Self-similar Optical Properties in Pascal-type Quasiperiodic Dielectric Multilayer

X.I. Saldaña
Instituto de Física, Universidad Autónoma de Puebla,
Apartado postal J-48, Puebla, Pue. 72570, México.
e-mail: xochitl@sirio.ifuap.buap.mx

D. A. Contreras-Solorio
Escuela de Física de la Universidad Autónoma de Zacatecas,
Apartado Postal C-580, 98060, Zacatecas, Zac., México.
e-mail: dacs@cantera.reduaz.mx

Elías López-Cruz
Instituto de Física, Universidad Autónoma de Puebla,
Apartado postal J-48, Puebla, Pue. 72570, México.
e-mail: elopez@sirio.ifuap.buap.mx

Recibido el 30 de noviembre de 2006; aceptado el 8 de octubre de 2007

Using the transfer-matrix method, we study the band structure of the propagating electromagnetic modes in dielectric quasiperiodic multilayers. The structure is formed with bilayer units. The first layer has a fixed width, whereas the second varies following a quasiperiodic Pascal sequence given by \( dp(x) = d_1 + D \times P \); \( d_1 \) is a chosen width, \( D \) is a fixed increment, and \( P \) represents the quasiperiodic sequence. In this case, \( 2^P \) is the number of bilayers forming the structure.

**Keywords:** Band structure; quasiperiodic dielectric multilayer.

Using el método de la matriz de transferencia se estudia la estructura de bandas de los modos electromagnéticos que se propagan en una estructura dieeléctrica de multicas cuasiperiódica. La estructura se forma con unidades constituidas por dos capas, una de ellas tiene un ancho fijo y la otra tiene un ancho que varía siguiendo una secuencia cuasiperiódica de Pascal dada por \( dp(x) = d_1 + D \times P \); \( d_1 \) es un ancho inicial elegido, \( D \) es un incremento fijo y \( P \) representa la secuencia cuasiperiódica. En este caso \( 2^P \) es el número de bicapas que forman la estructura.

**Descriptors:** Estructura de bandas; estructura dieeléctrica de multicas cuasiperiódica.

**PACS:** 41.20.Jb; 71.20.-b; 78.67Pt

### 1. Introduction

Quasiperiodic deterministic structures, which are intermediate between periodic and completely disordered ones, present properties not observed in the other two, such as the existence of electron and phonon critical states or the localization of light waves in dielectric quasiperiodic multilayers [1]. Of particular interest is the understanding of the propagation of optical waves in optical quasiperiodic systems not only for its theoretical interest but for its promising technological applications [2] first demonstrated by the fabrication of a dielectric Fibonacci superlattice by Merlin et al. [3]. Multilayers with other quasiperiodic profiles (Cantor, Rudin Shapiro, Thue-Morse) [4–6] have also been studied. In this work, we present a one-dimensional quasiperiodic dielectric multilayer constructed considering a numerical sequence obtained from the sum of all the odd entries of each row of Pascal’s Triangle [7]; this sequence has the property of self-similarity. We calculate the band structure of the electromagnetic waves propagating in the dielectric multilayer using the transfer matrix method and a special application of the Bloch theorem.

### 2. Infinite Quasiperiodic Multilayer

The self-similar sequence used in this study can be obtained by counting the odd entries in each row of Pascal’s Triangle. By doing so, we are left with an infinite set of numbers

\[ 1, 2, 4, 2, 4, 4, 8, 2, 4, 8, 16, \ldots \]

with a recursion relation given by

\[ f(n) = 2 \times f\left(n - 2^\left\lfloor \log_2 n \right\rfloor \right) \]

where \( f(x) \) means the integer part of \( x \) with \( n = 0, 1, 2, 3, \ldots \) and \( f(0) = 1 \). We observe that each element of the sequence is a power of two \( 2^P \) with \( P = 0, 1, 2, 1, 2, 3, 1, 2, 2, 3, 3, 4, \ldots \) For our purpose, we will use the sequence formed by the exponents which is also self-similar. The structure we propose is formed by an infinite stack of bilayer units. The first layer has a refractive index \( n_a \) and a fixed width \( d_a \). The second layer has a refractive index \( n_p \) and a variable width given by

\[ dp(P) = d_1 + D \times P \]

where \( d_1 \) is an initial width, \( D \) is a fixed increment, and \( P \) takes a different value for each bilayer unit following the sequence described above, so that the total width of each unit will be \( du(P) = da + dp(P) \). In Fig. 1a we show only the width profile \( du(P) \) as a function of the distance taken along the growth direction of the stacks with 8 bilayer units and 16 layers. In Fig. 1b the profile corresponds to a stack with 32 bilayer units. We see that the self-similarity property of the width profile becomes more apparent as the bilayer unit number increases.

