Time dependent correlation functions and polydispersity effects in bubbling flows

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We generalize previous works to calculate the volume fraction fluctuations correlation function for a size polydispersed bubble column. Theoretically, we adopt a statistical mechanistic point of view and use a stochastic description to calculate the above mentioned correlation function for a gas-liquid vertical bubble column. On the other hand, we also obtain this correlation function from the data obtained by an electrical impedance spectroscopy technique. We find that bubble size polydispersity affects several properties of the bubble column such as the gain factor, mean bubble velocity, kinematic and dynamic wave velocities. We find that bubble size polydispersity may enhance significantly the stability of void fraction waves by a factor which varies between 0.4 – 23% as a function of its frequency and for the specific bubble column considered in our experiments. The comparison between the calculated volume fraction correlation function and the same quantity obtained from measured data is then analyzed. We find that polydispersity effects yield a better comparison between theory and experiment than the monodispersed case and lead to an absolute mean error of 5%, as compared with the error of 10.6% obtained for the monodispersed case. Our results reinforce the point of view advocated in this work in the sense that a description of a bubble column based on the concept of randomness of a bubble cloud and average properties of the fluid motion, may be a useful approach that has not been exploited in engineering systems.

Keywords: Bubbly flows; void fraction waves; stability; kinetic theory.

Using the formalism of the hydrodynamics generalized we calculate the function of correlation of fraction volume of a column of bubbles polydispersa. Además, estas correlaciones se obtienen a partir de mediciones de la fracción volumen utilizando la técnica de impedancia eléctrica. Se encuentra que la polidispersidad en tamaño afecta diversas propiedades de la columna como son, el factor de ganancia, velocidad promedio, así como las velocidades de las ondas cinemática y dinámica. Se muestra que polidispersidad mejora la estabilidad de las ondas cinemáticas por un factor que varía entre 4.5% – 23%, dependiendo de la frecuencia considerada. Además, se encuentra una concordancia razonable entre las funciones de correlación teórica y experimental, con un error absoluto del 5% y del 10.6% para las distribuciones poli y monodispersas, respectivamente. Se discuten las limitaciones y posibles generalizaciones del modelo.

Descriptores: Flujos burbujeantes; ondas de fracción volumen; flujo multifásico.

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1. Introduction

Bubble columns are multiphase systems which have a wide range of applications in chemical industries, biotechnology or in nuclear reactors. Essentially, a bubble column is a reactor where a discontinuous gas phase in the form of bubbles moves relative to a continuous phase. They may also contain solid particles in the liquid stream. Many chemical heterogeneous reactors are bubble columns where reactions usually take place in the liquid phase, once a reactive chemical has been absorbed from a gaseous stream through the interfacial films between bubbles and liquid. Both, petrochemical and environmental applications are important examples of the use of these reactors. Transient phenomena are important at the start-up of these equipment and are of enormous importance in the operation of these reactors in order to prevent instabilities that could drive the system out of control and produce severe accidents. Thus, their analysis is important in order to characterize the dynamic behavior of the fluctuations occurring in the system and to get a better insight into the phenomena driving some events, such as the flow patterns transitions.

Transient analysis is built on the basis of data taken in records or time series, which are the relevant experimental information required. The usual description of instabilities of bubbly flows is in terms of the properties of the propagation of void fraction and pressure disturbances caused by natural or imposed fluctuations in the rate of air supply. From the experimental point of view, these features are detected by correlation techniques using impedance probes mounted in the walls of the column and which allow for the construction of the autocorrelation function of the fluctuations of the void fraction of the system, [1–5]. However, theoretical studies of these correlations is rather scarce in the literature. There exist, though, some studies dealing with models of autocorrelation functions for gas-liquid systems in a horizontal pipe [6] or for liquid-solid fluidized beds [7–12]. In previous works we have calculated void fraction fluctuations correlation functions in the case of size monodispersed bubble columns by using a fluctuating hydrodynamic approach [13]. Our basic purpose here is to generalize this work to calculate the same void fraction autocorrelation function for a size polydispersed bubble column. Instead of using the same fluctuating description as
before, in the present work we develop a different but equivalent stochastic description that allow us to calculate the above mentioned correlation function for a gas-liquid bubble column.

