Fictitious cavity approach to the casimir effect

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We obtain expressions for the Casimir energy and force following an approach which may be applied to cavities made up of arbitrary materials. In the case of planar cavities, we obtain the well known Lifshitz formula. The approach is easily generalized to other geometries.

Keywords: Casimir effect; quantum fluctuations; cooperative phenomena; nanostructures.

Obtenemos expresiones para la energía y la fuerza de Casimir siguiendo un procedimiento que puede ser aplicado a cavidades en materiales arbitrarios. En el caso de cavidades planares recuperamos la bien conocida fórmula de Lifshitz. El procedimiento puede ser generalizado fácilmente a otras geometrías.

Descriptores: Efecto Casimir; fluctuaciones cuánticas; fenómenos cooperativos; nanoestructuras.

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1. Introduction

Half a century ago, Casimir [1] predicted that the quantum fluctuations of the electromagnetic field within a planar cavity would produce an attractive macroscopic force on its boundaries. His prediction was based on the properties of the field when confined by perfectly reflecting mirrors. It is only recently that experimental studies have attained the necessary accuracy to test in detail the theoretical predictions [2]; the Casimir effect has now been measured [3–6] with uncertainties as small as 1% and at distances [7] down to \( \approx 60 \text{nm} \). Therefore, theories of the Casimir force that account for the properties of realistic cavities have become indispensable. In 1956, Lifshitz proposed a macroscopic theory for two semi-infinite homogeneous dielectric slabs [8] characterized by complex frequency dependent dielectric functions. The stress tensor was obtained from the self-correlation of the fluctuating electromagnetic field, whose source consists of fluctuating charge and current densities within each slab. Their autocorrelations are related through Kubo’s formalism [9], causality, and the fluctuation-dissipation theorem, to the complex dielectric response. In his calculation, Lifshitz considered only homogeneous and isotropic media, and it was assumed that fluctuating sources at a given position were completely uncorrelated with sources at nearby positions, so that the results were directly applicable to semi-infinite homogeneous flat local media, and did not cope with more complex systems, such as thin films, layered systems, superlattices, photonic crystals and metamaterials [10]. For the same reason, it seemed incapable of dealing with spatial dispersion and screening at realistic surfaces [11–13].

Several alternatives to the derivation of Lifshitz have been proposed. Barash and Ginzburg [14] determined the allowed frequencies \( \omega_{\ell} \) of the cavity modes by solving Maxwell’s homogeneous equations and imposing planar boundary conditions. The energy could be obtained from the resulting density of states if there were no dissipation. Nevertheless, dissipation yields complex frequencies, and the interpretation of the solutions as normal modes loses meaning. Barash and Ginzburg overcame this problem by introducing an auxiliary non-dissipative system. The use of an auxiliary system was further developed by Kupiszewska [15] in a 1D calculation in which the problem of quantizing a dissipative system is attacked by accounting both for the dynamics of the vacuum modes and of the atomic dipoles to which they couple and which make up the material, together with a thermal reservoir in which the atomic radiators dissipate the absorbed energy. A disadvantage of this approach is that it requires an explicit microscopic model for the walls of the cavity and for the thermal bath, thus appearing to restrict its generality. A similar approach [16] was based on a Green’s function method and Kubo’s theorem. In both cases, the stress tensor is obtained from the vacuum modes with an explicit dependence on the dielectric response \( \epsilon(\omega) \). An alternative treatment of the Casimir force was introduced by Jaekel and Reynaud [17], who calculated the radiation pressure within a cavity bordered by partially transmitting, but lossless, mirrors. Each mirror was replaced by an infinitesimally thin scatterer characterized by a unitary, energy conserving scattering matrix. Their calculation was later generalized to the case of lossy optical cavities [18] by complementing the cavity modes with noise modes, in such a way that the total scattering matrix was unitary. The scattering matrix corresponding to the cavity modes was then obtained through the optical theorem.

