On the transition probability for a quantum dot in a time dependent magnetic field

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In this work we analyze in detail the dynamical behavior of a quantum dot in presence of a uniform time-dependent magnetic field, in the effective mass approximation. An exact solution for the time-evolved wave function is obtained when the initial state is a particular Fock–Darwin state. Moreover, we calculate the transition probability from an initial Fock-Darwin state to a given final Fock-Darwin state for the case of a magnetic field that changes linearly in time, flipping its direction.

Keywords: Quantum Dot; Exact Solution; Time dependent Problems.

En este trabajo analizamos en detalle, en la aproximación de masa efectiva, la dinámica de un electrón en un punto cuántico en presencia de un campo magnético uniforme dependiente del tiempo. Se encuentra una solución exacta para la evolución temporal de la función de onda cuando el estado inicial es un estado de Fock-Darwin. Se calcula también la probabilidad de transición entre estados de Fock-Darwin específicos cuando el campo magnético invierte su dirección cambiando linealmente en el tiempo.

Descriptores: Cuánticos; soluciones exactas; problemas dependientes del tiempo.

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1. Introduction

Quantum dots (QD) are zero dimensional objects constructed by patterning and epitaxial growth techniques in semiconductor heterostructures [1]. An important characteristic of these systems is that the phase coherent length of the electron wave functions exceeds the size of the dots, and consequently, their motion is considered ballistic. There exists a vast amount of literature discussing diverse physical aspects of these systems, like spectroscopy, nonlinear optics, magnetotransport, many-body effects, etc [2]. In many theoretical studies of the QD, the electrons are considered independent, and their energy level structure is determined accordingly. In addition, it is usually assumed that the lateral confining potential has a parabolic shape. Apart from the obvious mathematical simplicity brought about by this assumption, self-consistent calculations, as well as experimental observations, have provided a support to the validity of this approach [3].

On the other hand, the dynamics of electrons in magnetic fields has played a fundamental role in physics and its application to technology in a wide spectrum of topics, such as the Aharonov – Bohm effect [4–7], the bidimensional electron gas at the interface of semiconductor heterostructures [8], and electromagnetic lenses with time-dependent magnetic fields [9–12]. All these situations are interesting by themselves, and present rather complex mathematical structures; therefore, exact analytic solutions have been found only in a few special cases [13]. Therefore, the study of electrons in a QD under the presence of the magnetic field is important and in order.

In the present work, we study the dynamics of electrons of a quantum dot interacting with a homogeneous time-dependent magnetic field together with the corresponding induced electric field, assuming a parabolic dot confining potential in the effective mass approximation. We find an exact solution for the corresponding propagator of the Schrödinger equation. An analytical expression for the time-evolved wave function is found when the initial state is a Fock-Darwin state. We also calculate the transition probability to a given Fock-Darwin state as a consequence of the interaction with the electromagnetic field. The arrangement of the paper is as follows: In Sec. 2, the theoretical model is presented. In Sec. 3, the application is performed. Finally, the conclusions are presented in Sec. 4.

2. Theoretical Model

In this section we will develop the quantum mechanical formalism to describe the dynamics of electrons in a quantum dot interacting with a homogeneous time-dependent magnetic field, B(t), and its corresponding induced electric field. Since the Hamiltonian is time-dependent, energy will not be conserved and there will not be an energy spectrum. Therefore, the problem must be approached by directly solving the time-dependent Schrödinger equation:

\[
\left\{ \frac{1}{2m} \left[ \mathbf{P} + \frac{e}{c} \mathbf{A} \right]^2 + U \right\} \Phi (\mathbf{r}, t) = i\hbar \frac{\partial}{\partial t} \Phi (\mathbf{r}, t),
\]