For this purpose the paper is organized as follows. In Sec. 2 we first briefly review the hydrodynamic model for bubbly flows introduced by Biesheuvel and Gorissen some years ago [14]. Then we generalize this model by introducing in it the effects of bubble size polydispersity through the drag force in the hydrodynamic equations [15]. In Sec. 3 we introduce fluctuations into the model and calculate the correlation function of fluctuations in the void fraction of the system. The experimental set up for the determination of this correlation is discussed and the main features of the data treatment are described in Sec. 4. Section 5 contains the main results of both, the theoretical and experimental analysis. We give the calculated values of the effects of bubble size polydispersity on the gain factor, mean bubble velocity, kinematic wave velocities as a function of void fraction for different wave frequencies of the void fraction wave. Although these properties were previously examined in general in Ref. 16, here we calculate them for the specific system where our experiments were performed. We find that size polydispersity may enhance significantly the stability of void fraction waves by a factor which varies between 0.4–23% as a function of its frequency and for the specific bubble column considered in our experiments. The comparison between the calculated void fraction correlation function and this same quantity obtained from the measured data is then analyzed. We find that bubble polydispersity effects yield a better comparison between theory and experiment than the monodispersed case and leads to an absolute mean error of 5%, as compared with the error of 10.6% obtained for the monodispersed case. Finally, in Section 6 we conclude the paper by emphasizing the predictions of our analysis; we discuss the limitations of our approach and point out that although our model predicts effects that might be verified experimentally, this remains to be assessed.

2. Dynamics of a bubble dispersion

The theoretical description of multiphase flows is essentially based on analyzing the response of a cloud of dispersed particles of different size ranges in a fluid. These particles constitute a dynamic phase and hence the system may be considered as a multiphase system. The equations of motion describing a swarm of bubbles in a bubble column have been derived in the literature by using standard methods of kinetic theory [14, 17, 18]; however, in these approaches a monodispersed bubble column is always considered. The bubbles are supposed to be small enough to remain spherical through the whole system and the air is assumed to be an incompressible fluid. No mass transfer is allowed between the bubbles and the water, which is also regarded as an incompressible Newtonian liquid.

In our previous work [13] we have briefly reviewed the main ideas and steps behind the formulation of the hydrodynamic model introduced by Biesheuvel and Gorissen [14]. They consider a dispersion of equally sized spherical air bubbles in a water column. The conservation equation for the mean number density of the gas bubbles, \( n(x, t) \), and for the mean bubble momentum, \( \rho_G v(x, t) \) (Kelvin impulse, [19]), read

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n v) = 0, \tag{1}
\]

\[
\frac{\partial}{\partial t} \left[ n \left( \frac{4}{3} \pi a^3 \rho_G v + I_L \right) \right] + v \cdot \nabla \left[ n \left( \frac{4}{3} \pi a^3 \rho_G v + I_L \right) \right]
- \nabla \cdot (T_G + T_L) = n F_D + n \frac{4}{3} \pi a^3 (\rho_L - \rho_G) g. \tag{2}
\]

Here the vector \( I_L \) denotes the fluid impulse, \( T_L(x, t) \) and \( T_G(x, t) \) are the tensor fluid stresses, \( F_D \) is the drag force exerted by the fluid on a bubble and \( g \) stands for the gravity field. \( \rho_L, \rho_G \) denote, respectively, the mass densities of water and air, \( \mu_L \) stands for the liquid viscosity and \( a \) is the bubble radius. It will be assumed that the flow of bubbles occurs only along the axial direction of the column. To describe the flow variables of the bubble swarm, Eqs. (1) and (2) should be expressed in terms of the void fraction \( \varepsilon \)

\[
\varepsilon(z, t) \equiv \frac{4}{3} \pi a^3 n(z, t). \tag{3}
\]

and their velocity field \( v \).

By assuming that the mean axial rise velocity of the bubbles is \( v_0(\varepsilon) \), the effect of hydrodynamic interactions between the bubbles on the mean frictional force is taken into account by phenomenologically introducing a function \( f_0(\varepsilon) \) into \( v_0(\varepsilon) \) as [14]

\[
v_0(\varepsilon) = f_0^{-1}(\varepsilon) v_\infty, \tag{4}
\]

where experiment suggest that [21]

\[
f_0(\varepsilon) = (1 - \varepsilon)^{-2}. \tag{5}
\]

The magnitude of the terminal velocity, \( v_\infty \), of a single bubble of radius \( a \) in a stagnant liquid is given by [20]

\[
v_\infty \equiv C_D^{-1}(\rho_L - \rho_G) g, \tag{6}
\]

where

\[
C_D \equiv 9 \mu_L / a^2 \tag{7}
\]

is the drag force factor.

