v}_2 - k_z \delta \tilde{v}_3 \right), \tag{106}
\]

\[
\dot{\delta \tilde{T}} = \frac{i}{\rho_{ss} c_p} k_2 \dot{\tilde{Q}}, \tag{107}
\]
The correlation functions of these stochastic terms are obtained from Eqs. (105)-(107) and the Fourier transform of the fluctuation dissipation theorems of the original noises \( \Sigma_{ij} \) and \( Q_i \) in the stationary state, which are given by Eqs. (A.8)-(A.10). In abbreviated form they read

\[
\langle \tilde{\sigma}_{v1} \left( \vec{k}, \omega \right) \tilde{\sigma}_{v1} \left( \vec{k}', \omega' \right) \rangle_{ss} = 2 (2\pi)^4 k_B T_0 \rho_{ss} \left( 1 + \text{i} \bar{B} \hat{e}_z \cdot \nabla_{\vec{k}} \right) \times \delta \left( \vec{k} + \vec{k}' \right) \delta (\omega + \omega') ,
\]

(108)

\[
\langle \tilde{\sigma}_{v2} \left( \vec{k}, \omega \right) \tilde{\sigma}_{v2} \left( \vec{k}', \omega' \right) \rangle_{ss} = 2 (2\pi)^4 k_B T_0 \rho_{ss} \left( 1 + \text{i} \bar{B} \hat{e}_z \cdot \nabla_{\vec{k}} \right) \times \delta \left( \vec{k} + \vec{k}' \right) \delta (\omega + \omega') ,
\]

(109)

\[
\langle \tilde{\sigma}_T \left( \vec{k}, \omega \right) \tilde{\sigma}_T \left( \vec{k}', \omega' \right) \rangle_{ss} = 2 (2\pi)^4 k_B T_0 \rho_{ss} \left( 1 + \text{i} \bar{B} \hat{e}_z \cdot \nabla_{\vec{k}} - \bar{B}^2 \left( \hat{e}_z \cdot \nabla_{\vec{k}} \right)^2 \right) \times \delta \left( \vec{k} + \vec{k}' \right) \delta (\omega + \omega') ,
\]

(110)

where the functions \( f_{v1}(\vec{k}, \vec{k}') \), \( f_{v2}(\vec{k}, \vec{k}') \) and \( f_T(\vec{k}, \vec{k}') \) are given by

\[
f_{v1}(\vec{k}, \vec{k}') = -\frac{1}{\rho_{ss}^2} k_i k_j' \left( \nu_{zij} - \nu_{zij} - \nu_{zij} - \nu_{yij} \right) ,
\]

(111)

\[
f_{v2}(\vec{k}, \vec{k}') = -\frac{1}{\rho_{ss}^2} \left( k_i k_j' \nu_{zij} - \frac{k_i' k_j' k_i k_j}{k^2} \nu_{zij} \right)
+ \frac{k_i k_j k_i' k_j'}{k^2} k_i k_j' \nu_{zij} + \frac{k_i k_j' k_i k_j}{k'^2} k_i k_j' \nu_{zij} ,
\]

(112)

\[
f_T(\vec{k}, \vec{k}') = -\frac{1}{\rho_{ss}^2} k_i k_j' k_i' k_j
\]

(113)

From Eqs. (103) and (108) we get an expression for the transverse velocity correlation function \( \langle \delta \nu_1(\vec{k}, \omega) \delta \nu_1(\vec{k}', \omega') \rangle_{ss} \). Taking the inverse Fourier transform of this expression and following a similar procedure to the one described in Sec. 3 of Ref. 32, we arrive at

\[
\lim_{\zeta \to 1} \langle \delta \nu_1(\vec{r}, t) \delta \nu_1(\vec{r}', t') \rangle_{ss} = \frac{k_B T_0(z)}{\rho_{ss}} \delta (\vec{r} - \vec{r}') ,
\]

(114)

where we have taken the limit of large distances or small times defined by \( \zeta = \nu |t - t'| / \rho_{ss} |\vec{r} - \vec{r}'|^2 \), since in this work we are only interested in the spatial behavior of the correlations. Notice that the non-equilibrium correlation function of transverse velocity fluctuations corresponds to a local version of its short-ranged equilibrium counterpart. This local equilibrium behavior is obtained from the assumption of the validity of the local version of the fluctuation dissipation relations.