where \( m \) is the electron effective mass, \( q = -e \) is the electron charge, \( c \) is the speed of light and the vector potential is chosen to be \( \mathbf{A} = -\mathbf{r} \times \mathbf{B}/2 \). In order to analyze a QD phenomenon, we will model the dot by a parabolic potential whose strength is related to the dot size \( l_d = (\hbar/m \varpi_0)^{1/2} \), where \( \varpi_0 \) is the corresponding oscillator frequency. There-
note that, if -τ ≤ t ≤ τ. Let (ζ₁, ζ₂) be two linearly independent solutions of Eq. (7) satisfying the initial conditions ζ₁ (−τ) = ζ₂ (−τ) = 0 and ζ₂ (−τ) = ζ₁ (−τ) = 1. We can easily express these solutions in terms of the solutions of Eq. (7) for the initial conditions μ₁ (0) = μ₂ (0) = 0, μ₂ (0) = μ₁ (0) = 1. The result is:

\[ \begin{align*}
\zeta_1 (t) &= μ_2 (τ) μ_1 (t) + μ_1 (τ) μ_2 (t) \\
\zeta_2 (t) &= μ_2 (τ) μ_1 (t) + μ_1 (τ) μ_2 (t)
\end{align*} \]  

where

\[ \begin{align*}
μ_1 (t) &= t \exp \left( -i \omega_L t^2 / 2τ \right) \\
μ_2 (t) &= \exp \left( -i \omega_L t^2 / 2τ \right) \\
μ_1 (0) &= t \exp \left( -i \omega_L t^2 / 2τ \right) \\
μ_2 (0) &= \exp \left( -i \omega_L t^2 / 2τ \right)
\end{align*} \]  

1F1 (a, b, z) being the usual Hypergeometric function; also, using the series expansion of this function it can be easily verified that the functions μ₁,₂ are real, and it is trivially proven that the functions {μ₁, μ₂} are odd and even functions of t, respectively. Figure 1 shows the time dependence of functions {ζ₁, ζ₂} in the interval −τ < t < τ for different values of the dimensionless ratio ω₀/ω_L, where we have chosen ω_L = 1 and τ = 10. Both functions have an oscillating behavior whose period decreases as the ratio ω₀/ω_L increases.

Now let us elucidate the structure of the propagator G(r_f, τ, r_i, −τ). Firstly, we choose θ (−τ) = 0, hence θ (τ) = 0; and using the parity of the functions {μ₁, μ₂} we find that

\[ \zeta_1 (τ) = 2μ₂ (τ) μ₂ (τ), \]
\[ \zeta_2 (τ) = 2μ₁ (τ) μ₂ (τ) − 1, \]
\[ ⌇_1 (τ) = ⌇_2 (τ) \]

and ⌇_2 (τ) = 2μ₁ (τ) μ₂ (τ). Therefore, Eq. (4) becomes

\[ G(r_f, τ, r_i, −τ) = \frac{m}{4πiℏμ₁ (τ) μ₂ (τ)} × \exp \left[ \frac{i m ζ₂ (τ)}{4ℏμ₁ (τ) μ₂ (τ)} \left( r_f^2 + r_i^2 − \frac{2}{ζ_2 (τ)} r_f ⋅ r_i \right) \right] \]  

In what follows we will analyze the temporal evolution of the QD wave function. The wave function at \( t = −τ \) can be expressed as a superposition of Fock–Darwin states [16–17]:

\[ \Psi (r_f, −τ) = \sum_{l,n} C_{l,n} \exp \left( i E_{l,n} \tau / h \right) ψ_{l,n} (r_i) \]  

where the C_{l,n} are constants,
and

\[ \psi_{1,n}(\mathbf{r}_i) = \left( \sqrt{p_{|n|}^i/|\pi \lambda_\omega^2|} \right) \times \exp \left[ i n \phi_i - m r_i^2 \sqrt{\omega_\omega^0 + \omega_\omega^2}/2h \right] \times (r_i/\lambda_\omega)^{|n|} L_i^{[n]} \left((r_i/\lambda_\omega)^2\right) \]

(14)

where \( p_{|n|}^i = i!/(l + |n|)! \), \( \lambda_\omega = \left( (\hbar/(m \omega_\omega^0))/\sqrt{1 + (\omega_L/\omega_\omega^2)^2} \right)^{1/2} \), \( L_i^{[n]}(x) \) are the generalized Laguerre polynomials, and \( \{ r_i, \phi_i \} \) are the polar coordinates of \( \mathbf{r}_i \). For simplicity, we will analyze the case \( l = 0 \). In such scenario the initial wave function can be cast as

\[ \Phi_{0,n}(\mathbf{r}_i, -\tau) = \tilde{C}_{0,n}(\tau) \exp \left[ i n \phi_i - A r_i^2 \right] r_i^{[n]} \]