The mean fluid impulse along the axial direction is modeled by

\[
n I_L = n \left( \frac{2}{3} \pi a^3 \rho_L \right) m_0(\varepsilon) v_0(\varepsilon), \tag{8}
\]

where \( m_0(\varepsilon) \) also takes into account the effect of the hydrodynamic interactions. According to Ref. [22], an expression for \( m_0(\varepsilon) \) that renders reliable results up to large values of \( \varepsilon \) as

\[
m_0(\varepsilon) = (1 + 2\varepsilon)(1 - \varepsilon). \tag{9}
\]

Since in a nonuniform bubbly flow the stress \( T = T_G + T_L \) play the role of an effective pressure, Biesheuvel and
Gorissen also assume that the kinetic contribution, $p_c(\varepsilon)$, is proportional to the effective density of the bubbles, $\varepsilon^{-1} \rho_{ef}(\varepsilon) \equiv \rho_G + \frac{1}{2} \rho_L m_0(\varepsilon)$, and to the mean square of their velocity fluctuations [20]

$$\Delta v^2 = H(\varepsilon) v_0^2(\varepsilon) = \frac{\varepsilon}{\varepsilon_{cp}} \left(1 - \frac{\varepsilon}{\varepsilon_{cp}}\right) v_0^2(\varepsilon).$$

(10)

Here $\varepsilon_{cp}$ stands for the limit of closest packing of a set of spheres, for bubbles with radius around $0.4\mu m$ it has been shown that $\varepsilon_{cp}$ is near to the value 0.62, [14]. Then, according to the above expressions, $p_c(\varepsilon) = \rho_{ef} \Delta v^2$. Furthermore, if the non-uniformity is the main cause of an additional transfer of bubble momentum and fluid impulse associated with stress, Biesheuvel and Gorissen [14] postulate that such a contribution to the stress should be given by the force $\mu_e(\varepsilon) \frac{\partial v}{\partial z}$. Thus, taking into account both contributions to the stress $T$ is rewritten as

$$T = -p_c(\varepsilon) + \mu_e(\varepsilon) \frac{\partial v}{\partial z},$$

(11)

where $v$ is the one dimensional nonuniform flow velocity and $\mu_e(\varepsilon) \equiv a_p \rho_{ef}(\varepsilon) v_0(\varepsilon) H^{1/2}(\varepsilon)$ is an effective viscosity.

On the other hand, the mean frictional force is enhanced by an effective diffusive flux of bubbles due to their fluctuating motion. This effect is similar to an steady drag force acting upon each one of the bubbles and proportional to the mean number density gradient. Therefore, (2) this force is represented by

$$nF_D = C_D \varepsilon f_0(\varepsilon) [v + \frac{\mu_e(\varepsilon) \partial \varepsilon}{\varepsilon}] v.$$ 

(12)

Substitution of (8), (11) and (12) into Eqs. (1), (2) leads to the following closed set of one-dimensional equations of motion for the bubbly flow in a zero volume flux reference frame,

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial z}(\varepsilon v) = 0,$$

(13)

$$\frac{\partial}{\partial t} \left[ \rho_{ef}(\varepsilon) v \right] + \frac{\partial}{\partial z} \left[ \rho_{ef}(\varepsilon) v^2 \right] - \frac{\partial T}{\partial z} = -C_D \varepsilon f_0 \left( v + \frac{\mu_e(\varepsilon) \partial \varepsilon}{\varepsilon \rho_{ef}} \right) - \varepsilon (\rho_G - \rho_L) g.$$  

(14)

These equations may be rewritten in a laboratory reference frame by considering the mean axial velocity of the dispersion, $U$, defined by

$$U(t) \equiv \varepsilon U_G + (1 - \varepsilon) U_L.$$ 

(15)

Here $U_G$ and $U_L$ are the mean bubble and fluid axial velocity in the laboratory reference frame. Note that due to the incompressibility of both, liquid and gas, $U$ is only a function of time. Therefore $v \equiv U_G - U$ and a Galilean transformation of Eqs. (13) and (14) yields

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial z} \varepsilon U_G = 0,$$

(16)

$$\frac{\partial}{\partial t} \left[ \rho_{ef}(\varepsilon) v \right] + \frac{\partial}{\partial z} \left[ \rho_{ef}(\varepsilon) v^2 \right] - \frac{\partial T}{\partial z} = -C_D \varepsilon f_0 \left( v + \frac{\mu_e(\varepsilon) \partial \varepsilon}{\varepsilon \rho_{ef}} \right) - \varepsilon (\rho_G - \rho_L) g.$$  

(14)