Mochán et al. [19,20] have obtained an expression for the Casimir force using both the scattering approach and a dissipationless ancillary system. They have argued that in thermal
equilibrium, all of the properties of the radiation field within a cavity are completely determined by the optical reflection amplitudes of the walls. Thus, the Casimir force may be obtained from the stress tensor of any system whose reflection amplitudes are identical to those of the real system. A dissipationless fictitious system with those properties was conceived: It had infinitely thin walls characterized by a unitary scattering matrix whose elements correspond to the optical reflection amplitudes from within the cavity were chosen to be identical to those of the real system. The transmission amplitudes were chosen in such a way that the energy that was not reflected was transmitted without loss to the vacuum outside of the cavity. This permitted a full quantum mechanical calculation of the fields, even when the real system is dissipative. The field modes were quantized and counted by adding perfect mirrors far away from the walls of the real cavity. These quantizing mirrors produce a field that mimics the incoherent radiation back into the cavity that is responsible for maintaining a detailed balance, and thus the thermodynamic equilibrium in the case of lossy or dissipative real mirrors. The field that enters the cavity after being reflected by the quantizing mirrors has a very large frequency dependent phase that becomes infinitely large as the mirrors are moved infinitely far away.

The main result from the work mentioned above is that if Lifshitz formula is written in terms of the reflection coefficients of the walls of the cavity, or equivalently, in terms of their exact surface impedance [21, 22], it becomes applicable to any system with translational invariance along the surfaces and isotropy around their normal and not only to semi-infinite, homogeneous, local mirrors. Thus, it may be employed to calculate the Casimir force between semi-infinite or finite, homogeneous or layered, local or spatially dispersive, transparent or opaque, finite or semi-infinite systems. Through a simple substitution of the appropriate optical coefficients, the formalism has allowed the calculation of the Casimir force between photonic structures [23, 24], non-local excitonic semiconductors [25], non-local plasmon-supporting metals with sharp boundaries [20, 26], and between realistic spatially dispersive metals with a smooth self-consistent electronic density profile [19, 27]. The relative simplicity of the formalism has allowed its generalization to non-isotropic systems and the calculation of Casimir torques [28]. With a few modifications, it has also been employed for the calculation of other macroscopic forces, such as those due to electronic tunneling across an insulating gap separating two conductors [29].

Nevertheless, there was a shortcoming in the derivation of the Casimir force presented in Refs. 19 and 20, as it was uncritically assumed that a unitary, energy-conserving scattering matrix could be built through a proper choice of transmission coefficients. Somewhat surprisingly, it turns out to be impossible to find a unitary scattering matrix for evanescent waves, i.e., for $Q > \omega/c$, where $Q$ is the projection of the wavevector parallel to the cavity walls, $\omega$ is the frequency and $c$ is the speed of light, as a single transmitted evanescent wave is unable to transport energy away from the surface, while an incident and a reflected evanescent wave do transport energy from the cavity towards the surface of lossy and dissipative systems [30]. Although the contribution of evanescent waves to the Casimir force could be obtained as an analytic continuation from the region of propagating waves, it is not obvious a priori that this extrapolation would yield the correct result.

Recently [31], the problem of energy transport in the evanescent region was dealt with by modifying the fictitious system introduced in Refs. 19 and 20, in such a way that evanescent waves within the cavity couple to propagating waves outside the cavity. This was accomplished by completely filling the region beyond the infinitesimally thin mirrors with a dispersionless and dissipationless fictitious dielectric with a large permittivity $\epsilon_f$. In this way, the fictitious light cone $Q \leq \sqrt{\epsilon_f} \omega/c$ extends beyond the light cone $Q \leq \omega/c$ of the vacuum cavity and in the limit $\epsilon_f \to \infty$ all of the evanescent waves within the cavity would be able to couple to propagating waves within the fictitious region. Lifshitz formula was thus proven to be valid both within and beyond the light cone [31].

One drawback of the calculation presented in [31] is its use of an extremely unrealistic dielectric, with a suspiciously large, real, frequency independent dielectric constant $\epsilon \to \infty$. As the Casimir force ought to be determined by the reflection coefficients of the real mirrors [19, 20], all of the details of the fictitious system beyond the walls of the cavity ought to be superfluous after their contribution to the reflection amplitudes has been accounted for, and it should be possible to set up the calculation without the need of specifying them.