(15)

where \( A = m \sqrt{\omega_\omega^0 + \omega_\omega^2}/2h \) and

\[ \tilde{C}_{0,n}(\tau) = C_0 \exp \left(i \tau E_{0,n}/\hbar \right) \]

with \( C_0 \) a normalization constant. Therefore, using Eqs. (3), (12) and (15), the time-evolved wave function is given by

\[ \Phi_{0,n}(\mathbf{r}, \tau) = \tilde{C}_{0,n}(\tau) \frac{\alpha(\tau)}{\pi i \zeta_2(\tau)} \times \exp \left[ i \alpha(\tau) r^2 \right] \eta_n(\mathbf{r}, \mathbf{r}; \tau) \]

(16)

where \( \alpha(\tau) = m \zeta_2(\tau)/(4 \hbar \mu_1(\tau) \mu_2(\tau)) \) and

\[ \eta_n(\mathbf{r}, \mathbf{r}; \tau) = \int d^2r_i \exp \left[ i n \phi_i - Z(\tau) r_i^2 \right] r_i^{[n]} \times \exp \left[-i \mathbf{s} \cdot \mathbf{r}_i \right] \]

(17)

with \( s(\tau) = 2 \alpha(\tau)/\zeta_2(\tau) \) and \( Z(\tau) = A - i \alpha(\tau) \). Let us first calculate the angular integral:

\[ I_\phi = \int_0^{2\pi} d\phi_i \exp \left[ i n \phi_i - i \mathbf{s} \cdot \mathbf{r}_i \cos(\phi_i - \phi) \right] \]

(18)

Using the integral representation of the Bessel function [18]:

\[ J_n(\xi) = \frac{1}{2\pi} \int_a^{2\pi + a} \exp \left[ i (nx - \xi \sin x) \right] dx \]

(19)

Equation (18) can be written as

\[ I_\phi = 2\pi \exp \left(i n (\phi - \pi/2) \right) J_n(s(\tau) r r_i) \]

(20)

Thus, we finally obtain the following expression for Eq. (11)

\[ \eta_n(\mathbf{r}, \mathbf{r}; \tau) = 2\pi \exp \left(i n (\phi - \pi/2) \right) \int_0^\infty d r_i r_i^{|n|+1} \times \exp \left(-Z(\tau) r_i^2 \right) J_n(s(\tau) r r_i) \]

(21)

The integral in Eq. (21) converges if \( A - \text{Re} [Z] > 0 \), and in such case its value is:

\[ \int_0^\infty d r_i r_i^{|n|+1} \exp \left(-Z(\tau) r_i^2 \right) J_n(s(\tau) r r_i) \]

\[ = \frac{1}{2Z(\tau)} \left( \frac{\alpha(\tau)}{\zeta_2(\tau) Z(\tau)} \right)^n \times \exp \left[-\frac{1}{Z(\tau)} \left( \frac{\alpha(\tau)}{\zeta_2(\tau)} \right)^2 \right] \]

(22)

where, for simplicity, we have assumed that \( n > 0 \). Therefore, Eq. (16) can be expressed as:

\[ \Phi(\mathbf{r}, \tau) = \tilde{C}_{0,n}(\tau) \left( \frac{\alpha(\tau)}{\zeta_2(\tau) Z(\tau)} \right)^n r^{[n]} \times \exp \left(i \alpha(\tau) - \frac{1}{Z(\tau)} \left( \frac{\alpha(\tau)}{\zeta_2(\tau)} \right)^2 \right) \]

(23)

Equation (23) is the exact expression for the time evolved wave function when the initial wave function is a particular Fock-Darwin state. We note that it has a similar structure to the initial wave function, the dissimilarity appearing in the coefficients and in a numerical phase.