2.1. Fluctuations

Consider a quiescent equilibrium state of the dispersion described by $\varepsilon = \varepsilon_0$. The deviations from this state will be described by the fluctuations $\delta \varepsilon(z, t) = \varepsilon(z, t) - \varepsilon_0$ and $\delta v(z, t) = v(z, t)$, where Galilean invariance has been taken into account. Linearization of Eqs. (13) and (14) around the reference state yields

$$\frac{\partial}{\partial t} \delta \varepsilon(z, t) + \varepsilon_0 \frac{\partial}{\partial z} \delta v(z, t) = 0$$

(19)

and

$$\left[ \rho_{ef} \frac{\partial}{\partial t} + \mu_e \frac{\partial^2}{\partial z^2} + C_D \varepsilon f_0 \right] \delta v(z, t) + \left[ \left( \rho_{ef} - \varepsilon(\rho_G - \rho_L) g \right) \right] \delta \varepsilon(z, t) = 0.$$ 

(20)

The primes (t) denote derivatives with respect to $\varepsilon$ evaluated at the unperturbed state $\varepsilon = \varepsilon_0$.

In the other hand, by linearizing Eqs. (16) - (18) and by rewriting the result in terms of $\delta \varepsilon$ we arrive at the wave-hierarchy equation

$$\tau_\varepsilon \left[ \left( \frac{\partial}{\partial t} + c^+ \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial t} + c^- \frac{\partial}{\partial z} \right) \right] \delta \varepsilon$$

$$- \nu_c \left( \frac{\partial}{\partial t} + U_G \frac{\partial}{\partial z} \right) \frac{\partial^2 \delta \varepsilon}{\partial z^2}$$

(21)

with lower wave velocity defined as

$$c_0 \equiv U_{G_0} + \varepsilon_0 v_0.$$ 

(22)
The higher-order wave velocities are defined by

\[ c^\pm = U_G - \frac{1}{2}\varepsilon_0\rho_L v_0 m_0 \rho_G + \frac{1}{2}\rho_L m_0 \pm \left[ \left( \frac{1}{2}\varepsilon_0\rho_L v_0 m_0 \rho_G + \frac{1}{2}\rho_L m_0 \right)^2 + \frac{p_f}{\rho_G + \frac{1}{2}\rho_L m_0} \right]^{1/2}. \]  

(23)

Also

\[ v_\varepsilon(\varepsilon_0) = v_0(\varepsilon_0) H^{1/2}(\varepsilon_0) \]  

(24)

and

\[ \tau_\varepsilon(\varepsilon_0) = [C_D f_0(\varepsilon_0)]^{-1} \left[ \rho_G + \frac{1}{2}\rho_L m_0(\varepsilon_0) \right]. \]  

(25)

Note that the lower wave velocity, \( c_0(\varepsilon_0) \), corresponds to the kinematic wave where \( \tau_\varepsilon(\varepsilon_0) = v_\varepsilon(\varepsilon_0) = 0 \). This means that the characteristic length and time scales of the disturbances are sufficiently large to neglect relaxation effects due to the inertia of the bubbles and the fluid, and the diffusive effects associated with the random bubble motion [14]. The dynamic wave velocities, \( c^\pm \), take into account these effects.

For relatively low radial frequencies the wave propagation (21) reduces to a linearized Burgers/Korteweg-de Vries equation

\[ \left( \frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial z} \right) \delta z \approx \left[ \tau_\varepsilon(c^+ - c_0)(c_0 - c^-) + v_\varepsilon \right] \frac{\partial^2 \delta z}{\partial z^2} + \tau_\varepsilon v_\varepsilon (U_G - c_0) \frac{\partial^3 \delta z}{\partial z^3}, \]

(26)

with a solution

\[ \delta z \propto \exp(\gamma z - i\omega t) \]  

(27)

where \( \omega \) is the frequency of the void fraction wave and

\[ \gamma(\varepsilon_0) \approx \frac{i\omega}{c_0} \left[ 1 - \frac{v_\varepsilon^2 \tau_\varepsilon \omega^2 (U_G - c_0)}{c_0^2} \right] - \frac{v_\varepsilon \omega^2 (U_G - c_0)}{c_0^2} \left[ \tau_\varepsilon(c^+ - c_0)(c_0 - c^-) + v_\varepsilon \right]. \]  

(28)

In terms of these quantities the so called gain factor which will determine the stability of the column is defined as [14]

\[ G_f(\varepsilon_0) \equiv \exp \left\{ \text{Re} \left[ \gamma(\varepsilon_0) \right] \omega^2 \Delta z \right\}, \]  

(29)

where Re denotes the real part and \( \Delta z \) the distance between two impedance probes in the experiments to measure \( G_f \) [1].