The purpose of the present paper is to develop yet another derivation of the Casimir force within cavities with walls made up of arbitrary materials characterized only by their optical reflection amplitudes. As in Refs. 19, 20, and 31, we introduce a dissipationless fictitious system with no degrees of freedom beyond those of the electromagnetic field and with a cavity whose walls have the same optical coefficients as the real system. However, unlike the calculations above, we avoid giving any detail of the fictitious system beyond the reasonable fact that it should be consistent with detailed balance so that thermodynamic equilibrium is satisfied, i.e., on the average, for each photon that is not coherently reflected at a cavity wall and is therefore either absorbed or transmitted beyond the system, an identical photon has to be incoherently injected back into the cavity with no phase relation to the lost photon. We believe that this derivation of the Casimir force is quite simple, and that it can be readily generalized to other geometries, allowing the calculation of the dispersion forces in cavities of varied shapes whose walls are made up of realistic materials.

The structure of the paper is the following: First, we briefly review the model employed in Ref. 31. In Sec. 2, we study the scattering matrix of a fictitious interface between vacuum and a dispersionless dielectric with an additional infinitesimally thin scatterer that forces the optical coefficients to be the same as those of the real system. In Sec. 3 we study the reflection amplitudes for both propagating and evanes-
cent waves after adding quantizing mirrors. Then, in Sec. 4 we eliminate the superfluos details from the calculation, keeping only the delay $T \to \infty$ before injecting back into the cavity the photons it loses through absorption or transmission in order to restore equilibrium. In Sec. 5, we obtain the electromagnetic normal modes of the cavity and the contributions of the cavity walls to the density of states which we employ in Sec. 6 to obtain their contribution to the thermodynamic properties. Finally, Sec. 7 is devoted to conclusions.

2. Energy flow

In Ref. 31, a fictitious system was introduced, consisting of a vacuum cavity bordered by two infinitesimally thin sheets followed by dispersionless dielectric slabs which are terminated by perfectly reflecting mirrors. It was argued that in equilibrium the electromagnetic field within the real cavity ought to coincide with the electromagnetic field within the fictitious cavity. The reflection amplitude of the infinitesimal sheet together with the dielectric was chosen to coincide with the reflection amplitude of the mirrors that make up the real cavity. The perfect mirrors were incorporated in order to quantize the normal modes and in order to inject back into the cavity any radiation that is not coherently reflected at the surface, guaranteeing thermodynamic equilibrium. The re-injected radiation acquires a large phase as reflected at the surface, guaranteeing thermodynamic equilibrium. In Sec. 5, we obtain the electromagnetic normal modes of the cavity and the contributions in order to quantize the normal modes and in order to restore equilibrium. In Sec. 6 to obtain their contribution to the thermodynamic properties.

Consider one of the mirrors of that fictitious cavity, as illustrated in Fig. 1, consisting of an infinitely thin reflector in a vacuum cavity and a dispersionless dielectric with permittivity $\epsilon_f$. Incoming $(i)$ and outgoing $(o)$ waves within vacuum $(v)$ and the dielectric $(d)$ are illustrated, as well as the projection of their wavevectors parallel $(Q)$ and normal $(k_v$ and $k_d)$ to the surface. The choice of coordinate axes is also shown.

Then $S$ should be unitary in the usual sense, i.e., $S^\dagger S = SS^\dagger = 1$, where $I$ is the unit matrix. We identify the components of $S$ as

\[ S = \begin{pmatrix} r_v & t_d \sqrt{k_v/k_d} \\ t_v \sqrt{k_d/k_v} & r_d \end{pmatrix}, \]

where $r_\alpha$ and $t_\alpha$ are the reflection and transmission amplitudes corresponding to incidence on the interface from medium $\alpha$. Unitarity then yields the relations

\[ |r_v|^2 + \frac{k_d}{k_v} |t_v|^2 = 1, \quad |r_d|^2 + \frac{k_v}{k_d} |t_d|^2 = 1, \]

\[ r_v^* t_d \sqrt{\frac{k_v}{k_d}} + r_d^* t_v \sqrt{\frac{k_d}{k_v}} = 0, \]

which imply

\[ R_v = R_d, \quad T_v = T_d, \quad R_v + T_v = R_d + T_d = 1, \]

where we identify as usual the reflectance ($R_v=|r_v|^2$, $R_d=|r_d|^2$) and transmittance ($T_v=(k_d/k_v)|t_v|^2$, $T_d=(k_v/k_d)|t_d|^2$) and we denote by $(\ldots)^*$ the complex conjugate of any quantity $(\ldots)$. Of course, Eqs. (6) are consistent with Fresnel relations [32]. However, we must remark that $r_\alpha$ and $t_\alpha$ above are not given by the Fresnel relations due to the presence of an additional scatterer at the interface, which forces the reflection amplitude from the vacuum side to agree with the reflection amplitude of a boundary of the real cavity [19, 20, 31].