We shall now construct the correlations of semi-slow longitudinal variables by solving Eq. (104) for \( \delta \hat{T} \), \( \delta \hat{v}_z \). Furthermore, these correlations will be expressed as the sum of two parts, namely, one arising from the dynamics of the variables of interest and the other coming from the coupling with the other variables. The first contribution will be identified by the superscript \( LE \) (local equilibrium) and is obtained by solving Eq. (104) when the off-diagonal elements of the reduced hydrodynamic matrix are neglected. The second contribution will be denoted by the superscript \( MC \) (mode coupling), because it contains the effects of the coupling terms of the reduced hydrodynamic matrix. The \( MC \) contribution will be calculated up to the smallest power in the temperature gradient. This procedure is similar to that used in Ref. [34] to compare the non-equilibrium long-range order effects produced by the local equilibrium version of the fluctuation dissipation theorems and the mode coupling mechanism.

For the temperature autocorrelation, this procedure leads to

\[
\langle \delta \hat{T}(\vec{k}, \omega) \delta \hat{T}(\vec{k}', \omega') \rangle_{ss} = \langle \delta \hat{T}(\vec{k}, \omega) \delta \hat{T}(\vec{k}, \omega') \rangle_{ss}^{LE} + \langle \delta \hat{T}(\vec{k}, \omega) \delta \hat{T}(\vec{k}', \omega') \rangle_{ss}^{MC} ,
\]

(115)

with

\[
\langle \delta \hat{T}(\vec{k}, \omega) \delta \hat{T}(\vec{k}', \omega') \rangle_{ss}^{LE}
= \frac{\langle \delta \hat{T}(\vec{k}, \omega) \delta \hat{T}(\vec{k}', \omega') \rangle_{ss}^{MC}}{[\text{i}\omega + \omega_T(\vec{k})]} [\text{i}\omega' + \omega_T(\vec{k}')] ,
\]

(116)

and

\[
\langle \delta \hat{T}(\vec{k}, \omega) \delta \hat{T}(\vec{k}', \omega') \rangle_{ss}^{MC}
= -\frac{k_i k_i'}{kk'} \left( \frac{dT_0}{dz} \right)^2 \times \frac{\langle \delta \nu_2(\vec{k}, \omega) \delta \nu_2(\vec{k}', \omega') \rangle_0}{[\text{i}\omega + \omega_T(\vec{k})]} [\text{i}\omega' + \omega_T(\vec{k}')] \times \frac{1}{[\text{i}\omega' + \omega_T(\vec{k}')]} ,
\]

(117)
where the subscript 0 indicates evaluation at \((dT_{ss}/dz) = 0\), that is,
\[
\langle \hat{\sigma}_{v2} \left( \vec{k}, \omega \right) \hat{\sigma}_{v2} \left( \vec{k}', \omega' \right) \rangle_0 = \langle \hat{\sigma}_{v2} \left( \vec{k}, \omega \right) \hat{\sigma}_{v2} \left( \vec{k}', \omega' \right) \rangle_{(dT_{ss}/dz)=0}.
\]

Similarly, for the temperature-velocity cross correlation we get
\[
\left\langle \delta \hat{v}_z \left( \vec{k}, \omega \right) \delta \hat{T} \left( \vec{k}', \omega' \right) \right\rangle_{ss}^{LE} = 0,
\]
and for the velocity autocorrelation we obtain
\[
\left\langle \delta \hat{v}_z \left( \vec{k}, \omega \right) \delta \hat{v}_z \left( \vec{k}', \omega' \right) \right\rangle_{ss}^{MC} = -\frac{k_k}{k} \left( \frac{dT_{ss}}{dz} \right) \left\langle \hat{\sigma}_{v2} \left( \vec{k}, \omega \right) \hat{\sigma}_{v2} \left( \vec{k}', \omega' \right) \right\rangle_0
\]
\[
\times \left[ k^2 \left[ -i\omega + \omega_{v2} \left( \vec{k} \right) \right] \left[ -i\omega + \omega_T \left( \vec{k} \right) \right] + \frac{k_k^2}{k^2} \left[ -i\omega + \omega_{v2} \left( \vec{k}' \right) \right] \left[ -i\omega + \omega_T \left( \vec{k}' \right) \right] \right].
\]

It should be pointed out that in order to arrive at expressions (116)-(121) we have used the same typical parameter values for a thermotropic nematic and taken into account relations (101) and (102).