3. Size polydispersity effects

To take into account the fact that we are dealing with a collection of bubbles of different sizes, we use a result derived by Tam long ago [15]. He showed that for a low Reynolds number viscous flow past a large collection of spheres of a given size distribution, the drag experienced by one of the particles depends only on the first three moments of the particle size distribution function. His treatment relies on calculating the average flow field around the test particle and on using a point force approximation, which essentially consists in replacing the disturbance produced by a sphere in the slow flow by that of a point force located at the centre of the sphere. He then uses the concept of randomness of the bubble cloud and derives equations describing the average properties of the fluid motion. These averages are taken over a statistical ensemble of particle configurations. For a large collection of spheres of a given size distribution, \( n(a) \), this method essentially leads to the correction drag force factor given by

\[ C_D^p = \lambda C_D \equiv \left[ 1 + \phi \pi + \frac{1}{3} \left( \phi \pi \right)^2 \right] C_D, \]  

(30)

where

\[ \phi = \frac{6\pi M_2 + \left( 6\pi M_2 \right)^{1/2} + 12\pi M_1 (1 - 3c)}{(1 - 3c)} \]  

(31)

and \( \pi \) denotes the mean bubble radius. The moments \( M_n \) of the size distribution \( n(a) \) are

\[ M_n = \int n(a) a^n da \]  

(32)

and \( c \equiv \frac{4}{3} \pi M_3 \). Note that since the terminal velocity of a bubble depends on \( C_D \), it is reasonable to assume that in the polydisperse case \( v_0(\varepsilon) \) should be replaced by \( v_0^p \equiv \lambda^{-1} v_0 \).

Substitution of this assumption into Eqs. (16) - (18), carrying out the linearization procedure described in the last section and using the explicit expressions of the set of quantities \( \beta \equiv \left\{ U_{G_0}, c_0, c^\pm \right\} \), one can show that they scale as \( \beta^p \equiv \left\{ \lambda^{-1} U_{G_0}, \lambda^{-1} c_0, \lambda^{-1} c^\pm \right\} \). If these polydispersed properties of the bubbles are substituted back into Eq. (29), one arrives to an expression for the polydispersed gain factor,

\[ G_f^p(\varepsilon_0) \equiv \exp \left\{ \text{Re} \left[ \gamma^p(\varepsilon_0) \right] \omega^2 \Delta z \right\}. \]  

(33)

\( \gamma^p \) is given by Eq. (28) with \( \tau_\varepsilon \) and \( v_\varepsilon \), defined by Eqs. (24) and (25), are now dependent on the polydispersity through \( C_D^p \) given by (30).

4. Volume fraction correlation function

To deal with the dynamics of the void fraction fluctuations \( \delta_\varepsilon \), in previous works we have used a fluctuating hydrodynamic description [13,23]. In this approach random fluxes are added to the linearized equations around equilibrium, which are then converted into stochastic equations of the Langevin type. The connection between the spectrum of spontaneous fluctuations and the transport properties of the system is provided by the fluctuation-dissipation relations. In the present work we shall calculate the void fraction autocorrelation function by restoring to a method proposed by Mountain long ago, [24–26]. The basic idea of this approach is to construct the correlation functions directly from the linearized hydrodynamic equations without restoring to the addition of fluctuating terms into the constitutive equations. Actually, there exist in the literature many applications of this method to a
variety of systems such as viscoelastic fluids near equilibrium [27, 28], or under temperature gradients [29]; suspensions on viscoelastic solvents under density and velocity gradients [30]; suspensions in liquid crystalline solvents [31,32], or colloidal suspensions with memory effects [33]. However, in spite of the apparent differences between both methods it is important to emphasize that both approaches lead to the same results, as has been confirmed many times by light scattering experiments in many systems.

The linearized equations for the fluctuations \( \{ \delta c(t), \delta v(t) \} \), (19) and (20) are most easily solved by using Fourier-Laplace analysis. Defining the Fourier transform of an arbitrary one dimensional scalar field \( A(z,t) \) by

\[
\delta \tilde{A}(q,s) = \int_0^\infty e^{-st} dt \int e^{iqz} \delta A(z,t) dz
\]

the Fourier transforms of (19) and (20) are

\[
\begin{align*}
\frac{\partial}{\partial t} \left( \delta \tilde{\varepsilon}(q,t) \right) &= \left( \begin{array}{cc} 0 & a(q) \\ b(q) & 0 \end{array} \right) \left( \begin{array}{c} \delta \tilde{\varepsilon}(q,t) \\ \delta \tilde{\nu}(q,t) \end{array} \right) \\
&\quad + \left( \begin{array}{c} 0 \\ d(q) \end{array} \right),
\end{align*}
\]