In the case of evanescent waves, i.e., when $Q > \omega/c$ the normal component of the wavevector $k_v = i \kappa_v$ is imaginary and the energy flux is not given by Eq. (1) but by

\[ S_{vz} = \frac{e^2}{8 \pi \omega} \kappa_v (i_r^* o_v)'' \]

instead of Eq. (1), where we denote by $(\ldots)'$ and $(\ldots)''$, or equivalently, by $\text{Re}(\ldots)$ and $\text{Im}(\ldots)$ the real and imaginary parts of any quantity $(\ldots)$. If $Q$ were so large $Q > \sqrt{T \omega}/c$ so that waves were also evanescent in the dielectric, we would also have

\[ S_{dz} = \frac{e^2}{8 \pi \omega} \kappa_d (i_d o_d)'' \]

\[ x \quad o_v \quad i_v \quad k_v \quad -k_d \quad i_d \quad o_d \]

Vacuum: $\epsilon = 1$ I Dielectric: $\epsilon = \epsilon_f > 1$

Figure 1. Infinitely thin scattering interface (I) between vacuum and a dispersionless dielectric with permittivity $\epsilon_f$. Incoming $(i)$ and outgoing $(o)$ waves within vacuum $(v)$ and the dielectric $(d)$ are illustrated, as well as the projection of their wavevectors parallel $(Q)$ and normal $(k_v$ and $k_d)$ to the surface. The choice of coordinate axes is also shown.


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so that although \( r_v \) is complex, \( r_t \) is a real quantity. According to Eq. (8), this means there is no net energy flux towards the interface so that energy balance would be achieved within the vacuum cavity also for evanescent waves.

As argued in Refs. 20, 19, and 31, the properties of the electromagnetic field within a cavity made up of lossless mirrors such as those described in this section ought to agree with those within the real cavity. Thus, equations equivalent to (11) and (12) were employed in Ref. 31 to calculate the normal modes of the cavity, and from them, its thermodynamic properties, including the Casimir force. The purpose of this paper is to discard the unnecessary and dubious details of the fictitious system, such as the dispersionless dielectric with a large permittivity.

4. Delay

Equations (11) and (12) contain elements from the real system, such as the reflection amplitude \( r_v \) of the cavity walls. They also contain quantities that relate to fictitious quantities, such as the dielectric constant \( \epsilon_f \) of the dielectric, its width \( L_d \) and the reflection amplitude \( r_d \) for light impinging on the interface from the dielectric. To eliminate those quantities from the model, we first notice that Eqs. (11) and (12) contain terms which are relatively slowly varying functions of the frequency, such as the reflection amplitudes \( r_v \) of the real cavity walls. On the other hand, they contain extremely fast varying functions of the frequency such as \( e^{2ikdL_d} \), which are due to the long time taken by the field to transverse twice the width of the \( L_d \) dielectric after entering it across the interface in order to return after reflection in the perfect mirror. This long delay is precisely what allows the fictitious system to replenish those photons that are lost from the cavity with such a large phase that they mimic the incoherent thermal photons that would be radiated back into the cavity in the real lossy system. It is actually the essence of the fictitious cavity model: in the real cavity, photons that are not coherently reflected are lost through transmission or absorption at the walls; the cavity walls would radiate photons incoherently and inject them into the cavity to sustain thermodynamic equilibrium. In the fictitious system, photons are not lost, but they are delayed a time \( T = 2kdL_d/\omega \to \infty \) before they reach the cavity again, so they are essentially indistinguishable from incoherently radiated thermal photons. We can keep the delay \( T \) in the model, eliminating all other details by postulating that the reflection amplitude takes the form

\[
    r_t = \frac{r_v + ae^{i\omega T}}{1 + be^{i\omega T}},
\]

where \( a \) and \( b \) are slowly varying functions of the frequency to be determined. The term \( r_v \) in the numerator accounts for the coherent reflection of the real system. The term \( ae^{i\omega T} \) corresponds to re-radiation by the cavity walls or to thermal photons entering the system from outside to replenish the cavity losses, and it includes the long delay \( T \), which mimics incoherence. Finally, it may happen that such a photon
is reflected back from the interface into the walls and doesn’t enter the cavity on its first attempt. Thus, it could be absorbed and re-emitted some time later. The possibility of multiple re-injection attempts is accounted for by the term $be^{i\omega T}$ in the denominator, analogous to the denominator in Eq. (10).