We now write the above correlation functions in \( \vec{r} - t \) space. For this purpose we first substitute the fluctuation-dissipation relations (108)-(110) in Eqs. (116)-(121) and take the inverse Fourier’s transform according to (9). Moreover, if we also take the limit \( \zeta \rightarrow 0 \), the \( LE \) part Eq. (116) reduces to
\[
\lim_{\zeta \ll 1} \left\langle \delta \hat{T} \left( \vec{r}, t \right) \delta \hat{T} \left( \vec{r}', t \right) \right\rangle_{ss}^{LE} = \frac{k_B T_0}{\rho_s c_p} \delta \left( \vec{r}' - \vec{r} \right)
\]
\[
\times \left( \frac{dT_{ss}}{dz} \right)^2 \frac{k_B}{4\pi \rho_s c_p} \left\{ \frac{1}{\rho_s \left( \vec{r} \right)} \hat{T}_{ss} \left( z \right) \right\}^{1/2}
\]
\[
\times \left[ \frac{1}{\rho_s \left( \vec{r} \right)} + \frac{1}{\rho_s \left( \vec{r} \right)} \right]^{1/2},
\]
which shows a short-range contribution proportional to \( \delta \left( \vec{r}' - \vec{r} \right) \) and an algebraically decaying anisotropic long-range part arising from the thermal gradient through the assumption of the validity of a local version of the fluctuation dissipation relations.

It can also be shown that, in the same limit, the mode coupling contribution Eq. (117) reads
\[
\lim_{\zeta \ll 1} \left\langle \delta \hat{T} \left( \vec{r}, t \right) \delta \hat{T} \left( \vec{r}', t \right) \right\rangle_{ss}^{MC} = \frac{\left( dT_{ss} \right)^2}{2\pi \rho_s D_T \left( \vec{r}_3 + D_T \right)}
\]
\[
\times \left[ \frac{I_{a_1} \left( \vec{r}, \vec{r}' \right)}{(a_1 - a_2)(a_1 - a_3)} + \frac{I_{a_2} \left( \vec{r}, \vec{r}' \right)}{(a_2 - a_1)(a_2 - a_3)} + \frac{I_{a_3} \left( \vec{r}, \vec{r}' \right)}{(a_3 - a_1)(a_3 - a_2)} \right],
\]
where \( \vec{r}_a = \nu_a / \rho_{ss}, \alpha = 1, 2, 3 \), are kinematic viscosity coefficients, \( a_3 = D_T / D_0 \), and \( a_1 \) and \( a_2 \) depend only on material parameters through
\[
\left\{ \begin{array}{l}
\frac{a_1}{a_2} \frac{1}{\vec{r}_3 + D_T} \left\{ \frac{2 (\vec{r}_1 + \vec{r}_2 - \vec{r}_3)}{\vec{r}_3 + D_T} + D_T^2 + 4 (D_T^2 + \vec{r}_3) \left( D_T^2 + \vec{r}_3 \right)^{1/2} \right\}. \end{array} \right.
\]
\[ I_a (\vec{r}', \vec{r}'') = \frac{1}{\sqrt{a}} \times \left[ \left| \vec{r}'_\perp - \vec{r}''_\perp \right|^2 + a (z - z')^2 \right]^{1/2}. \]  

Eqs. (123) and (125) show that the mode coupling contribution to the temperature autocorrelation increases anisotropically with the distance \( |\vec{r} - \vec{r}'| \) with a simple power law. This result is similar to the one obtained for an isotropic fluid [43].