where we have used the following abbreviations

\[
\begin{align*}
a(q) &\equiv i\varepsilon_0 q, \\
b(q) &\equiv \frac{1}{\rho_e' f} \left( \rho_c q^2 - \varepsilon_0 C_D f_0 \right), \\
c(q) &\equiv -\frac{g}{\rho_e' f} \left( \rho_c - \rho_L \right) i q + \frac{1}{\rho_e' f} \left( \rho_e' f q - \rho_c C_D f_0 \right) \\
&\equiv e^R(q) + ic^l(q), \\
d(q) &\equiv \frac{g\varepsilon_0}{\rho_e' f} (\rho_c - \rho_L) \delta(q).
\end{align*}
\]

Taking now the Laplace transform of (35) we get

\[
\delta \tilde{\varepsilon}(q,s) = \frac{1}{(s-s_1)(s-s_2)} \left\{ [s - b(q)] \delta \tilde{\varepsilon}(q,0) + a(q) \delta \tilde{\nu}(q,0) + \frac{a(q)d(q)}{s} \right\}.
\]

From (45)-(47) we obtain \( \delta \tilde{\varepsilon}(q,s) \)

\[
\delta \tilde{\varepsilon}(q,s) = \frac{1}{(s-s_1)(s-s_2)} \left\{ [s - b(q)] \delta \tilde{\varepsilon}(q,0) + a(q) \delta \tilde{\nu}(q,0) + \frac{a(q)d(q)}{s} \right\}.
\]

The void fraction correlation function \( (\delta \varepsilon(q,s)\delta \varepsilon(q,0)) \) can be found by multiplying (48) by \( \delta \tilde{\varepsilon}(q,0) \) and by taking an ensemble average of the product. The resulting equation can be greatly simplified because in the limit \( q \to 0 \), \( (\delta \varepsilon(q,0)\delta \varepsilon(q,0)) = 0 \) and \( (\delta \varepsilon(q,0)) = 0 \). The resulting correlation function is then

\[
C(q,s) = \frac{(\delta \varepsilon(q,s)\delta \varepsilon(q,0))}{(\delta \varepsilon(q,0)\delta \varepsilon(q,0))} = \frac{s-b(q)}{(s-s_1)(s-s_2)}.
\]

Since \( s = i\omega \)

\[
C(q,\omega) = \frac{R(q,\omega) T(q,0)}{R(q,0) T(q,\omega)}.
\]

where we have defined

\[
R(q,\omega) = [\omega^2 - \varepsilon_0 q^2 c^l(q)] - \omega [b(q)\omega + \varepsilon_0 q^2 R(q)],
\]

\[
T(q,\omega) = [\omega^2 - \varepsilon_0 q^2 c^l(q)]^2 + [b(q)\omega + \varepsilon_0 q^2 R(q)]^2.
\]

5. Experimental

The experiments were performed in a Plexiglas vertical column, with inner diameter 0.20 and 2.6 m height. Bubbles where released to a liquid stream, just below the distributor. The mean section of the column was adapted with a set of flushed electrodes to the wall inside of the column. Eleven pairs of electrodes were thus placed and numbered from the bottom to the top. Odd number electrodes were denoted as guard electrodes, while the even number ones were designated as measuring electrodes. In this way a set of five measuring electrodes were supplied. Each electrode pair is formed by two stainless steel fine sheets 74mm×157mm, flushed mounted and glued to the inner pipe wall, placed at the same height and on opposite sides of the wall. The center line of the first measuring electrode was 0.914 m above the distributor, as can be seen in Fig. 1 [5].
at a selected sampling frequency. The dynamic characterization of the two-phase flow is therefore an excellent tool for the prevention of instabilities. The homogeneous bubbling regime in a bubble column is characterized by a spectrum of void fraction signals with just one dominant frequency peak, below 1.0 Hz. Experimental autocorrelation functions or finite-time autocorrelograms, \( C^{*\exp}(t) \), that is, those computed from the measured void fraction time series in the measuring volumes of the column, were found according to the definition

\[
C^{*\exp}(j) = \frac{1}{1001-j} \sum_{i=1}^{1001-j} \delta \varepsilon_G(i) \delta \varepsilon_G(i+j),
\]

for \( j = 0, 1, \ldots, M \) (53)