### 3. Theory

For the numerical calculations, we use the transfer matrix theory as presented in [8]. The infinite quasiperiodic multilayer...
SELF-SIMILAR OPTICAL PROPERTIES IN PASCAL-TYPE QUASIPERIODIC DIELECTRIC MULTILAYER

311

In (a) we see the width profile of a unit cell with 8 bilayers $P = 3$, in (b) we see the profile for a unit cell with 32 bilayers $P = 5$, here the self-similar property is more apparent. The bilayer’s parameter values are: first refractive index $n_a = 4.5$, and width $d_a = 2$; second refractive index $n_p = 1.5$ and variable width $d_p(P) = d_1 + D \times P$, with $d_1 = 1$, $D = 1$ and $P = 0, 1, 2, 3, 4, 5, \ldots$

can be studied by using the rational approximation method [9], which consists of a periodic sequence of unit cells containing $2^P$ bilayer units with a quasiperiodic profile as shown in Fig. 1a. In this form, the structure’s self-similar properties will become apparent. The band structure for $TE$ and $TM$ polarizations is then calculated by imposing the condition that $\frac{(1/2)|Tr(M)|}{\sqrt{Tr(M^2)}} \leq 1$; here, $M$ is the total transfer matrix corresponding to the unit cell. If $P = 0$, the bands will correspond to a periodic superlattice with only one bilayer unit in its unit cell; if $P = 1, 2, 3, \ldots$, each band of the periodic superlattice will split into $2^P$ bands due to the quasiperiodicity of the new unit cell structure.

4. Numerical Results

The energy band structure for $TE$ and $TM$ polarizations, as a function of the component of the propagation vector parallel to the interfaces, was calculated with parameter values $d_a = 2$, $d_1 = 1$, $D = 1$, $n_a = 4.5$, $n_p = 1.5$. When $P = 0$, the number of bilayer units is one; so, we recover the periodic structure. In Fig. 2 we show the first allowed band for both polarizations. Using the same parameters, and $P = 1, 2, 3, \ldots$, the allowed band for the periodic structure separates into $2^P$ minibands with $2^P - 1$ gaps. In Fig. 3 we present the energy band structure for $P = 3$ where the unit cell of the multilayer has 8 bilayer units and 16 layers (see Fig. 1a). We notice that the bands for both polarizations separate in 8 minibands; for $TM$ polarization, the forbidden gaps become zero for certain values of the parallel wave vector $b$, which is associated with the Brewster angle for the structure.

Figure 1. In (a) we see the width profile of a unit cell with 8 bilayers $P = 3$, in (b) we see the profile for a unit cell with 32 bilayers $P = 5$, here the self-similar property is more apparent. The bilayer’s parameter values are: first refractive index $n_a = 4.5$, and width $d_a = 2$; second refractive index $n_p = 1.5$ and variable width $d_p(P) = d_1 + D \times P$, with $d_1 = 1$, $D = 1$ and $P = 0, 1, 2, 3, 4, 5, \ldots$

Figure 2. First allowed energy band for a periodic system with $P = 0$ and one bilayer, (a) $TM$ modes, (b) $TE$ modes.

Figure 3. The same energy band as in previous figure with $P = 3$, now we have $2^3 = 8$ minibands for both polarizations, in (a) we observe that the gap becomes zero for values of the propagation wave vector parallel to the interfaces related to the Brewster angle.

The calculations are made for normalized frequency $W=\omega(da+d1)/2\pi c$ and normalized parallel wave vector $b=b_z(da+d1)/2\pi$. Although it is possible to do these calculations with $dp$, $da$ and $na$ constants, and variable refractive index $np(P)=n1+D\times P$, with similar results for the band structures, the existing technological limitations for the preparation of multilayers with very high values of the refractive index makes the structure with fixed refractive index and variable width, more suited for technological applications.

5. Conclusions

We have presented for the first time a calculation of the energy band diagrams for a multilayer, where the width of the layers is modulated by a self-similar sequence obtained from Pascal’s Triangle. This property is observed in the width profile shown in Fig. 1 and by comparing the band structures obtained for the periodic multilayer and the multilayers obtained for each value of the sequence $P$ described before, where we observe (Figs. 2 and 3) that each band for the periodic superlattice separates into $2^P$ minibands for the new quasiperiodic one, as $P$ increases. The dielectric structures proposed here exhibit properties which make them potentially useful for technological applications and can be prepared in the laboratory with the available technology.

6. Acknowledgment

This work was partially supported though grant 37-EXC-G-06 from VIEP-BUAP.