Similarly, in \( \vec{r} - t \) space Eqs. (120) and (121) are, respectively,

\[ \lim_{\zeta \ll 1} \langle \delta v_z (\vec{r}, t) \delta v_z (\vec{r}', t) \rangle_{ss}^{LE} = \frac{k_B T_0}{4\pi \rho_{ss}} \left( \frac{3 (z - z')^2}{|\vec{r} - \vec{r}'|^3} - \frac{1}{|\vec{r} - \vec{r}'|^3} \right) \]

\[ - \frac{k_B T_0}{4\pi \rho_{ss}} \frac{B}{2} \left( \frac{3 (z - z')^2}{|\vec{r} - \vec{r}'|^3} - \frac{1}{|\vec{r} - \vec{r}'|^3} \right), \]

\[ \lim_{\zeta \ll 1} \langle \delta v_z (\vec{r}, t) \delta v_z (\vec{r}', t) \rangle_{ss}^{MC} = - \frac{g a k_B T_0}{4\pi \rho_{ss} \nu_3 (\nu_3 + D^T)} \left( \frac{dT_{ss}}{dz} \right) \frac{1}{(b_2 - b_1) (a_1 - a_2)} \]

\[ \times \left\{ \left[ I_{b_2} (\vec{r}, \vec{r}') \right]_{a_1 - b_2} - \left[ I_{b_2} (\vec{r}, \vec{r}') \right]_{a_2 - b_2} + \left[ I_{b_1} (\vec{r}, \vec{r}') \right]_{a_1 - b_1} - \left[ I_{b_1} (\vec{r}, \vec{r}') \right]_{a_2 - b_1} \right. \]

\[ \left. + \left[ I_{a_2} (\vec{r}, \vec{r}') \right]_{a_2 - b_2} - \left[ I_{a_2} (\vec{r}, \vec{r}') \right]_{a_1 - b_2} \right\}. \]

Here

\[ b_1, b_2 \] are given by (125). Note that in contrast to Eq. (122), (126) apparently does not show a short-range behavior proportional to \( \delta (\vec{r} - \vec{r}') \), but decays as \( |\vec{r} - \vec{r}'|^{-3} \).

Finally, in configuration space the cross-correlation Eq. (119) turns out to be

\[ \lim_{\zeta \ll 1} \langle \delta v_z (\vec{r}, t) \delta T (\vec{r}', t') \rangle_{ss}^{MC} = \left( \frac{dT_{ss}}{dz} \right) \frac{k_B T_0}{4\pi \rho_{ss} \nu_3 (\nu_3 + D^T)} (a_2 - a_1) \]

\[ \times \left\{ \frac{1}{\sqrt{a_1} \left[ \left| \vec{r}'_\perp - \vec{r}'_\perp '' \right|^2 + a_1 (z - z')^2 \right]^{1/2}} - \frac{1}{\sqrt{a_2} \left[ \left| \vec{r}'_\perp - \vec{r}'_\perp '' \right|^2 + a_2 (z - z')^2 \right]^{1/2}} \right\}. \]

These terms arise because we have extrapolated the correlation (120) for \( t < t', \) to \( t = t' \). But in doing this we have neglected the effects due to the fast variables; this is actually the physical significance of approximation (99). However, it can be shown that, if the fast variable \( \delta v_3 \) is also taken into account in the construction of the correlation (120), the first term in (126) reduces to \( \delta (\vec{r} - \vec{r}') \) [42].

The limit (130) guarantees that the calculated correlations are produced in the bulk of the nematic and in a region where the changes to the material parameters induced by the thermal gradient are not significant.