where \( \delta \varepsilon_G(i) = \varepsilon_G(i) - \langle \varepsilon_G \rangle \), is the difference between the \( i \)-th void fraction datum and the mean value of the time series for a given electrode. \( j \) denotes a time lag index which divided by the sampling frequency (100 Hz), gives the time lag in seconds. It should be stressed that these autocorrelations were further normalized with respect to the value \( C_0 \equiv C^{*\exp}(j = 0) \). The finite Fourier transform of the autocorrelogram is the frequency density of the time averaged power that an electrical signal (a voltage) would dissipate in a resistance and it is also known as the empirical finite-time power spectral density of the time series. \( C^{*\exp}(\omega) \) was computed by the trapezoidal rule as

\[
C^{*\exp}(p) = 2\Delta t \left( E_0 + 2 \sum_{j=1}^{M-1} E^{exp}(j) \cos \frac{\pi j p}{M} \right)
\]

\[+ E^{exp}(M) \cos \pi p \]  (54)

for \( p = 0, 1, \ldots, P \), where \( p \) is a frequency index which divided by twice the total time lag window \( 2M\Delta t \), gives the frequency in Hertz.

6. Results

We calculate \( G_f \) and \( G_{fp}^p \) as defined by Eqs. (29) and (33), to compare the monodispersed (\( MD \)) and polydispersed (\( PD \)) results on the gain factor (\( G_f, G_{fp}^p \)), mean bubble velocity (\( U_{G0} \)), kinematic wave velocity (\( c_0 \)) and dynamic wave velocities (\( c^\pm \)) as functions of the void fraction \( \varepsilon_0 \), we used the material parameter values for an air-water bubble column given in Table I.

In Fig. 2 we plot both, the gain factor \( G_f \) for the monodispersed (\( MD \)) and \( G_{fp}^p \) for the polydispersed (\( PD \)) cases vs. \( \varepsilon_0 \) for different frequencies \( \omega = 1.5, 2, 2.5, 3 \) Hz. The bubble size polydispersity is described by a log-normal distribution \( n(\alpha) \) with average \( \overline{\alpha} = 0.04 \) cm and dispersion 0.5.
This means that stability is larger in about 23% for the latter case, a change that is significant in bubble reactors [39]. The mean percentage difference defined by

$$\frac{1}{N} \sum_{i=1}^{N} \left( G_{fi} - G_{fi}^p \right)$$

for a frequency of 1.5 Hz, is 0.60%, whereas for a frequency of 3 Hz is 11.35%.

The sets of quantities $\beta \equiv \{ U_{G0}, \epsilon_0, c^\pm \}$ and $\beta^p \equiv \{ \lambda^{-1} U_{G0}, \lambda^{-1} \epsilon_0, \lambda^{-1} c^\pm \}$ are plotted as functions of $\epsilon_0$ in Fig. 3.

Observe that the curve for $\epsilon_0$ is always between that for $c^+(PD)$ and $c^-(PD)$. Note that according to the Whitham stability criterion [40], when $\epsilon_0 < c^-$ the uniform flow is unstable. In our case this occurs for both distributions; however, for the monodispersed case it occurs for $\epsilon_0 > 0.35$, whereas for the polydispersed case the system is stable up to a larger value of the void fraction, e.g. $\epsilon_0 > 0.36$.

As in gas dynamics, dissipation occurs due to random motions of the constituent particles. For a dispersion of bubbles with radii of 0.4 mm, the diffusivity $\nu_e(\epsilon_0)$ is 0.182 cm$^2$/s and 0.173 cm$^2$/s, for monodisperse and polydisperse cases, respectively. On the other hand, the relaxation time $\tau_e(\epsilon_0)$ is 0.97 and 0.92 s for the monodisperse and polydisperse cases, respectively. The action of these parameters indicate the contribution of the inertia of the bubbles and the surrounding fluid as well as the fluctuating bubble motion due to hydrodynamic interactions on the wave dynamics, as can be seen from Eqs. (27) and (28).

To compare our theoretical predictions for $G_f$ for the monodispersed $(MD)$ and $G_f^p$ for the polydispersed $(PD)$ cases, with the experimental measurements of the spectral density of the void fraction correlation function, in Fig. 4 we plot $C(q, \omega)$ as a function of $\omega$ for a fixed wave number $q = 2\pi/z$, as given by Eq. (49). Here $z = \Delta z$ denotes the height of level 1 in the column and we used the parameter values given in Table I. In Fig. 4 we plot together the theoretical fraction volume correlations $C^{MD}$, $C^{PD}$ and the experimental results for $C^{exp}(q)$ vs. $\omega$, for $\Delta z$ and $\epsilon_0 = 0.0833$. $C^{exp}$ is the normalized experimental correlation, $C^{exp}(q, \omega) \equiv (C^{exp}(p))/C_0$ that is obtained in the homogeneous bubbling regime.