In equilibrium, all the energy that leaves the cavity has to enter it again. Thus, for propagating waves, the total reflection amplitude must obey

$$|r_t|^2 = 1,$$  \hspace{1cm} (14)

which yields

$$|r_v|^2 + |a|^2 + 2\text{Re}(r_v^*ae^{i\omega T}) = 1 + |b|^2 + 2\text{Re}(be^{i\omega T})$$  \hspace{1cm} (15)

after substitution of Eq. (13). Separating the slowly and rapidly varying functions of $\omega$ in the previous equation, we obtain the two equations

$$b = r_v^*a, \quad |r_v|^2 + |a|^2 = 1 + |b|^2$$  \hspace{1cm} (16)

from which we obtain

$$a = e^{i\delta}, \quad b = r_v^*e^{i\delta},$$  \hspace{1cm} (17)

where the real phase $\delta = \delta(\omega)$ is some slowly varying function of $\omega$. Substituting Eqs. (17) in (13), we finally obtain

$$r_t = \frac{r_v + e^{i(\delta + \omega T)}}{1 + r_v^*e^{i(\delta + \omega T)}}.$$  \hspace{1cm} (18)

For evanescent waves, energy conservation requires

$$r''_t = 0$$  \hspace{1cm} (19)

instead of Eq. (14), which implies

$$\text{Im}[r_v + ab^* + (a - r_v^*)b]e^{i\omega T} = 0.$$  \hspace{1cm} (20)

As done above for Eq. (15), we separate fast- and slow- varying functions of $\omega$ to obtain

$$a = r_v^*b, \quad \text{Im}(r_v + ab^*) = 0,$$  \hspace{1cm} (21)

from which we obtain

$$b = e^{i\delta}, \quad a = r_v^*e^{i\delta},$$  \hspace{1cm} (22)

where $\delta$ is again a slowly varying function of the frequency. Substituting Eqs. (22) in (13) we finally obtain

$$r_t = \frac{r_v + r_v^*e^{i(\delta + \omega T)}}{1 + e^{i(\delta + \omega T)}} = \frac{2\text{Re}(r_v + r_v^*e^{i(\delta + \omega T)})}{|1 + e^{i(\delta + \omega T)}|^2}.$$  \hspace{1cm} (23)

Notice that Eqs. (11) and (12) have the form of Eqs. (18) and (23), respectively, where we identify $e^{i\delta} = r_v/r_d^*$ for propagating waves and $e^{i\delta} = r_d^*$ for evanescent waves, as $|r_v/r_d^*| = 1$ in the former case according to (6), and $|r_d| = 1$ in the latter case according to (9). However, we didn’t have to postulate any extraneous fictitious dielectric slab nor the reflection amplitude $r_d$ from within it. We will take the limit $T \to \infty$, and we will show that the unknown phase $\delta$ is irrelevant.

5. Normal modes

Given a real dispersive and dissipative system with a coherent reflection amplitude $r_v$, in the previous section we obtained an expression for the total reflection amplitude $r_t$ that accounts for the coherent reflection and mimics the incoherent reflection in thermodynamic equilibrium through terms that oscillate rapidly as a function of frequency. Consider now a planar cavity bordered by two arbitrary material slabs with reflection amplitudes $r_1$ and $r_2$ (Fig. 3).