In order to illustrate the non-equilibrium behavior of these correlations, we consider the long-range order contributions to the temperature auto-correlation function as given by the second term on the right hand side of Eq. (122) and by Eq. (123). We define the normalized long-range order

\[ \frac{a g}{\nu_3 D^T} \left( \frac{dT_{ss}}{dz} \right) ^{-1/4}. \]
contributions by
\[ F^{LE}(\vec{r}, \vec{r}') = \beta'^2 \frac{d}{\sqrt{[\vec{\nabla} - \vec{\nabla}_r']^2 + \frac{D}{\beta} (z - z')^2}^{1/2}, \] (132)
where $\beta' = d\beta$ is the normalized temperature gradient and, accordingly,
\[ F^{MC}(\vec{r}, \vec{r}') = \beta'^2 \frac{2dT_{0c}}{D}(\vec{\nabla}_r + \vec{\nabla}_r' \cdot \vec{\epsilon}_z) \left(\frac{I_a(\vec{r}, \vec{r}')}{(a-b)(a-c)} + \frac{I_b(\vec{r}, \vec{r}')}{(b-a)(b-c)} + \frac{I_c(\vec{r}, \vec{r}')}{(c-a)(c-b)}\right). \] (133)

It should be emphasized that the former arises from the local version of the fluctuation dissipation relations while the latter occurs due to the coupling between velocity and temperature fluctuations in the presence of the thermal gradient. In Fig. 4 we plot these contributions for $\vec{r} = 0$, $\vec{r}' = z'\vec{\epsilon}_z$, values of the material parameters involved for a typical thermotropic nematic, $\gamma_T \sim 10^{-4}$ and $d \sim 1$ cm, as a function of the normalized distance $\vec{z}' = z'/d$. Fig. 4 shows that, for values of $z'$ such that $\vec{z}' \sim 10^{-6} \ll \Lambda/d$, we have $F^{MC} \gg F^{LE}$, that is, in the region where the calculated correlations are valid, the mechanism of mode coupling is much stronger than the long-range order induced by the local equilibrium version of the fluctuation-dissipation relations.

6. Discussion

In summary, by using a fluctuating hydrodynamic description, in this work we have investigated theoretically the spatial behavior of all the fluctuation correlation functions for a thermotropic nematic liquid crystal. To this end we have explicitly used the fact that for a thermotropic nematic the modes associated with the director relaxation are much slower than the visco-heat and sound modes. The method described in Refs. 37 to 39 allowed us to find, on the slow time-scales, a contracted description in terms of the slow variables only, with a reduced dynamic matrix which can be constructed by the perturbation procedure. To clarify and elaborate on some of our results, the following comments may be useful.

First, we showed that the director, temperature and velocity autocorrelations, as well as the temperature-velocity cross correlation functions exhibit long-range order when the system is in a nonequilibrium steady state ($NESS$) induced by the action of an external temperature gradient and the gravity field. This is shown in Eqs. (88), (91), (122), (123), (126), (127) and (129), respectively. These results also show explicitly that these correlation functions contain two contributions, namely, a local equilibrium and a mode coupling contribution. For the temperature-temperature correlation, these contributions are plotted in Fig. 4 for typical material parameter values and for a value of the gradient which is used experimentally for simple fluids, $(dT_{\text{ex}}/dz) = 50$ K cm$^{-1}$ [43]. From these curves one can clearly see that the contribution due to the mode coupling mechanism is dominant over the spatial inhomogeneities in the fluctuation-dissipation relation. We also estimated the influence of the stationary heat flux on the light scattering spectrum of a thermotropic nematic. The analysis carried out in this work included only the orientation correlation functions and the model’s geometry has been constructed so that it corresponds to those used in an experimental arrangement appropriate for detecting the so-called mode 2 of the spectrum [44]. However, it should be emphasized that this is a model calculation and since to our knowledge there are no experimental results to compare with, the correctness of our choice of experimental parameters such as $d$, $\theta$ or $\beta'$, remains to be assessed. However, as found in other nonequilibrium states for liquid crystals [33], their magnitude suggests that they could be experimentally detected.

Secondly, this behavior of the correlation functions is in agreement with the one obtained for the director-fluctuation correlation function obtained in previous work [32]. Therefore, all functions for a liquid crystal in $NESS$ exhibit long-range behavior. Thus these results confirm the existence of the so-called generic scale invariance in a liquid crystal, a property shared by systems in nonequilibrium steady states, which has been proposed as the origin of the long-range nature of the correlation functions [18–20].