Now let us calculate the probability transition to different Fock-Darwin states due to the particular time dependence of the magnetic field. Let us consider the states \( \Phi_{0,k}(-\tau) \) and \( \Phi_{0,n}(\tau) \). The transition amplitude is:

\[ \langle \Phi(-\tau) | \Phi(\tau) \rangle = \int d^2 \mathbf{r} \Phi_{0,k}^*(\mathbf{r}, -\tau) \Phi_{0,n}(\mathbf{r}, \tau) \]

(24)

Equation (24) can be written as

\[ \langle \Phi(-\tau) | \Phi(\tau) \rangle = X_{n,k}(\tau) \int_0^{2\pi} d\phi \exp \left(i \phi (n - k) \right) \times \int_0^\infty d r r^{|n|+|k|+1} \exp \left[-\Delta(\tau) r^2 \right] \]

(25)

with

\[ X_{n,k}(\tau) = \frac{\tilde{C}_{0,n}(\tau) \tilde{C}_{0,k}^\ast(\tau)}{i} \left( \frac{\alpha(\tau)}{\zeta_2(\tau) Z(\tau)} \right)^{n+1} \times \exp \left(-i n \pi/2 \right) \]

(26)
and
\[ \Delta(\tau) = Z(\tau) + \frac{1}{Z(\tau)} \left( \frac{\alpha(\tau)}{\zeta_2(\tau)} \right)^2 \]

(27)

Since
\[ \int_0^{2\pi} d\phi \exp(i\phi(n-k)) = 2\pi\delta_{n,k} \]

(28)

where \(\delta_{n,k}\) is the Kronecker delta, the only possible transition is that which conserves the \(z\)-component of the angular momentum. Hence, Eq. (25) is reduced to
\[ \langle \Phi(-\tau) | \Phi(\tau) \rangle = 2\pi X_{n,n}(\tau) \]
\[ \times \int_0^\infty dr r^{2|n|+1} \exp[-\Delta(\tau)r^2] \]

(29)

Since
\[ \Re \left\{ Z + \frac{\alpha^2}{\zeta_2^2} Z \right\} = A + \frac{\alpha^2A}{\zeta_2^2(\alpha^2+A^2)} > 0, \]

the integral converges. Its value is
\[ \int_0^\infty dr r^{2|n|+1} \exp[-\Delta(\tau)r^2] = \frac{|n|!}{2(\Delta(\tau))^{|n|+1}}. \]

(30)

Therefore,
\[ \langle \Phi(-\tau) | \Phi(\tau) \rangle = \frac{2\pi}{\nu} \left| \tilde{C}_{0,n}(\tau) \right|^2 \left( \frac{m}{4\hbar\mu_1(\tau)\mu_2(\tau)} \right)^{n+1} \]
\[ \left| n \right|! \exp(-in\pi/2) \left( Z(\tau)^2 + (\alpha/\zeta_2)^2 \right)^{|n|+1} \]

(31)

The transition probability is of course proportional to \(|\langle \Phi(-\tau) | \Phi(\tau) \rangle|^2\). Figure 2 shows the un-normalized transition probability as a function of \(\tau\) (in arbitrary units) and the ratio \(\varpi_0/\omega_L\) for the \(n = 1\) case. We note that it has a non-trivial structure, with an oscillatory dependence on the length of the time interval \(\tau\), and a peak as a function of \(\varpi_0/\omega_L\). As a consequence of these behaviors, the transition probability presents a line of maximums in certain regions of the parameter space; besides, for a fixed \(\tau\), the probability transition decays asymptotically for large \(\varpi_0/\omega_L\), away from the line of maximums.

4. Conclusions

The dynamics of electrons in a quantum dot interacting with a uniform time-dependent magnetic field has been analyzed when the magnetic field changes linearly in time, flipping its direction in a given time interval. An exact analytical solution for the time-evolved wave function has been obtained when the initial state corresponds to a specific Fock-Darwin level. The classical dynamics of this system has been found; it presents a non trivial oscillatory behavior through its dependence on the hypergeometric functions in Eq. (12). Both the amplitude and the period of the oscillation decrease as the strength of the dot potential, and therefore its confining properties, increases. The evolved wave function has a structure similar to that of the initial state, but with different time-dependent coefficients. In addition, the probability transition has been calculated for these states; we found that this probability is appreciably different from zero only in some regions of the parameter space.

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