It is apparent that qualitatively, the theory reproduces the decay at low frequencies observed in the experiment. Furthermore, $C^{PD}$ adjusts better than $C^{MD}$ to $C^{exp}$. To quantify the differences between both curves for each $\omega$, we introduce the relative error,

$$\Delta = \left| \frac{C^{exp}(z, \omega) - C(z, \omega)}{C^{exp}(z, \omega)} \right|.$$

The monodisperse

$$\Delta^{MD} = \left| \frac{C^{exp}(z, \omega) - C^{MD}(z, \omega)}{C^{exp}(z, \omega)} \right|.$$
varies between 5.33% for \( \omega \) and polydisperse theoretical and experimental values, \( \Delta \) values, between the monodisperse theoretical and experimental values are given in Table II, errors values are given in Table II.

Note that for the range 0 - 0.4 Hz, the difference between the monodisperse theoretical and experimental values, \( \Delta^{MD}(\%) \), is larger than the difference between the polydisperse theoretical and experimental values, \( \Delta^{PD}(\%) \). For the former \( \Delta \) amounts to a percentage difference that varies between 5.33% for \( \omega = 0.1 \) Hz and 47.32 for \( \omega = 0.4 \) Hz, whereas for the latter interval this difference may be as low as 0.96% for \( \omega = 0.3 \) Hz and grows up to 12.27% for \( \omega = 0.4 \) Hz.

The standard deviation or average error estimates the statistical average error of our theoretical autocorrelation functions with the experimental data. It is defined as [42]

\[
\sigma \equiv \left[ \frac{1}{N} \sum_{i=1}^{N} (C_i - C_i^{exp})^2 \right]^{1/2} \quad (58)
\]

and it turns to be \( \sigma^{MD} = 0.106 \) and \( \sigma^{PD} = 0.052 \), respectively.

Also note that from Eqs. (49) - (52) one verifies that

\[
C(q, \omega) \sim q^{-4}. \quad (59)
\]

This is an interesting result since shows that in spite of the complexity of Eq. (49) as a function of \( q \), an explicit overall \( q \) dependence for \( C(q, \omega) \) is obtained and which is of the same type as the one obtained for correlation functions of other complex fluids, such as a liquid crystal in near equilibrium states [41].

### 7. Discussion

In this work we have introduced a statistical physics approach to describe some of the properties of a two-phase bubble column. The following comments may be useful to clarify and elaborate on some of our results.

First, in this work we have analyzed the effects of bubble size polydispersity and several features of the void fraction waves and their stability properties. We found that the presence of this distribution in sizes reinforces the stability of the waves, as shown in Figs. 2 and 3. Furthermore, the percentage difference \( \Gamma \equiv |\beta - \beta^o|/\beta \) turns out to be \( \Gamma = 4.9\% \).

Secondly, it is convenient to emphasize once again, that the hydrodynamic model used in this work [14] is idealized in many aspects. For instance, compressibility and hydrodynamic interactions between bubbles and with the boundaries, have not been taken fully into account. The liquid phase is incompressible implying that its density is not a function of space and time. However, the bubble dispersion is compressible because the local hold-up is not a constant. The compressibility could be accounted for by the spatial variation of the fractional gas hold-up. Although the effect of hydrodynamic interactions between bubbles on the mean frictional force have been partially accounted for through the function \( f_0(\varepsilon) \) in Eq. (5), other hydrodynamic interactions have been neglected.

Thirdly, the results obtained with this model show that the agreement between the experimental and theoretical values for the void fraction correlation function is reasonably good with an absolute error which may vary between 0.052 for the polydisperse case and 0.106 for the monodisperse case.

We should also mention that in this work we have assumed an initial polydisperse bubble size distribution and the coalescence and fragmentation of bubbles have been entirely neglected [5, 43]. Although this assumption is reasonable in many cases its validity of this assumption remains to be assessed [16, 43]. Nevertheless the mean field approach considered here by including the influence of the distribution through the drag effects, is an attempt to set a first framework to incorporate bubble size effects in future studies.

Our analysis illustrates how some of the methodology and concepts of kinetic theory and statistical mechanics may be used to deal with complex phenomena in engineering systems.

![Figure 4. Comparison between theoretical fraction volume correlations \( C^{MD} \), \( C^{PD} \) and the experimental fraction volume correlation \( C^{exp}(\bullet) \) vs. \( \omega \), for a given \( \Delta z \).](image-url)
TIME DEPENDENT CORRELATION FUNCTIONS AND POLYDISPERSITY EFFECTS IN BUBBLING FLOWS

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