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A Appendix: Fluctuating Nematodynamics

The hydrodynamic equations for the small deviations of the state vector \( \vec{a} \equiv (\delta p, \delta T, \delta v_x, \delta v_y, \delta v_z, \delta n_x, \delta n_y, \delta n_z)^T \) in the geometry under consideration shown in Fig. 1, are derived by linearizing the general conservation equations of mass, momentum, energy and the relaxation equation for the director field of a thermotropic nematic liquid crystal given by Eqs. (A.1)-(A.9) in Ref. 33. The result is

\[
\frac{\partial}{\partial t} \delta p = \frac{\alpha}{\chi_T} \left( D_T^T \nabla_x^2 + D_T^T \nabla_y^2 \right) \delta T - \frac{\gamma}{\chi_T} \nabla_i \delta v_i + \rho_{ss}(z) \delta v_x + D_T^T \frac{\alpha}{\chi_T} \frac{d T}{d z} \nabla_i \delta n_i - \frac{\alpha}{\chi_T c \rho_{ss}(z)} \nabla_i Q_i, \tag{A.1}
\]

\[
\frac{\partial}{\partial t} \delta T = \frac{\gamma}{\alpha} \left( D_T^T \nabla_x^2 + D_T^T \nabla_y^2 \right) \delta T - \frac{1}{\alpha} \nabla_i \delta v_i - \frac{d T}{d z} \delta v_x + \frac{d T}{d z} \frac{D_T^T}{d z} \nabla_i \delta n_i - \frac{1}{c_v \rho_{ss}(z)} \nabla_i Q_i, \tag{A.2}
\]

\[
\rho_{ss}(z) \frac{\partial}{\partial t} \delta v_x = [(\nu_2 + \nu_4) \nabla_x^2 + \nu_2 \nabla_y^2 + \nu_3 \nabla_z^2] \delta v_x + (\nu_3 + \nu_5) \nabla_x \nabla_z \delta v_z + \nu_4 \nabla_x \nabla_y \delta v_y - \nabla p - \frac{1 + \lambda}{2} (K_1 - K_2) \nabla_x \nabla_y \nabla_z \delta n_z - \nabla \cdot \rho_{ss}(z) \delta v_x, \tag{A.3}
\]

\[
\rho_{ss}(z) \frac{\partial}{\partial t} \delta v_y = [(\nu_2 + \nu_4) \nabla_y^2 + \nu_2 \nabla_x^2 + \nu_3 \nabla_z^2] \delta v_y + (\nu_3 + \nu_5) \nabla_y \nabla_z \delta v_z + \nu_4 \nabla_x \nabla_y \delta v_x + \nabla p - \frac{1 + \lambda}{2} (K_1 - K_2) \nabla_x \nabla_y \nabla_z \delta n_z - \nabla \cdot \rho_{ss}(z) \delta v_y, \tag{A.4}
\]

\[
\rho_{ss}(z) \frac{\partial}{\partial t} \delta v_z = [(2 \nu_1 + \nu_2 + 2 \nu_5 - \nu_4) \nabla_x^2 + \nu_3 \nabla_z^2] \delta v_z + (\nu_3 + \nu_5) (\nabla_x \nabla_z \delta v_x + \nabla_y \nabla_z \delta v_y) - \nabla \cdot \rho_{ss}(z) \delta v_z.
\]

\[
\frac{\partial}{\partial t} \delta n_x = -\frac{1}{\gamma_1} \left( K_1 \nabla_x^2 + K_2 \nabla_y^2 + K_3 \nabla_z^2 \right) \delta n_x + \frac{1 + \lambda}{2} \nabla_x \delta v_x - \frac{1 - \lambda}{2} \nabla_x \delta v_z
\]

\[
+ \frac{1}{\gamma_1} (K_1 - K_2) \nabla_x \nabla_y \delta n_y - \Upsilon_x, \tag{A.6}
\]

\[
\frac{\partial}{\partial t} \delta n_y = \frac{1}{\gamma_1} \left( K_2 \nabla_x^2 + K_1 \nabla_y^2 + K_3 \nabla_z^2 \right) \delta n_y + \frac{1 + \lambda}{2} \nabla_x \delta v_y - \frac{1 - \lambda}{2} \nabla_y \delta v_z
\]

\[
+ \frac{1}{\gamma_1} (K_1 - K_2) \nabla_x \nabla_y \delta n_x - \Upsilon_y. \tag{A.7}
\]