For a given $\bar{Q}$ and $\omega$, the field within the cavity would be proportional to $A_r e^{ik_0z} + A_l e^{-ik_0z}$, with $A_r$ and $A_l$, the amplitudes of the right and left moving contributions to the field. Applying boundary conditions at $z_1(A_r e^{ik_0z_1} = r_1 A_r e^{-ik_0z_1})$ and at $z_2(A_l e^{-ik_0z_2} = r_2 A_l e^{-ik_0z_2})$ would yield the usual condition for the normal modes of the cavity,

$$D \equiv 1 - r_1 r_2 e^{2ik_0L} = 0.$$  \hspace{1cm} (24)

Due to the losses at the mirrors, we expect that after externally exciting any of the modes obtained from Eq. (24), it would decay in time (Fig. 3). Thus, for a given real $\bar{Q}$, Eq. (24) would yield a set of complex frequencies whose imaginary part is related to the finite lifetime of each mode. Nevertheless, at thermodynamic equilibrium, the energy of each mode would be replenished through re-radiation, so that the equilibrium modes would be stationary instead of decaying. We may obtain the stationary modes by replacing the reflection amplitudes $r_1$ and $r_2$ by the corresponding total reflection amplitudes $r_{1t}$ and $r_{2t}$, obtained through equations such as (18) and (23),

$$D_t \equiv 1 - r_{1t} r_{2t} e^{2ik_0L} = 0.$$  \hspace{1cm} (25)
As \( r_{1t} \) and \( r_{2t} \) are lossless, the frequencies \( \omega_\ell \) that solve Eq. (25) for any real value of \( Q \) are necessarily real. Thus, we can straightforwardly apply quantum mechanical methods to the modes derived from (25). In particular, the contribution of the \( \ell \)-th mode to the energy of the system is simply
\[ E_\ell = \left( \langle n_\ell \rangle + 1/2 \right) h \omega_\ell, \]
where \( \langle n_\ell \rangle \) is the average occupation number of the mode.

For propagating waves \( |r_{1e}| = |r_{2e}| = 1 \) (Eq. (14)), so that Eq. (25) may be recast as
\[ \arg(r_{1t} r_{2e} e^{2i k_e L}) = 2 \pi \ell \quad (26) \]
with \( \ell \) an integer. Rewriting (18) as
\[ r_{\ell t} = \frac{(1 + r_e e^{-i(\delta_1 + \omega T)} + 2 r_e e^{-i(\delta_2 + \omega T)} + \omega T + \delta + 2 k_e L = 2 \pi \ell , \quad (27) \]
where we introduce the subindex \( i = 1, 2 \) to denote each of the two mirrors, Eq. (26) becomes

\[ 2 \arg(1 + r_e e^{-i(\delta_1 + \omega T)}) + 2 \arg(1 + r_e e^{-i(\delta_2 + \omega T)}) \]
\[ + \omega T + \delta + 2 k_e L = 2 \pi \ell , \quad (28) \]
where \( T = T_1 + T_2 \) and \( \delta = \delta_1 + \delta_2 \). Notice that as \( \omega \) varies, \( 1 + r_e e^{-i(\delta_1 + \omega T)} \) moves counterclockwise in the complex plane around a circle centered at 1 whose radius \( |r_e| < 1 \). Thus, it does not encircle the origin and its contribution to the phase in (28) is bounded. This happens with the second term in (28). Thus, the eigenfrequencies \( \omega_\ell \) never get far away from the eigenfrequencies
\[ \omega_{00} = (2 \pi \ell - 2 k_e L - \delta) / T \quad (29) \]
corresponding to vacuum, for which \( r_1 = r_2 = 0 \). Notice that as \( T \to \infty \), successive frequencies approach each other, and the number of states diverges.

Consider now a small frequency range \( \Omega \) of size \( \Delta \omega \), centered at a given frequency \( \omega \), where \( \Delta \omega \) is much smaller than any characteristic frequency of the system. Nevertheless, as we will take the limit \( T_1, T_2 \to \infty \), we may assume that \( \Delta \omega T \gg 1 \), so that the number \( N(\Omega) \) of normal modes \( \omega_\ell \) within \( \Omega \) is large \( N(\Omega) \gg 1 \). Using Cauchy’s argument principle, we obtain
\[ N(\Omega) = \frac{1}{2 \pi i} \int_{\gamma} \frac{d}{d \omega} \log f(\omega), \quad (30) \]
where \( \gamma \) is a clockwise closed path that encircles \( \Omega \) and
\[ f(\omega) = (1 + \rho_1 e^{i(\delta_1 + \omega T)}) (1 + \rho_2 e^{i(\delta_2 + \omega T)}) \]
\[ - (r_1 + e^{i(\delta_1 + \omega T)}) (r_2 + e^{i(\delta_2 + \omega T)} e^{2i k_e L}) \quad (31) \]
has within \( \gamma \) the same zeroes as \( D_\ell \) (Eq. (25)) and no poles. Here, we introduced linearized and therefore analytical approximations \( \rho_1, \rho_2 \) to \( r_1, r_2 \), taking advantage of the smallness of \( \Delta \omega \), so that \( f \) is an analytical function even if \( D_\ell \) is not. Choosing \( \alpha \) as a path that goes from \( \omega \) to \( -\Delta \omega / 2 \) to \( \omega + \Delta \omega / 2 \) a small distance \( \eta \to 0 \) below the real axis and returns a distance \( \eta \) above the real axis, we can rewrite Eq. (30) as
\[ N(\Omega) = \frac{1}{2 \pi i} \int_{\omega + \Delta \omega / 2}^{\omega - \Delta \omega / 2} d \omega \frac{d}{d \omega} \log \left( f(\omega - i \eta) f(\omega + i \eta) \right). \quad (32) \]