In these equations, \( \alpha = -(1/\rho) (\partial \rho / \partial T) \) is the thermal expansion coefficient, \( \chi_T \) is the isothermal compressibility, \( D_T^T \) and \( D_T^T \) are the thermal diffusivities in the parallel and perpendicular directions with \( \nabla_{ss} = \vec{D}_{ss} = \vec{D}_{ss}^T - \vec{D}_{ss}^T \) is the corresponding anisotropy, and \( \gamma = c_p/c_v \) is the specific heat at constant volume and pressure with \( \gamma = c_p/c_v \). Eqs. (A.1)-(A.7) can be written in terms of the isentropic sound speed, \( c \), by using the thermodynamic relation \( c^2 = \gamma / \rho \chi_T \). \( \nu_1, \nu_2, \nu_3, \nu_4, \nu_5 \) denote the five nematic viscosity coefficients of a nematic, \( \gamma_1 \) is the orientational viscosity and \( K_1, K_2, K_3 \) are the splay, twist and bend elastic constants [35].

\( \nabla \cdot \Sigma_{ij}(\vec{r}, t), \nabla \cdot Q_i(\vec{r}, t), \Upsilon_i(\vec{r}, t) \) are the fluctuating components of the momentum current, the heat current and the relaxation quasi-current of the orientation of the nematic, respectively. These stochastic currents are chosen so that they are zero averaged stochastic processes \( \langle \Sigma_{ij}(\vec{r}, t) \rangle = Q_i(\vec{r}, t) = \langle \Upsilon_i(\vec{r}, t) \rangle = 0 \) satisfying fluctuation-dissipation relations of the form [31]

\[
\langle \Sigma_{ij}(\vec{r}, t) \Sigma_{kl}(\vec{r}', t') \rangle = 2k_B T_{ss}(z) \kappa_{ij}^{ss} \delta (\vec{r} - \vec{r}'), \tag{A.8}
\]

\[
\langle \Upsilon_i(\vec{r}, t) \Upsilon_j(\vec{r}', t') \rangle = 2k_B T_{ss}(z) \delta_{ij} \delta (\vec{r} - \vec{r}'), \tag{A.9}
\]

\[
\langle Q_i(\vec{r}, t) Q_j(\vec{r}', t') \rangle = 2k_B T_{ss}(z) \kappa_{ij}^{ss} \delta (\vec{r} - \vec{r}'), \tag{A.10}
\]

Here \( k_B \) is Boltzmann’s constant, \( \delta (t - t') \) denotes Dirac’s delta function, \( \delta_{ss} \) is a linearized projection operator and \( \delta_{ij} \) is the usual Kronecker’s delta. \( \kappa_{ij}^{ss} \) is the linearized thermal conductivity tensor, \( \kappa_{ij}^{ss} = \kappa_{ss} \delta_{ij} + \kappa_{ss} n_{ss,i} n_{ss,j} \),

where $\kappa_a = \kappa_{\parallel} - \kappa_\perp$ is the anisotropy in the thermal conductivity of the nematic where $\kappa_\perp$, and $\kappa_{\parallel}$ denote its perpendicular and parallel components with respect to the director field respectively. The linearized stress tensor $\nu_{ijkl}^n$ is given by

$$
\nu_{ijkl}^n = \nu_2 (\delta_{j,k} \delta_{i,l} + \delta_{i,k} \delta_{j,l}) + 2 (\nu_1 + \nu_2 - 2 \nu_3) n_{ss,i} n_{ss,j} n_{ss,k} n_{ss,l} +
$$

$$
\nu_3 (\nu_1 + \nu_2 - 2 \nu_3) n_{ss,j} (n_{ss,k} \delta_{il} + n_{ss,l} \delta_{ik}) + (\nu_4 - \nu_2) \delta_{ij} \delta_{kl} + (\nu_5 - \nu_2) \delta_{ik} n_{ss,l} + \delta_{il} n_{ss,j}).
$$

\text{(A.11)}

**References**