Approaching the limit \( T_1, T_2 \to \infty \), so that the limit \( \eta \to 0 \), we may assume that \( \eta T_1, \eta T_2 \gg 1 \) even if \( \eta \ll \Delta \omega \), so that \( f(\omega - i \eta) \to e^{i(\delta + \omega T)} + \eta T e^{i(\delta + \omega T)} \) and \( f(\omega + i \eta) \to 1 - r_1 r_2 e^{2i k_e L} \). Subtracting the number of modes \( N_0(\Omega) \) corresponding to vacuum, which may be obtained from Eq. (32) by replacing \( r_1 \) and \( r_2 \) by zero, we obtain the contribution \( \Delta N(\Omega) \) of the cavity walls to the number of modes
\[ \Delta N(\Omega) = \frac{1}{\pi} \int_{\omega - \Delta \omega / 2}^{\omega + \Delta \omega / 2} d \omega \Im \frac{d}{d \omega} \log(1 - r_1 r_2 e^{2i k_e L}). \quad (33) \]
Notice that all the terms in Eq. (33) are slowly varying and, as we chose a very small \( \Delta \omega \), the integral becomes trivial. Dividing the result by \( \Delta \omega \), we obtain the contribution of the walls to the density of states
\[ \rho = \frac{1}{\pi} \int_{\omega - \Delta \omega / 2}^{\omega + \Delta \omega / 2} d \omega \Im \frac{d}{d \omega} \log(1 - r_1 r_2 e^{2i k_e L}). \quad (34) \]

A similar procedure may be employed for evanescent waves by substituting Eq. (23) instead of (18) in (25). The number of modes within the frequency range \( \Omega \) is again given by Eq. (30), but choosing
\[ f(\omega) = (1 + e^{i(\delta_1 + \omega T)}) (1 + e^{i(\delta_2 + \omega T)}) \]
\[ - (r_1 + \rho_1 e^{i(\delta_1 + \omega T)}) (r_2 + \rho_2 e^{i(\delta_2 + \omega T)} e^{2i k_e L}), \quad (35) \]
where we took \( k_e = i k_e \). Now we substitute \( f(\omega - i \eta) \to e^{i(\delta + \omega T)} + \eta T e^{i(\delta + \omega T)} \) and \( f(\omega + i \eta) \to 1 - r_1 r_2 e^{-2i k_e L} \) in Eq. (30) and subtract from the resulting expression the number of modes corresponding to vacuum. The result is again given by Eq. (33), so that the contribution of the walls to the density of states for evanescent waves is also given by Eq. (34), which is therefore valid both in the propagating (\( \omega > Q/c \)) and in the evanescent (\( \omega < Q/c \)) sectors.

Finally, from Eq. (34) we can obtain an expression for the contribution of the walls to the average of any real quantity \( W_\mu(Q, \omega) \), namely
\[ \langle W \rangle = -A \Im \sum_{\mu} \frac{1}{4 \pi^3} \int d^2 Q \]
\[ \times \int d \omega W_\mu(Q, \omega) \frac{d}{d \omega} \log(1 - \zeta_\mu^{-1}), \quad (36) \]
where
\[ \zeta_\mu = (r_{\mu 1} r_{\mu 2} e^{2i k_e L})^{-1}, \quad (37) \]
we incorporated the fact that light has two independent polarizations \( \mu = s, p \) over which we summed, and we performed the usual sum over parallel wavevectors \( \sum \vec{q} \ldots \rightarrow A/(4\pi^2) \int d^2Q \ldots \) where \( A \) is the area of the mirrors.

6. Thermodynamic quantities

As simple applications of Eq. (36) we calculate the ground state energy

\[
U_0 = \langle \hbar \omega /2 \rangle = -A \text{Im} \sum_{\mu} \frac{\hbar}{8\pi^2} \int d^2Q \\
\times \int d\omega \frac{d}{d\omega} \log(1 - \zeta_\mu^{-1}). \tag{38}
\]

Performing the angular part of the wavevector integration and integrating by parts over frequency we obtain

\[
U_0 = A \text{Im} \frac{\hbar}{4\pi^2} \int dQ Q \\
\times \int d\omega \log[(1 - \zeta_s^{-1})(1 - \zeta_p^{-1})]. \tag{39}
\]

Similarly, the internal energy at finite temperature

\[
U = \langle \hbar \omega g \rangle = A \text{Im} \frac{\hbar}{2\pi^2} \int dQ Q \\
\times \int d\omega g \frac{d}{d\omega} \log[(1 - \zeta_s^{-1})(1 - \zeta_p^{-1})], \tag{40}
\]

where

\[
g - \frac{1}{2} = \frac{1}{2} \coth \left( \frac{\beta \hbar \omega}{2} \right) - \frac{1}{2} \tag{41}
\]

is the occupation number of a state with frequency \( \omega \) at temperature \( 1/k_B\beta \), with \( k_B \) Boltzmann’s constant. The free energy

\[
F = -\langle \log z \rangle / \beta = A \frac{\hbar}{2\pi^2} \text{Im} \int dQ Q \\
\times \int d\omega g \log[(1 - \zeta_s^{-1})(1 - \zeta_p^{-1})], \tag{42}
\]

where

\[
z = \frac{1}{2} \text{csch} \left( \frac{\beta \hbar \omega}{2} \right) \tag{43}
\]

is the partition function of a single mode of frequency \( \omega \). From (42) and (40), we may obtain the entropy

\[
S = A \frac{\hbar k_B\beta}{2\pi^2} \text{Im} \int dQ Q \\
\times \int d\omega \left( \omega \frac{d}{d\omega} g \right) \log[(1 - \zeta_s^{-1})(1 - \zeta_p^{-1})]. \tag{44}
\]

Finally, deriving the free energy with respect to \( L \), we may obtain the Casimir force

\[
F = A \frac{\hbar}{2\pi^2} \text{Re} \int dQ Q \\
\times \int d\omega g k_v \left( \frac{1}{\zeta_s - 1} + \frac{1}{\zeta_p - 1} \right), \tag{45}
\]

which agrees with Lifshitz formula when written in terms of the reflection amplitudes. For the actual evaluation of the frequency integrals in Eqs. (39)–(45), the integration path can be conveniently deformed from the real into the imaginary axis, yielding the usual Matsubara summations.

7. Conclusions

We have obtained expressions for the thermodynamic properties of a cavity formed by two flat mirrors. We derived our expressions without making any assumption about the mirrors, which were completely characterized by their reflection amplitudes \( r_1 \) and \( r_2 \). The mirrors could have been conducting or dielectric, opaque or transparent, dispersive or non-dispersive, lossless or dissipative, semi-infinite or finite, homogeneous or layered, with abrupt or smooth boundaries, local or spatially dispersive, etc. To derive our results, we postulated a fictitious system which has the same reflection amplitudes as the real system, but such that any energy that leaves the cavity eventually comes back but after a very long delay which is taken to infinity. Thus, we mimic in a closed system the incoherent field radiated by the walls of the cavity and the thermal radiation of the environment, which replenishes the energy lost by the cavity in thermal equilibrium. The requirement of equilibrium allowed us to find total, lossless reflection amplitudes for the fictitious system, which allowed a full quantum mechanical description of the system, from which we identified and counted the normal modes, obtaining an expression for the contribution of the walls of the cavity to the density of states of the system. With the density of states, obtaining expressions for all of the thermodynamic quantities becomes a straightforward task. The expressions we obtained agree with those found in the literature, although our derivation shows they are more general than implied by most other derivations. We believe that the procedure is simple enough to be easily extended to non-planar cavities made up of arbitrary materials.

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